The McKay correspondence, mutation and related topics.

July 17, 2020

"On the G-Hilbert scheme of the closure of the regular nilpotent orbit of type A"

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Summary

- Nilpotent orbits in $\mathfrak{sl}_n = \mathrm{Lie}(SL_n)$.
- The G-Hilbert scheme of the Cox realization
- The Springer's resolution via the Hilbert-Chow morphism

Nilpotent orbits in $\mathfrak{sl}_n$

Nilpotent orbit = Adjoint orbit of a nilpotent element

Example. ($n = 2$)

- Nilpotent elements are conjugate to

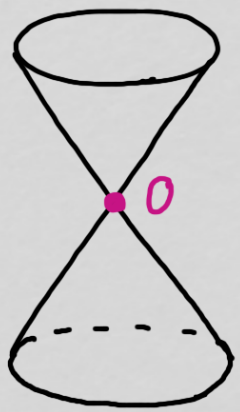
$$A_{[2]} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad A_{[1,1]} = 0$$

- $\mathcal{O}_{[2]} = SL_2 \cdot A_{[2]}$, $\mathcal{O}_{[1,1]} = \{0\}$

  ↑ regular nilpotent orbit.

- $$\mathfrak{sl}_2 \supset \overline{\mathcal{O}_{[2]}} = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} : x^2 + yz = 0 \right\}$$

$$\overline{\mathcal{O}_{[2]}} = \mathcal{O}_{[2]} \cup \mathcal{O}_{[1,1]}$$



A_1 - singularity

There is a one to one correspondence ([Collingwood-McGovern, 1993]):

$$\{\text{Nilpotent orbits in } \mathfrak{sl}_n\} \overset{1:1}{\longleftrightarrow} \{\text{Partitions of } n\}.$$

$$\mathcal{O}_{\mathbf{d}} := SL_n \cdot A_{\mathbf{d}}, \quad \longleftarrow \quad \mathbf{d} = [d_1, \cdots, d_k]$$

where

- $A_{\mathbf{d}} := \begin{pmatrix} J_{d_1} & & 0 \\ & \ddots & \\ 0 & & J_{d_k} \end{pmatrix}$,

$d_1 + \cdots + d_k = n$

i.e. $d_1 \geq \cdots \geq d_k \geq 1$

- $J_{d_i} := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$

Jordan block of size d_i

Remark.

① $\#\{\text{Nilpotent orbits in } \mathfrak{sl}_n\} < \infty$

② An order relation is defined on the set of nilpotent orbits by the inclusion of closures.

③ The maximal nilpotent orbit is called the regular nilpotent orbit.

corresponds to $\mathbf{d} = [n]$

Regular nilpotent orbit.

- $\mathcal{O} := \mathcal{O}_{[n]} = SL_n \cdot \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}$
- $\overline{\mathcal{O}} = \bigcup_{\lambda} \mathcal{O}_{\lambda} \subset \mathfrak{sl}_n$
- $\mathrm{Sing}(\overline{\mathcal{O}}) = \overline{\mathcal{O}} \setminus \mathcal{O} = \bigcup_{\lambda \neq [n]} \mathcal{O}_{\lambda}$

Notations.

- $B = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \in SL_n \right\}$
- $\mathfrak{n} = \left\{ \begin{pmatrix} 0 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ & & & 0 \end{pmatrix} \in \mathfrak{sl}_n \right\}$
- $SL_n \times^B \mathfrak{n} := (SL_n \times \mathfrak{n})/B$,
  where $b \cdot (g, A) = (gb^{-1}, bAb^{-1})$

Springer's resolution

Thm (Fu 2003, Fu-Namikawa 2004)

(i) $\exists!\ \tilde{Y} \to Y = \overline{\mathcal{O}}$ symplectic resol.

(ii) $\tilde{Y} \simeq SL_n \times^B \mathfrak{n}$

Remark

① For $\overline{\mathcal{O}}$, a resol. of singularities is symplectic $\Leftrightarrow$ crepant.

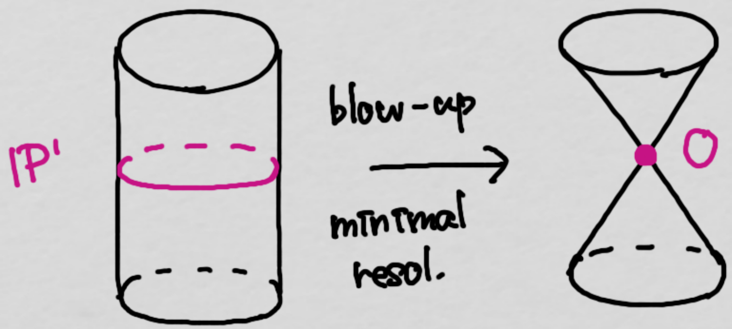
② $SL_n \times^B \mathfrak{n} \longrightarrow \overline{\mathcal{O}}$ is given by
$(g, A) \longmapsto gAg^{-1}$

Today Construct the Springer's resolution as a G-Hilbert scheme.

i.e. $\exists\, G \curvearrowright X$ s.t. $\gamma : G\text{-Hilb}(X) \to X/G$
finite g.p

$$\begin{array}{ccc} G\text{-Hilb}(X) & \xrightarrow{\gamma} & X/G \\ \wr\!\mid & & \wr\!\mid \\ SL_n \times^B \mathfrak{n} & \to & \overline{\mathcal{O}} \end{array}$$

Case $n=2$ $\overline{\mathcal{O}} = (x^2 + yz = 0) \subset \mathbb{C}^3$



$$Bl_0(\overline{\mathcal{O}}) \longrightarrow \overline{\mathcal{O}}$$

[Ito-Nakamura, 1996]

$$\begin{array}{ccc} Bl_0(\overline{\mathcal{O}}) & \longrightarrow & \overline{\mathcal{O}} \\ \wr\!\mid & & \wr\!\mid \\ \mu_2\text{-Hilb}(\mathbb{C}^2) & \xrightarrow{\gamma} & \mathbb{C}^2/\mu_2 \\ \cup & & \Downarrow \\ \mathbb{P}^1 \simeq \gamma^{-1}(0) & & 0 \end{array}$$

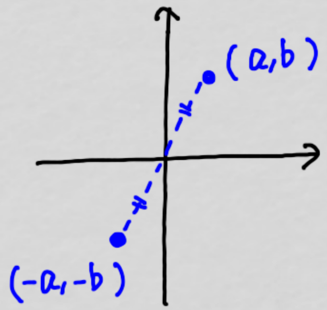
μ_2-orbits space

$$\| $$

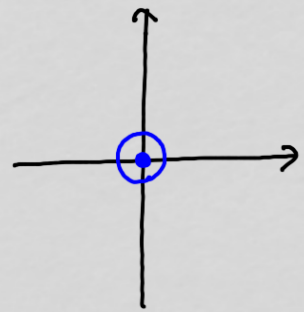
$$\{ I = (bt_0 - at_1, t_0^2, t_0t_1, t_1^2) \mid [a:b] \in \mathbb{P}^1 \}$$

μ_2 - orbits.

① $(a,b) \neq 0$



② $(a,b) = 0$



$\boxed{\text{Therefore, } n=2 \Rightarrow \gamma = \text{Springer's resol.}}$

Question What if $n \geq 3$?

Observation. When $n=2$,

1. $\gamma : \mu_2\text{-Hilb}(\mathbb{C}^2) \longrightarrow \mathbb{C}^2/\mu_2$
$\quad\quad \rotatebox{90}{$\supseteq$} \quad\quad\quad\quad\quad \rotatebox{90}{$\supseteq$}$
$\quad SL_2 \times^B \mathfrak{n} \longrightarrow \overline{\mathcal{O}}$

   i.e. $\mathrm{Cox}(\overline{\mathcal{O}}) \simeq \mathbb{C}[t_0, t_1]$ ↺ $\mathrm{Spec}(\mathbb{C}[\mathrm{Cl}(\overline{\mathcal{O}})]) \simeq \mu_2$

$$\begin{cases} t_0 \mapsto -t_0 \\ t_1 \mapsto -t_1 \end{cases}$$

2. $\mathbb{C}^2 \to \mathbb{C}^2/\mu_2 \simeq \overline{\mathcal{O}}$ is the Cox realization of $\overline{\mathcal{O}}$

Question If we put $G = \mathrm{Spec}(\mathbb{C}[\mathrm{Cl}(\overline{\mathcal{O}})])$, $X = \mathrm{Spec}(\mathrm{Cox}(\overline{\mathcal{O}}))$,

is $G\text{-Hilb}(X) \simeq SL_n \times^B \mathfrak{n}$?

$\boxed{\text{Thm (K, 2020) Question is true for } \forall n.}$

Generalize the framework.

$\rightsquigarrow$ Question on

"The invariant Hilbert scheme of the Cox realization."

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Def (Alexeev-Brion, 2005 for G: connected, Brion 2013 for $\forall G$).

Let G: a red. alg. group, X: an affine G-var., h: Hilbert function.

$h: \mathrm{Irr}(G) \to \mathbb{Z}_{\geq 0}$
the set of isom. classes of irred. representations of G.

- The invariant Hilbert scheme associated with (G, X, h) is:

$$\mathrm{Hilb}^G_h(X) = \{ \underset{\text{closed } G\text{-subsch.}}{Z \subset X} \mid \mathbb{C}[Z] \simeq \bigoplus_{M \in \mathrm{Irr}(G)} M^{\oplus h(M)} \text{ as } G\text{-modules} \}$$

- For a well-chosen h, $\exists$ analogue of the Hilbert-Chow morphism:

$$\gamma: \mathrm{Hilb}^G_h(X) \to X/\!/G = \mathrm{Spec}(\mathbb{C}[X]^G), \ [Z] \mapsto Z/\!/G.$$

Remark. ① The invariant Hilbert scheme is a generalization of the G-Hilbert scheme:

$$\begin{cases} \cdot \ \# G < \infty, \\ \cdot \ h = \text{Hilbert function of the regular representation} \end{cases}$$

i.e. $\mathbb{C}[G] \simeq \bigoplus_{M \in \mathrm{Irr}(G)} M^{\oplus \dim M}$.
$h(M) = \dim M$.

$\Rightarrow \mathrm{Hilb}^G_h(X) = G\text{-}\mathrm{Hilb}(X)$.

② γ is a candidate for a resol. of sing. of $X/\!/G = Y$.
$\rightsquigarrow$ depends on the quotient description of Y!

$\rightsquigarrow$ Question What would be a "special" choice for a quotient construction of Y?

Cox ring

Let Y be a normal var.

s.t. $\begin{cases} \cdot\ Cl(Y) \text{ is fin. gen.} \\ \cdot\ \mathbb{C}(Y)^* = \mathbb{C}^* \end{cases}$ —— $(\star)$.

The Cox ring of Y is

$$Cox(Y) = \bigoplus_{[D] \in Cl(Y)} \Gamma(Y, \mathcal{O}_Y(D))$$

(!) $Cox(Y)$ is a $Cl(Y)$-graded $\Gamma(\mathcal{O}_Y)$-mod.

(!) Multiplication on $Cox(Y)$ is defined by that of rational functions under the following identification:

$\Gamma(Y, \mathcal{O}_Y(D)) = \{ f \in \mathbb{C}(Y) \mid D + (f) \geq 0 \}$.

Example $Y = \mathbb{P}^n$, $Cl(Y) \simeq \mathbb{Z}$

$$Cox(Y) \simeq \mathbb{C}[x_0, \cdots, x_n] = \bigoplus_{d \in \mathbb{Z}} \mathbb{C}[x_0, \cdots, x_n]_d$$

Cox realization

Let $\begin{cases} \cdot\ X = Spec(Cox(Y)) \\ \cdot\ G = Spec(\mathbb{C}[Cl(Y)]) \end{cases}$

Then, $G \curvearrowright X$ by the $Cl(Y)$-grading.

(e.g. $Cl(Y) \simeq \mathbb{Z}^m \oplus \mathbb{Z}/m_1 \oplus \cdots \oplus \mathbb{Z}/m_r$
$\Rightarrow G \simeq (\mathbb{C}^*)^m \times \mu_{m_1} \times \cdots \times \mu_{m_r}$)

Thm (Arzhantsev-Derenthal-Hausen-Laface, 2015).
Y: affine $\Rightarrow X /\!/ G \simeq Y$.

Def (Arzhantsev-Gaifullin, 2010)
If Y is affine,
$$\pi : X \to X /\!/ G \simeq Y$$
is called the Cox realization of Y.

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The invariant Hilbert scheme of the Cox realization.

Let

$$\begin{cases} \bullet\ Y \text{ a normal var. satisfying } (\star). \\ \bullet\ X \to X/\!/G \text{ the Cox realization of } Y \\ \bullet\ \gamma : \mathrm{Hilb}^G_h(X) \to X/\!/G \simeq Y. \end{cases}$$

We refer to $\mathrm{Hilb}^G_h(X)$ as the invariant Hilbert scheme of the Cox realization of Y

Question (K).

Does γ give a resolution of singularities of Y ?

Consider the question for the case $Y = \overline{\mathcal{O}}$.

Main result

Setting

$$\begin{cases} \bullet\ Y = \overline{\mathcal{O}} = \bigcup_{d|} \mathcal{O}_{d|} \subset \mathfrak{sl}_n \\ \quad \Rightarrow \mathrm{Cl}(Y) \simeq \mathbb{Z}/n \text{ by [Fu, 2003]} \\ \bullet\ X = \mathrm{Spec}(\mathrm{Cox}(Y)) \\ \bullet\ G \simeq \mu_n \end{cases}$$

$$\rightsquigarrow \begin{array}{ccc} \mu_n\text{-}\mathrm{Hilb}(X) & \xrightarrow{\gamma} & X/\mu_n \simeq \overline{\mathcal{O}} \\ \cup & & \cup \\ \gamma^{-1}(\mathcal{O}) & \simeq & \mathcal{O} = SL_n \cdot A_{[n]} \\ \Downarrow & & \Downarrow \\ \exists!\, [I_{[n]}] & \mapsto & A_{[n]} \end{array}$$

Main Theorem

γ coincides with the Springer's resol.

i.e. $\mu_n\text{-}\mathrm{Hilb}(X) \simeq SL_n \times^B \mathfrak{n}$.

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Main result

Setting

$$\begin{cases} \bullet\ Y = \overline{\mathcal{O}} = \bigcup_{d|l} \mathcal{O}_{d l} \subset \mathfrak{sl}_n \\ \quad \Rightarrow Cl(Y) \simeq \mathbb{Z}/n \text{ by [Fu, 2003]} \\ \bullet\ X = \mathrm{Spec}(\mathrm{Cox}(Y)) \\ \bullet\ G \simeq \mu_n \end{cases}$$

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Sketch of proof

STEP 1. Calculate $\mathrm{Cox}(\overline{\mathcal{O}}) =: R = \bigoplus_{d \in \mathbb{Z}/n} R_d$.

1-(1) By [Graham, 1992],

$$R \simeq \mathbb{C}[SL_n]^U \otimes R_0$$
$$(U \subset B,\ R_0 \simeq \mathbb{C}[\overline{\mathcal{O}}])$$

1-(2) [Grosshans, 1997] gives explicit generators of $\mathbb{C}[SL_n]^U$ in terms of "std. Young bitableaux"

STEP 2. Construct an equiv. morphism (Use STEP 1 & descriptions of the generators of $I_{[n]}$)

$$\begin{array}{ccccc} \eta : \mu_n\text{-Hilb}(X) & \twoheadrightarrow & SL_n/B & \hookrightarrow & \prod_{1 \le d \le n-1} \mathbb{P}(\Lambda^d (\mathbb{C}^n)^\vee) \\ \cup & & \Downarrow & & \\ N := \eta^{-1}(\mathbb{1}) & & \mathbb{1} & & \end{array}$$

$$\rightsquigarrow \mu_n\text{-Hilb}(X) \simeq SL_n \times^B N$$

Main result

Setting

$$\begin{cases} \bullet\ Y = \overline{\mathcal{O}} = \bigcup_{\mathrm{dl}} \mathcal{O}_{\mathrm{dl}} \subset \mathfrak{sl}_n \\ \quad \Rightarrow \mathrm{Cl}(Y) \simeq \mathbb{Z}/n \text{ by [Fu, 2003]} \\ \bullet\ X = \mathrm{Spec}(\mathrm{Cox}(Y)) \\ \bullet\ G \simeq \mu_n \end{cases}$$

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Main Theorem

γ coincides with the Springer's resol.

i.e. $\mu_n\text{-Hilb}(X) \simeq SL_n \times^B \mathfrak{n}$.

Sketch of proof

STEP 3 Show $N \simeq \mathfrak{n}$

(!) $\mathfrak{n} = \left\{ \begin{pmatrix} 0 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ & & & 0 \end{pmatrix} \right\} = \bigcup_{\mathrm{dl}} B \cdot A_{\mathrm{dl}}$

(!) $N = \bigcup_{\mathrm{dl}} \big(\underbrace{N \cap \gamma^{-1}(\mathcal{O}_{\mathrm{dl}})} \big)$

↑ may be a union of more than one B-orbit.

WTS: $\exists\ I_{\mathrm{dl}} \subset R$ ideal

s.t. • $\gamma([I_{\mathrm{dl}}]) = A_{\mathrm{dl}}$

• $N \cap \gamma^{-1}(\mathcal{O}_{\mathrm{dl}}) = B \cdot [I_{\mathrm{dl}}]$

3-(1) Find a "suitable" candidate for I_{dl} by using STEP 1 & STEP 2

3-(2). Show "="

($\supset$) easy

($\subset$) Use relations among generators of R.

Conclusion

$Y = \overline{\mathcal{O}}$ gives a positive answer to

The invariant Hilbert scheme of the Cox realization.

Let

- Y a normal var. satisfying $(\star)$.
- $X \to X/\!\!/G$ the Cox realization of Y
- $\gamma : \mathrm{Hilb}^G_h(X) \to X/\!\!/G \simeq Y$.

We refer to $\mathrm{Hilb}^G_h(X)$ the invariant Hilbert scheme of the Cox realization of Y

Question $(\star)$.
Does γ give a resolution of singularities of Y ?

Consider the question for the case $Y = \overline{\mathcal{O}}$.