01 Kavli IPMU lecture: **Title page**

Saturday, 1 August 2020 09:05

Trihedral groups: old and new ideas and methods (with many problems still to solve)

Section 1. What groups are we talking about?
G = A x) T with A diag in SL(3) and T the 3-cycle (x,y,z)
Section 2. Affine pieces of G-Hilb corr. to combinatorics (a) Leng partitions (b) trihedral boats
Section 3. (a) and (b) by computer algebra: Running my Magma code is a fun, easy do-it-yourself game: follow the instructions on
https:/homepages.warwick.ac.uk/~masda/McKay/tri
Section 4. Compute some affine pieces in the baby case A = V4 = mu2 + mu2 by generators and relations (or as quiver moduli, somewhat implicitly).
My title says ideas, methods and open problems, not results or theorems.

02 Figure 1: the Trihedral Plane

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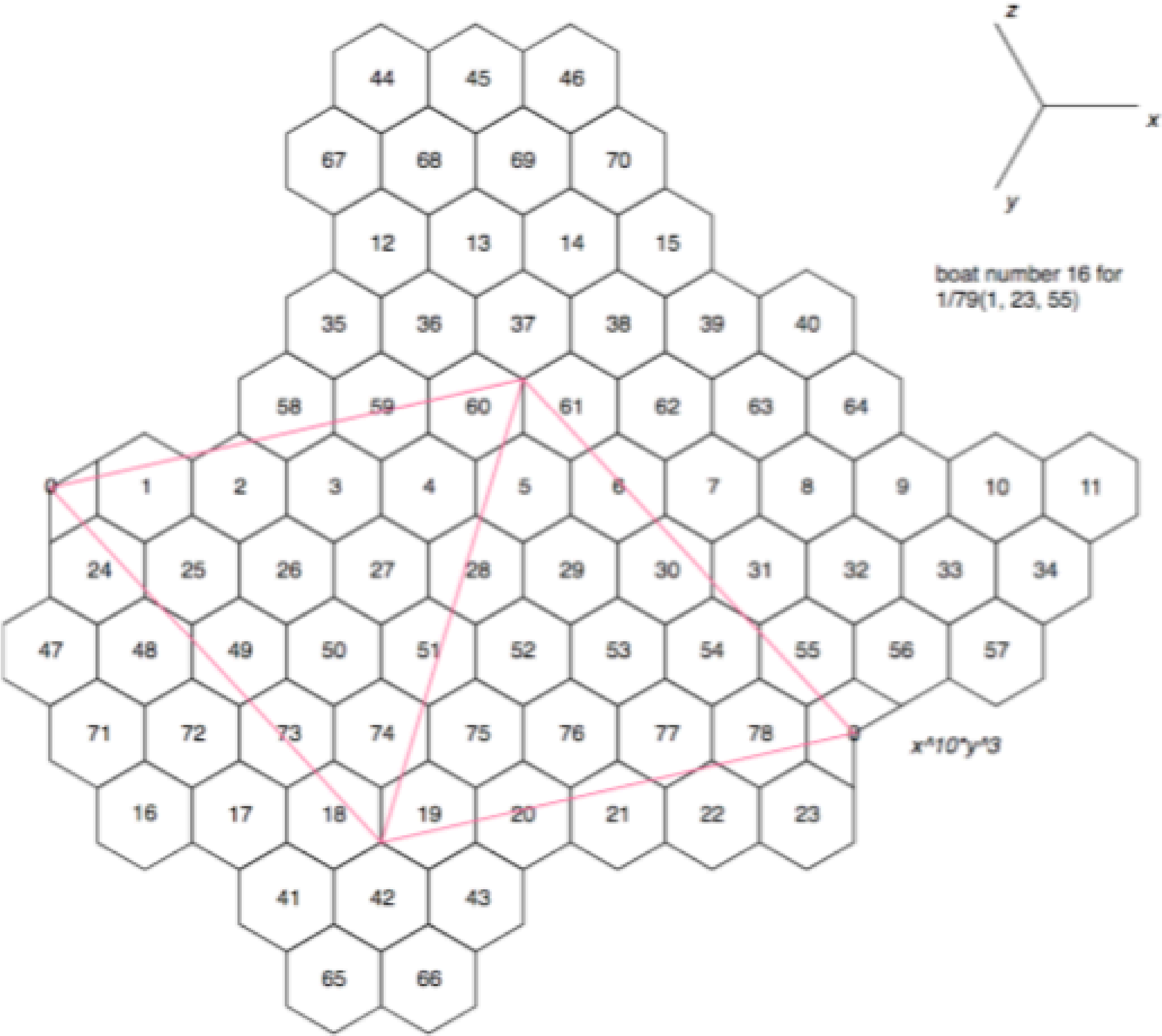
Picture in trihedral plane illustrates almost every point of my argument



03 Part 1. What groups am I talking about?

Saturday, 1 August 2020 09:22

Trihedral refers to the 3-cycle T = (x,y,z) or permutation matrix in SL(3,CC).
Trihedral group in SL(3,CC) has a diagonal subgroup A normalised by T, then G = A x) T.
(1) If A is cyclic it is = 1/r(1,s,s^2) with r | 1+s+s^2.
(See (a,b) below for better treatment).
(2) Cyclic group A1 normalised by T has **inflation** An = { diag g | g^n in A1} in SL(3,CC), which is A1 extended by mu_n + mu_n.
Inflation refines the hex lattice of Figure 1 to its regular tesselation by 1/n size hexagons.

04 Trihedral plane and the (a,b) form

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More discussion: (1) A cyclic group 1/r(1,s,s^2) relates to coprime (a,b) starting from trihedral matrix [a b \\ a-b a]. Two steps:

Get r as the **determinant** r = det [a b \\ a-b a] = a^2 - a*b + b^2.

Get s as the **kernel** mod r, that is: a*s + b == (a-b)*s + a == 0 mod r.

Exc. These formulae imply r | 1+s+s^2.

The **trihedral plane** is RR^3/RR*(1,1,1) with the Euclidean metric:

x^2 - x*y + y^2

- x*z - y*z   In the (x,y) triant, (x-1/2)^2 + (sqrt3/2*y)^2

z^2   with 1/2, sqrt3/2 = cos, sin 60^deg.

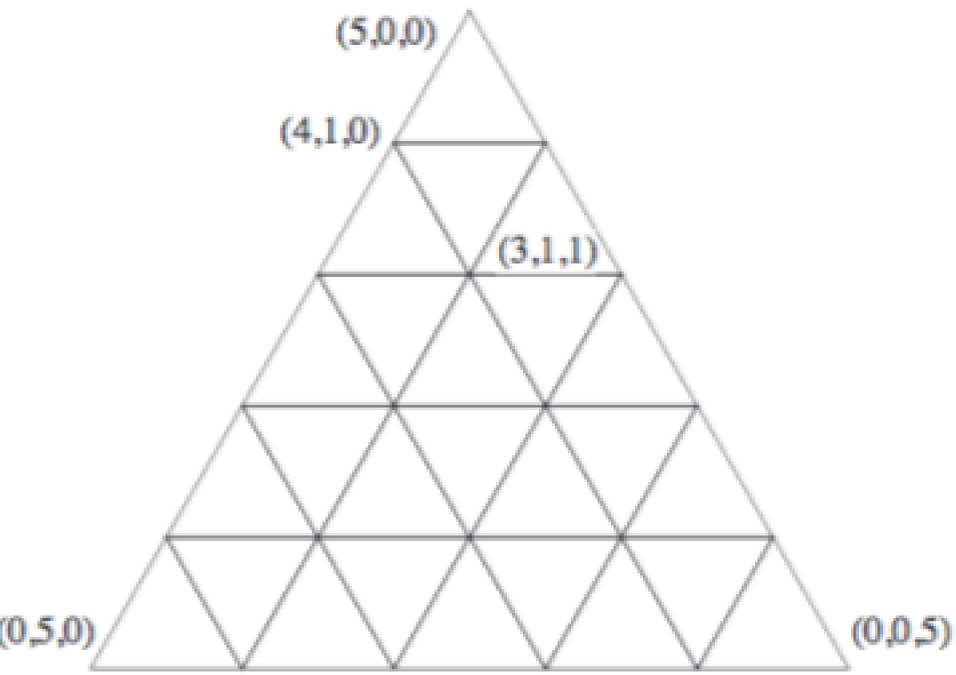
The lattice of invariant monomials has **shortest generators** x^a*y^b and x^(a-b)*y^a with length^2 = r. Triangle < 1, x^a*y^b, x^(a-b)*y^a > has 3 equal sides.

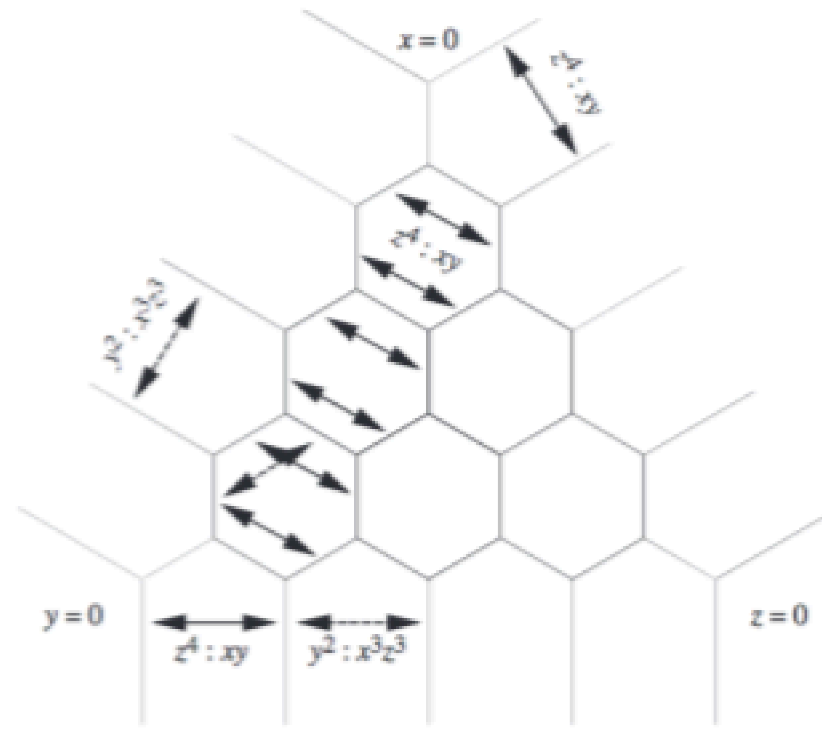
Exc. For Figure 1, check a=10, b=3 gives 1/79(1,23,55) and
10*1 + 23*3 == 7*1 + 23*10 == 0 mod 79.

05 Inflated G

Saturday, 1 August 2020 10:04

Inflation (2) is already interesting in trivial case A1 = (1). Then
An = mu_n x mu_n in SL(3, CC) or 1/n(1,-1,0) + 1/n(0,1,-1).
Then CC^3/An is the hypersurface t^n = x*y*z, and has crepant resolution the regular subdivision of the junior simplex.





This case already featured in the first papers on mirror symmetry.
The group An is obviously normalised by T, which gives the trihedral group G = An x) T. Computing its G-Hilb is already challenging for n = 2. I go through some of this in the final Part 4, illustrating some features of calculating G-Hilb beyond the combinatorics.

06 Part 2. Affine pieces of G-Hilb

Saturday, 1 August 2020 10:56

G in SL(3, CC). Write Q = CC^3 for the **given representation** (the action is matrix times column vector).

G-Hilb is the moduli space parametrising finite subschemes Z in CC^3, G-invariant, with H^0(OZ) = regular repn CC[G].

Each Z has ideal IZ in CC[x,y,z], with coordinate ring H^0(OZ) = CC[Z] = CC[x,y,z]/IZ.

The polynomial ring CC[x,y,z] is based by monomials (of course), so each quotient H^0(OZ) has some **monomial basis**. The affine pieces of G-Hilb correspond to a **combinatorial choice** of monomial basis. Two ways of viewing this choice: Leng partitions in the x,y triant and boats in the trihedral plane.

For a given monomial basis, writing out the affine piece as a variety is a problem in commutative algebra that is usually fairly tricky. (See Part 4.)

07 Quasi- and twinning

Saturday, 1 August 2020   11:23

A monomial basis for CC[Z] appropriate for the G-action involves two subtleties: **quasi-monomials** and **twinning**. Both relate to the three 1-dim repns L0, L1, L2 of G on which A acts trivially, and T acts by 1, om, om^2 (characters of ZZ/3).
Quasi: m an A-invariant monomial, then m + T(m) + T^2(m) is G-invariant, so not basic: it must be a scalar multiple la*1 where 1 in L0 is basic.
However, the Kummer cyclotomic combinations m + om^2*T(m) + om*T^2(m) (ditto with om <-> om^2) are needed to base L1 and L2. In Leng's thesis she says quasi-monomial and quasi-monomial basis to include these.
Twinning also arises from the relation m + T(m) + T^2(m) = la*1: the products x*m, x*T(m) and x*T^2(m) cannot all be viewed as basic. In some cases a basis needs one, but then x*m, x*T(m) are zero or linearly dependent in CC[Z].

08 McKay quiver

Saturday, 1 August 2020 12:20

The McKay quiver of G = A x) T is the **orbifold quotient** of A^dual, which it is convenient to view as a domain in the trihedral plane, for example a regular parallelogram: the unit parallelogram <x^10*y^3, x^7*y^10> in Figure 1 contains exactly one monomial in each of the character spaces [0..78].
Orbifold quotient means that a T-orbit of size 3 induces up to a 3-dim irreducible, whereas a T-fixed point breaks up into 3 1-dim repns where A acts as scalars and T by 1, om, om^2. (cf. the horns of D_{n+3} as ZZ/2 orbifold quotient of A_{2n+1}.)
For brevity, assume throughout A is cyclic 1/r(1,s,s^2) and r == 1 mod 6. (**Exc.** This holds iff {a,b} not {1,2} mod 3.) Then G has order 3*r, its only 1-dim repns are L0, L1, L2, and it has (r-1)/3 irreducibles 3 dim repns Ri induced fromT-orbits {1/r(i,s*i,s^2*i)} of characters of A (as i runs through a slice of the T-orbits).
Away from the 1-dim repns, the arrows of the quiver multiply by x,y,z, giving 3 arrows x: Ri -> R_{i+1}, y: R_i -> R_{i+s}, and z: R_i -> R_{i+s^2}.

09 Leng partitions

Sunday, 2 August 2020 11:53

From Rebecca Leng's 2002 thesis. The regular repn of G has 3 copies of each R_i, and a single L0, L1, L2. A monomial basis {x^i*y^j*z^k} for CC[Z] has 9 monomials in each R_i, 3 each with A-characters i, s*i and s^2*i. These 9 characters divide up as 3 T-orbits, 3 each in the (x,y), (y,z) and (z,x) triants, but not necessarily one each of i, s*i and s^2*i -- one triant commonly get one duplicated and one missing.
A **Leng partition** is a decreasing partition L = [a0, a1,..] of r such that set of monomials Mon(L) = { x^i*y^j | 0 < i <= aj } in (x,y) triant satisfy
(1) A-characters of all Mon(L) include 3 items in the list {i, s*i, s^2*i} for each 3-dim repn Ri (not nec. one in each).
(2) Include exactly one A-invariant monomial.
(3) More involved "twinning" condition around the A-invariant monomial (omit).

10 Exa of Leng partition

Tuesday, 4 August 2020 11:50

Example A = 1/19(1,7,11)

```
Leng Partition [5, 5, 5, 2, 2] gives
set of monomials                                 with A-characters
 xy^4 x^2y^4                                     10 11
 xy^3 x^2y^3                                      3  4
 xy^2 x^2y^2 x^3y^2 x^4y^2 x^5y^2                15 16 17 18  0
   xy   x^2y   x^3y   x^4y   x^5y                 8  9 10 11 12
1   x    x^2    x^3    x^4    x^5             0   1  2  3  4  5
```

In R1 = 1/19(1,7,11), the 3 monomials x,x^4y and x^2y^2 have A-char 1, 11, 11.

In R2 = 1/19(2,14,3), the 3 monomials x^2, x^3 and xy^3 have A-char 2, 3 ,3.

11 Trih boats

Sunday, 2 August 2020 08:18

Definition. **Trihedral boat** is a figure B satisfying:
(1) fundamental domain for translation by sublattice of invariant monomials.
(2) Union (B, T(B), T(B^2)) in trihedral plane has no essential overlap, and
(3) is a monomial basis of Artinian ring in each triant.
These are jigsaw and convexity properties:
(1) means translations of B tesselate the plane. Every A-char appears once, so regular repn of A.
(2) B and T(B) fit together as jigsaw pieces around the origin.
(3) In each triant, convexity of a decreasing partition.
The idea is to replace rectangular Cartesian coordinate system by trihedral coordinates. Can cut-and-paste monomial basis from Leng partition to a trihedral boat. See Figure 1 and Exa on next slide.
Pro: Monomial basis becomes 3 x A^, that is, a T-orbit of A-constellations.
Cons: Boat is in (x,y) triant union (x,z) triant, so coord geometry is harder. Ready-made software for Partitions not available.

12 Boat building

Sunday, 2 August 2020 12:15

```
Leng Partition [5, 5, 5, 2, 2] gives
set of monomials                          with A-chars
 xy^4 x^2y^4                              10 11
 xy^3 x^2y^3                               3  4
 xy^2 x^2y^2 x^3y^2 x^4y^2 x^5y^2         15 16 17 18  0
   xy   x^2y   x^3y   x^4y   x^5y          8  9 10 11 12
1   x    x^2    x^3    x^4    x^5       0  1  2  3  4  5

Rotate duplicates to the (x,z) triant by T^2.
xy^2, xy^3, xy^4 with 15, 3, 10 rotate to
x^2z, x^3z, x^4z with 13, 14, 15
after which  x^2y^3, x^2y^4 with 4, 11 rotate to
             x^3z^2, x^4z^2 with 6, 7, giving

    16 17 18  0
   8  9 10 11 12
 0  1  2  3  4  5
     13 14 15
       6  7
```

In nautical terminology, cut away the foresail to provide a keel.

Plain text gives the optical illusion of regular triangles.

13 Doodle

Monday, 3 August 2020 05:54

NB. Two duplicate monomials are translates by invariant monomials, so at distance^2 a multiple of r. Rotating shortens distances.
e.g. in the example, duplicate pair x^3 and xy^3 has quotient
x^-2*y^3 equiv y^5*z^2 of length^2 = 5^2 - 2*5 + 2^2 = 19.
After rotation x^3 and x^3*z have quotient z^-1 equiv x*y, length^2 =1.
I believe this leads to a proof that boat building always works (following suggestion by Ben Wormleighton).

14 Part 3. Computer algebra

Sunday, 2 August 2020 15:58

The combinatorics of the monomial bases for CC[Z] described in Part 2 are obviously made for computer algebra. From a conceptual point of view, to find all Leng partitions for 1/r(1,s,s^2) x) T, I just need to list the partitions of r and count how many of the resulting I*r + J belong to each of the 3-dim representations Ri. Keep the partitions that score 3 for each i and satisfy another couple of fairly minor numerical conditions.
My code is in Magma, but you can use without any prior knowledge, as I show. Or you can read the output from my files.
The first routine has input the (a,b) of Part 1, and its output is a list L of Leng partitions. It works instantly for r <= 61. (It uses partitions, that grow double exponentially, so is unusable for r >= 100.)
The second routine takes (a,b) and a Leng partition, and makes it into a trihedral boat. It works instantly even on quite big case.

15 Magma scripts

Sunday, 2 August 2020 16:34

The lectured material describes a fun game. Please play it for yourself. The online Magma calculator is free and very easy to use. Open page:

magma.maths.usyd.edu.au/calc {}.

Now go to my page:

homepages.warwick.ac.uk/~masda/McKay/tri/ {},

(currently unlinked, from a previous lecture Stockholm, Dec 2019). Scroll down 2 pages to **Magma script for Leng partitions**. Copy the first 72 lines of code, plug into the Magma calculator and Submit. Then, go back 15 lines and input your own choice of coprime values of (a,b), preferably with a <= 8.

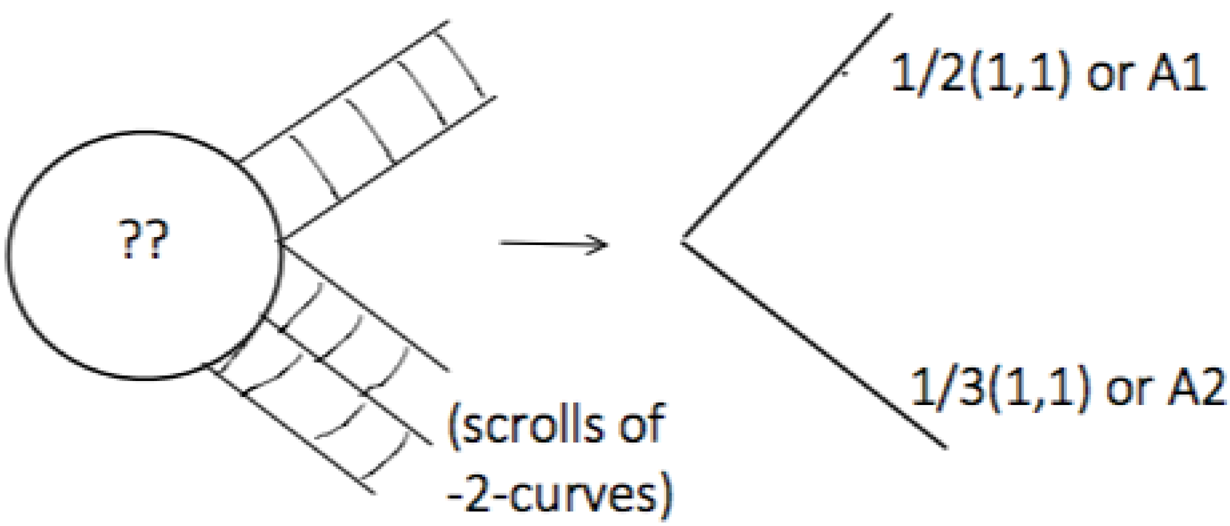
Next, scroll down to **Magma script for trihedral boats**. Copy the first 174 lines of code, plug into Magma and Submit. This second routine assumes (for programming convenience) that (a,b), (r,s) and Irr3 are input correctly. It takes a list L of Leng partitions and forms them into boats. The output page linked a few lines below gives a few hundred more cases of boats.

Problem. My list L for 1/91(1,9,81) from my first script took several months of computing (on an antique Unix machine), because of searching over partitions. (I need a cleverer programmer).

16 Part 4. V4 x) ZZ/3

Sunday, 2 August 2020 17:08

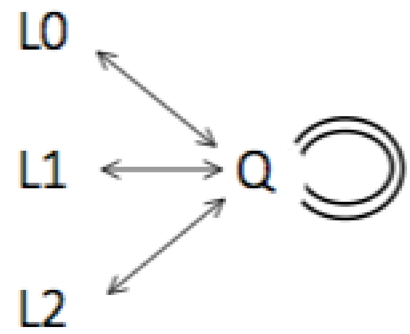
The trihedral group G = A x) ZZ/3 with A = V4 = mu2 +mu2.
Health warning: this section is new research over the last 2 weeks.
I first describe the geometry of the group and its orbifold quotient.
Write x,y,z for coordinates on CC^3. Each element of V4 fixes a coordinate axis, and acts at 1/2(0,1,1). The quotient CC^3/A is the hypersurface point XYZ = t^2, where X=x^2 (etc.), t = xyz are generators of the ring of invariants.
The trihedral rotation T conjugates the elements of V4, and cycles the 3 coordinate axes, so that the quotient CC^3/G has a single axis of 1/2(1,1) orbifold points. Also, T fixes the main diagonal x=y=z, acting there are 1/3(0,1,2), so CC^3/G also has an axis of 1/3(1,2).
(The more interesting group always have nonisolated fixed locus.)
Ignoring what happens over the origin, the quotient X has crepant resolution G-Hilb = Y -> X of this form:



17 McKay quiver

Sunday, 2 August 2020 18:51

The McKay quiver of G is



With less onerous graphics,

L0 - Q 2 loops Q -> Q
L1 /
L2 /

with the given representation Q the only 3-dim irreducible.
I study the 3 affine pieces of G-Hilb with q-monomial basis in the (x,y) triant is

x*y
1 x [x^2] together with (1) x^3 or (2) x^2*y or (3) x*y^2.

These cases illustrate the problems at the most basic level.
I don't know for certain whether there are other affine pieces.

18 Conclusion in Case 1

Sunday, 2 August 2020 21:07

CC[Z] has monomial basis

1, x, y, z, x*y, y*z, z*x, x^2, y^2, [not z^2], x^3, y^3, z^3

The generators of IZ have parameters a,b,c,d,e with a,c dependent.

a and c are given by

a = (c+b*e)*d/(1+e+e^2); c = -(e^2*b+d^2)/(1+2*e);

The affine piece of G-Hilb is AA^3<b,d,e> with (1+2*e)*(1+e+e^2) invertible. The ideal IZ is generated by the 5 relations

x*y*z-a

x^2+y^2+z^2-b

x*z^2-e*x^3-c*x-d*y*z

y*x^2-e*y^3-c*y-d*x*z

z*y^2-e*z^3-c*z-d*x*y

At some point I need (1+e) invertible -- IZ also contains

x*y^2 + (1+e)*x^3 + d*y*z + (-b+c)*x

y*z^2 + (1+e)*y^3 + d*x*z + (-b+c)*y

z*x^2 + (1+e)*z^3 + d*x*y + (-b+c)*z

19 Derivation

Tuesday, 4 August 2020 11:04

The monomial basis implies the first 5 generators for IZ with some coefficients abcde. Playing with them gives 6 linear relations

x^2*z^2 - e*x^4 - c*x^2 - a*d

x^2*y^2 - e*y^4 - c*y^2 - a*d

y^2*z^2 - e*z^4 - c*z^2 - a*d

x^2*y^2 + (1+e)*x^4 + (-b+c)*x^2 + a*d

y^2*z^2 + (1+e)*y^4 + (-b+c)*y^2 + a*d

x^2*z^2 + (1+e)*z^4 + (-b+c)*z^2 + a*d

for the 6 biquadratic monomials x^2*y^2, x^4 etc., with Det (1+2*e)*(1+e+e^2).
Then x multiplying monomial basis is matrix X, and sim., y is matrix Y. Several rows of these matrix have a single entry 1 (when x * basis element is another basis element). This is a stability condition, that **threads** the quiver repn.

a = (c+b*e)*d/(1+e+e^2) and c = -(e^2*b+d^2)/(1+2*e)

are among the entries of the commutator X*Y-Y*X.
Similar results for the other affine pieces still require checking.