Limit mixed Hodge structures of families of algebraic varieties and their applications

(joint work with Kiyoshi TAKEUCHI)

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- $\mathbb{C}(t)$: the field of rational functions of t.

$f(t,x) = \sum_{\alpha \in \mathbb{Z}^n} C_\alpha(t) x^\alpha \in \mathbb{C}(t)[x_1^{\pm}, \cdots, x_n^{\pm}]$: Laurent polynomial over $\mathbb{C}(t)$.

or $f(t,x) = \sum_{\alpha \in \mathbb{Z}_+^n} C_\alpha(t) x^\alpha \in \mathbb{C}(t)[x_1, \cdots, x_n]$: polynomial over $\mathbb{C}(t)$.

- We regard it as a family of polynomials parametrized by t.

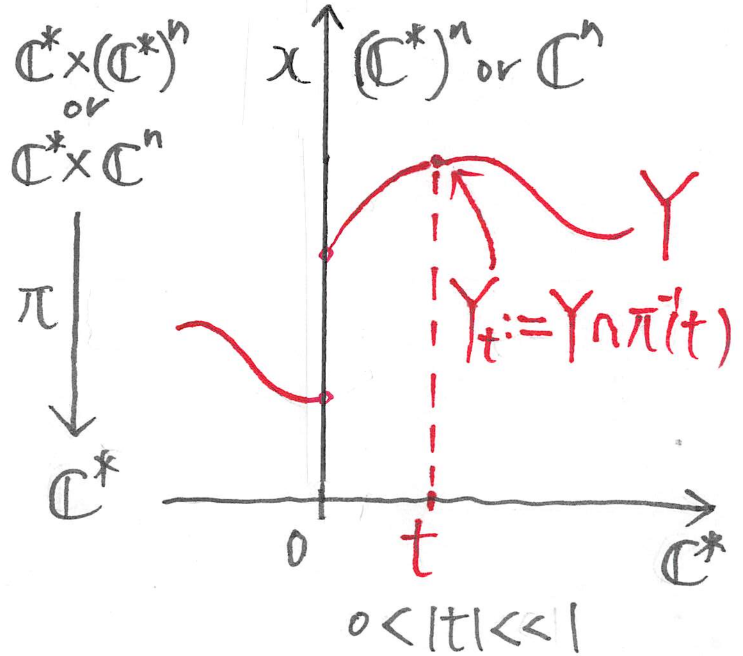
- In the case where f : Laurent poly. : $Y := f^{-1}(0) \subset \mathbb{C}_t^* \times (\mathbb{C}^*)_x^n$

  ($\rightsquigarrow$ family of hypersurfaces in $(\mathbb{C}^*)_x^n$)

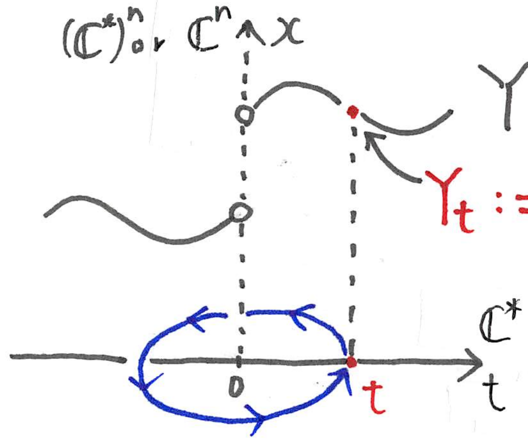
- In the case where f : poly. : $Y := f^{-1}(0) \subset \mathbb{C}_t^* \times \mathbb{C}_x^n$

  ($\rightsquigarrow$ family of hypersurfaces in $\mathbb{C}_x^n$)

$\xrightarrow{\exists \varepsilon > 0}$ We get a locally trivial fibration on $B(0,\varepsilon)^* \subset \mathbb{C}^*$.



$$\begin{array}{c} \mathbb{C}^* \times (\mathbb{C}^*)^n \\ \text{or} \\ \mathbb{C}^* \times \mathbb{C}^n \\ \Big\downarrow \pi \\ \mathbb{C}^* \end{array}$$



$Y = f^{-1}(0) \subset \mathbb{C}^* \times (\mathbb{C}^*)^n$ or $\mathbb{C}^* \times \mathbb{C}^n$

$Y_t := Y \cap \pi^{-1}(t)$

Taking a path along a small circle $\subset B(0, \varepsilon)^*$ $(\subset \mathbb{C}^*)$
we get an automorphism of Y_t

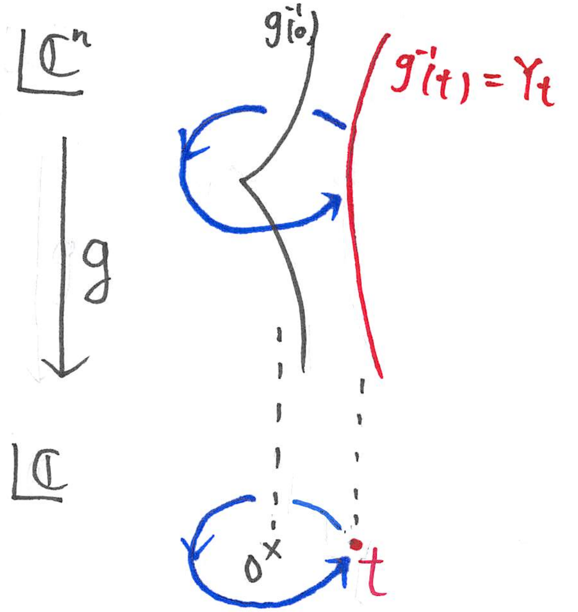
$\rightsquigarrow$ $H^j_c(Y_t ; \mathbb{C}) \circlearrowleft \Phi_j$ <u>monodromy of Y</u>

$\updownarrow$

<u>limit mixed Hodge structure of $H^j_c(Y_t ; \mathbb{C})$</u>

※ In many cases, $H^{n-1}_c(Y_t ; \mathbb{C}) \circlearrowleft \Phi_{n-1}$ is the most complicated one.

Ex1 $g(x) \in \mathbb{C}[x_1, \cdots, x_n]$



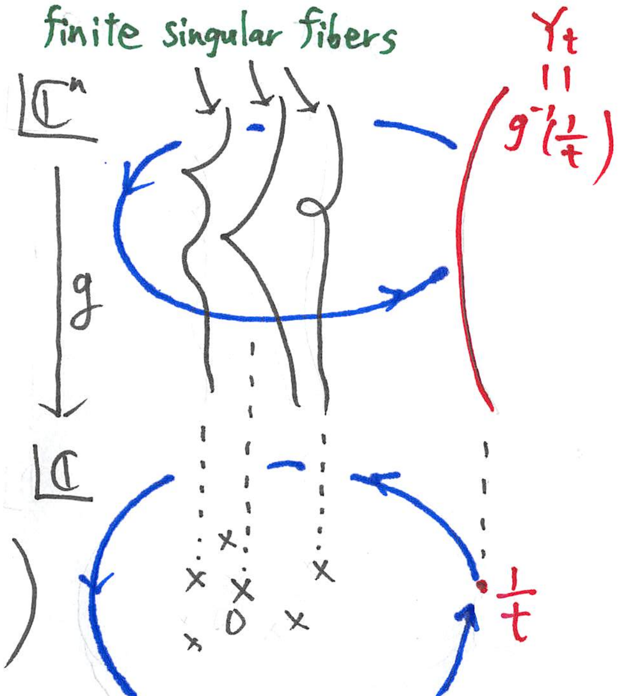
$\rightsquigarrow f(t,x) := g(x) - t \in \mathbb{C}(t)[x_1, \cdots, x_n]$

$\overset{0<|t|\ll 1}{\rightsquigarrow} Y_t = g^{-1}(t) \subset \mathbb{C}^n$

$\rightsquigarrow H^j_c(g^{-1}(t); \mathbb{C}) \circlearrowleft \Phi_j$

Ex2 (monodromy at infinity)

$g(x) \in \mathbb{C}[x_1, \cdots, x_n]$



$\rightsquigarrow f(t,x) := g(x) - \frac{1}{t} \in \mathbb{C}(t)[x_1, \cdots, x_n]$

$\overset{0<|t|\ll 1}{\rightsquigarrow} Y_t = g^{-1}(\frac{1}{t}) \subset \mathbb{C}^n$

big! $(\frac{1}{|t|} \gg 1)$

$\rightsquigarrow H^j_c(g^{-1}(\frac{1}{t}); \mathbb{C}) \circlearrowleft \Phi_j$ (monodromy at infinity)

<u>Aim</u> Describe the monodromy and the limit mixed Hodge structure of $H^j_c(Y_t;\mathbb{C})$.

More precisely,

① Improve Stapledon's description of the motivic nearby fiber of f.

$\rightsquigarrow$ Compute the limit mixed Hodge str. of $H^j_c(Y_t;\mathbb{C})$.

② Describe the Jordan normal form of the monodromy $H^{n-1}_c(Y_t;\mathbb{C}) \circlearrowleft \Phi_{n-1}$.

③ Extend the results to families of complete intersection varieties.

§ limit mixed Hodge structures

- $H_c^j(Y_t;\mathbb{Q})$ has Deligne's canonical mixed Hodge structure:

$$\begin{cases} F^{\cdot} : \text{Hodge filtration} \\ W_{\cdot} : \text{weight filtration} \end{cases}$$

- $H_c^j(Y_t;\mathbb{Q}) \circlearrowleft \Phi_j$ has another filtrations $F_\infty^{\cdot}$, $M_{\cdot}$, which have a part of data of monodromy Φ_j.

- $(F_\infty^{\cdot}, M_{\cdot})$ defines another mixed Hodge structure of $H_c^j(Y_t;\mathbb{Q})$ (limit mixed Hodge str.)

- For $\lambda \in \mathbb{C}$, $H_c^j(Y_t;\mathbb{C})_\lambda \subset H_c^j(Y_t;\mathbb{C})$: the generalized eigenspace for λ

$$\rightsquigarrow h^{p,q}(H_c^j(Y_t;\mathbb{C}))_\lambda := \dim \mathrm{Gr}_{F_\infty}^p \mathrm{Gr}_{p+q}^M H_c^j(Y_t;\mathbb{C})_\lambda .$$

(properties of $M_\bullet$) $H_c^{n-1}(Y_t;\mathbb{C}) \circlearrowleft \Phi_{n-1} = \Phi_s \Phi_u$ $\left(\begin{cases} \Phi_s : \text{semisimple part} \\ \Phi_u : \text{unipotent part} \end{cases} \right)$

The filtration on $\mathrm{Gr}_\ell^W H_c^{n-1}(Y_t;\mathbb{C})$ $(\ell \in \mathbb{Z})$ induced by $M_\bullet$ is the monodromy filtration associated to $N = \log \Phi_u$ centered at ℓ.

(This has the data of the nilpotent part of $\Phi_{n-1} \circlearrowright \mathrm{Gr}_\ell^W H_c^{n-1}(Y_t)$)

For an eigenvalue $\lambda \in \mathbb{C}$ of $\Phi_{n-1} \circlearrowright H_c^{n-1}(Y_t;\mathbb{C})$,

suppose $\mathrm{Gr}_\ell^W H_c^{n-1}(Y_t)_\lambda = 0 \quad (\ell \neq n-1)$

$H_c^{n-1}(Y_t)_\lambda$ is the generalized eigenspace for λ

$\left(\rightsquigarrow \mathrm{Gr}_{n-1}^W H_c^{n-1}(Y_t)_\lambda = H_c^{n-1}(Y_t)_\lambda \right)$.

Then we have

$$\# \left\{ k \left\{ \overbrace{\boxed{\begin{matrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{matrix}}}^{k} \right. \in \Phi_{n-1} \right\}$$

$$= \dim \mathrm{Gr}_{n-k}^M H_c^{n-1}(Y_t;\mathbb{C})_\lambda - \dim \mathrm{Gr}_{n-2-k}^M H_c^{n-1}(Y_t;\mathbb{C})_\lambda .$$

equivariant E-polynomial

For $\lambda \in \mathbb{C}$,

the dimension of the (p,q)-part of the limit mixed Hodge str.

$$E_\lambda(Y_t; u, v) = \sum_{p,q} \left(\sum_j (-1)^j h^{p,q}(H_c^j(Y_t))_\lambda \right) u^p v^q,$$

$$E_\lambda(Y_t; u, v, w) = \sum_{p,q,r} \left(\sum_j (-1)^j h^{p,q}(Gr_r^W H_c^j(Y_t))_\lambda \right) u^p v^q w^r$$

Rem

- $E_\lambda(Y_t; u, v, 1) = E_\lambda(Y_t; u, v)$.

- We will compute two E_λ-polynomials very explicitely by using the motivic nearby fiber $\psi_t([Y])$.

For $\lambda \in \mathbb{C}$ $\quad E_\lambda(Y_t; u, v, w) = \sum_{p,q,r} \left(\sum_j (-1)^j h^{p,q}(\mathrm{Gr}^W_r H^j_c(Y_t))_\lambda \right) u^p v^q w^r$

Our strategy for the computation of the Jordan normal form of Φ_{n-1}

<u>Step1</u> We define a finite subset $R_f \subset \mathbb{C}$ of "bad" eigenvalues of Φ_j.

<u>Step2</u> Assuming that Y is <u>schön</u> ($\leftarrow$ generic condition), we show

<u>Theorem A</u> For $\lambda \notin R_f$, $E_\lambda(Y_t; u, v, w) = \left(\sum_{p,q} * u^p v^q\right) \cdot w^{n-1}$.

<u>Theorem B</u> For $\lambda \notin R_f$, $\mathrm{Gr}^W_\ell H^j_c(Y_t)_\lambda = 0 \quad (\ell \neq j)$.

$\longrightarrow$ <u>Cor</u> For $\lambda \notin R_f$, $H^j_c(Y_t)_\lambda = 0 \quad (j \neq n-1)$.

<u>Step3</u> For $\lambda \notin R_f$, we describe the Jordan normal form of $\Phi_{n-1} \circlearrowright H^{n-1}_c(Y_t)$ for the eigenvalue λ in terms of $E_\lambda(Y_t; u, v, w)$!!!

Motivic nearby fiber of the family Y

· Motivic nearby fiber $\psi_t([Y])$ is ...

$$\psi_t([Y]) = [V_1 \circlearrowleft \mu_{d_1}] + \cdots + [V_k \circlearrowleft \mu_{d_k}] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} = K_0(Var_{\mathbb{C}}^{\hat{\mu}})[\mathbb{L}^{-1}]$$

: a formal sum of alg. varieties with cyclic group actions.

· $H_c^j(V_i;\mathbb{Q}) \circlearrowleft \mu_{d_i}$: Deligne's mixed Hodge str.

For $\lambda \in \mathbb{C}$ ⇝

$$\sum_i [V_i \circlearrowleft \mu_{d_i}] \xrightarrow[\text{(Hodge realization)}]{\text{taking } E_\lambda\text{-poly}} \sum_i \sum_{p,q} \sum_j (-1)^j h^{p,q}(H_c^j(V_i))_\lambda u^p v^q$$

the dimension of (p,q)-Deligne's mixed Hodge number

$$\parallel$$

By Denef-Loeser 1998 and Guibert-Loeser-Merle 2006

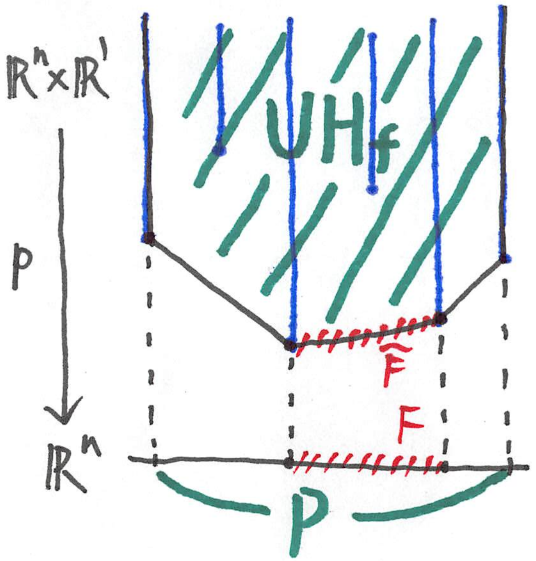
$$E_\lambda(Y_t; u, v)$$

(E_λ-poly for the limit mixed Hodge str.)

preparation $\quad f(t,x) = \sum_{\alpha}\left(\sum_j c_{\alpha,j} t^j\right) x^{\alpha} \in \mathbb{C}(t)[x_1^{\pm}, \cdots, x_n^{\pm}]$

Laurent expansion at 0.

We define the following



- an polyhedron $UH_f \subset \mathbb{R}^n \times \mathbb{R}^1$
- a polytope $P := p(UH_f) \subset \mathbb{R}^n$

  (in what follows, we assume $\dim P = n$)
- a subdivision of P

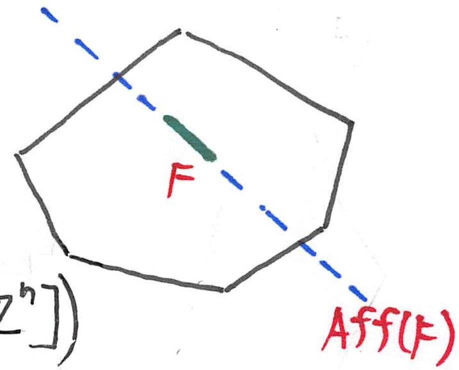
  $\mathcal{S} = \{F \subset P \mid \exists \widetilde{F}$: a bottom face of UH_f s.t $p(\widetilde{F}) = F\}$

For $F \in \mathcal{S}$,

$$T_F := \mathrm{Spec}(\mathbb{C}[\mathrm{Aff}(F) \cap \mathbb{Z}^n]) \simeq (\mathbb{C}^*)^{\dim F}$$

$$\cup$$

$$V_F := \{I_f^F = 0\} \quad \left(I_f^F = \sum_{(\alpha,j)\in \widetilde{F}} c_{\alpha,j} x^{\alpha} \in \mathbb{C}[\mathrm{Aff}(F) \cap \mathbb{Z}^n]\right)$$



V_F has an action of μ_{m_F} defined by UH_f.

Thm (Stapledon 2014) Assume that the family Y of hypersurfaces in $(\mathbb{C}^*)^n$ is schön (non-degenerate). Then we can describe the motivic nearby fiber of Y by

$$\psi_t([Y]) = \sum_{\substack{\mathrm{rel.int} F \subset \mathrm{Int} P \\ F \in \mathcal{S}}} [V_F \circlearrowleft \mu_{m_F}] \cdot (1-\mathbb{L})^{n-\dim F}.$$

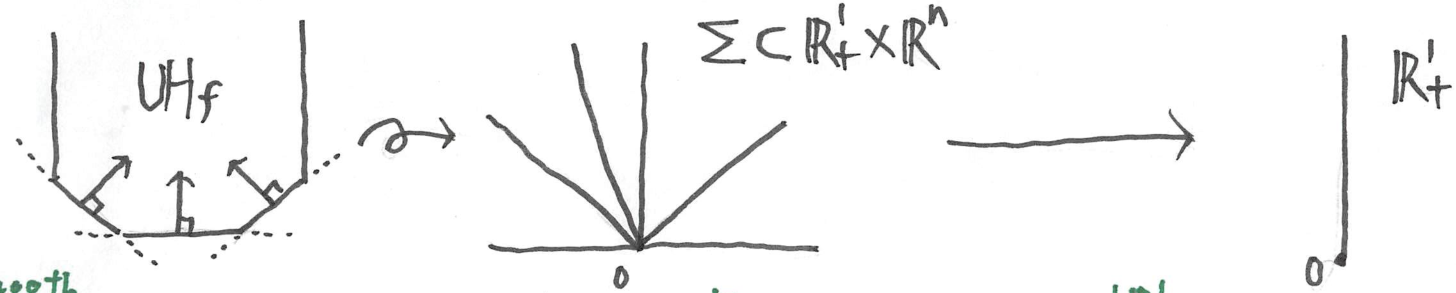
$$\left(\begin{array}{l} 1 = [pt] \\ \mathbb{L} = [\mathbb{C}^1] \end{array} \right)$$

· Therefore, the E_λ-polynomial of the limit mixed Hodge str. can be expressed by the Hodge realization of the right hand side:

$$E_\lambda(Y_t; u, v) = E_\lambda(\psi_t([Y]); u, v) \quad (\forall \lambda \in \mathbb{C})$$

· We gave a simpler proof than Stapledon's one.

Our proof Σ : a smooth subdivision of the dual fan of UH_f.



smooth toric var. ass. to Σ $\longrightarrow$ X_Σ

toric var. ass. to $\mathbb{R}^1_+$ $\longrightarrow$ $\mathbb{C}$

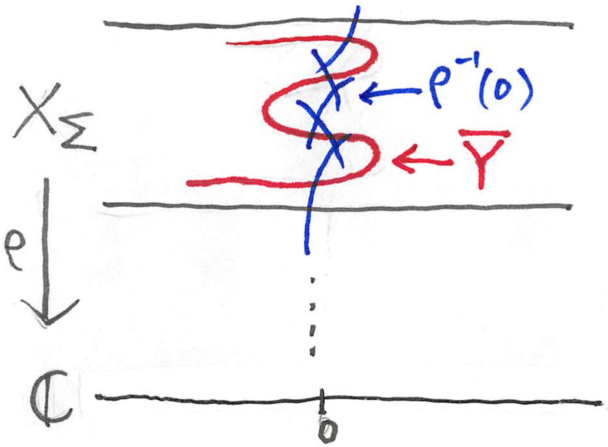
$$\begin{tikzcd}
 & X_\Sigma \arrow[rr, "\rho:\text{proper}"] & & \mathbb{C} \\
\mathbb{C}^* \times (\mathbb{C}^*)^n \arrow[ur, bend left] \arrow[dr, "\pi"'] & Y \arrow[l, hook'] \arrow[u, "\iota"] \arrow[d, "\pi_Y"] & & \\
 & \mathbb{C}^* \arrow[uurr, bend right, "j"'] & &
\end{tikzcd}$$

As mixed Hodge structures,

$$\begin{aligned}
H^j_c(Y_t;\mathbb{Q}) &\simeq H^j\psi_t(j_! R\pi_{Y!}\mathbb{Q}_Y) \\
&\simeq H^j\psi_t(R\rho_*(\iota_!\mathbb{Q}_Y)) \\
&\simeq H^j(\rho^{-1}(0);\psi_\rho(\iota_!\mathbb{Q}_Y))
\end{aligned}$$

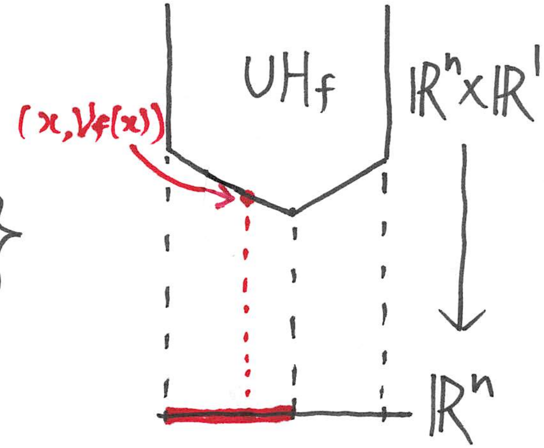
by Denef-~~Loeser~~ 1998 and Guibert-Loeser-Merle 2006

☆ the E-polynomial of the final term can be expressed by the Hodge realization of a formal sum of alg. var. constructed by using the data of the irreducible components of $\rho^{-1}(0) \cap \overline{Y}$. □



Step 1 The set of "bad" eigenvalues

We define a piecewise linear function $V_f : P \to \mathbb{R}$ by the bottom part of UH_f.



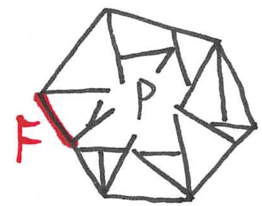
$\rightsquigarrow$ For $F \in \mathcal{S}$ $(F \subset P)$,

$$m_F = \max\left\{ s \in \mathbb{Z}_{\geq 1} \;\middle|\; \begin{array}{l} \frac{t}{s} \in V_f(\mathrm{Aff}(F) \cap \mathbb{Z}^n) \\ \gcd(t,s) = 1 \end{array} \right\}$$

$\Downarrow$

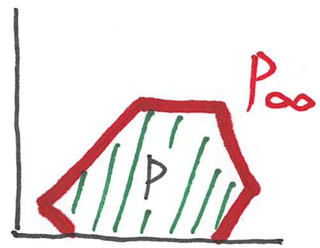
- In the case of

$$Y_t \subset (\mathbb{C}^*)^n : \quad R_f := \bigcup_{\substack{F \in \mathcal{S} \\ F \subset \partial P}} \{ \lambda \in \mathbb{C} \mid \lambda^{m_F} = 1 \}$$



- In the case of

$$Y_t \subset \mathbb{C}^n : \quad R_f := \bigcup_{\substack{F \in \mathcal{S} \\ F \subset P_\infty}} \{ \lambda \in \mathbb{C} \mid \lambda^{m_F} = 1 \}$$



Step2 Theorem A and B

Theorem A (follows from Stapledon 2014 and Matsui-Takeuchi 2013)

Assume that the family Y of the hypersurfaces in $(\mathbb{C}^*)^n$ or $\mathbb{C}^n$ is schön. Then for $\lambda \notin R_f$, we have

$$E_\lambda(Y_t; u, v, w)$$
$$= \left(\frac{(-1)^{n-1}}{uv} \sum_{\substack{F \in \mathcal{S} \\ \text{rel.int}F \\ \subset \text{Int}P}} v^{\dim F} l_\lambda^*(F, V_f; uv^{-1}) \cdot l_P(\mathcal{S}, F; uv) \right) \cdot w^{n-1}$$

(l_λ^* and l_P are polynomials with coefficients in $\mathbb{Z}$ defined by $P, \mathcal{S}, V_f$)

For the proof of this theorem, we use

- the purity of the MHS of the intersection cohomology groups of singular compact varieties.
- Some results on the combinatorics of polynomials ass. to polytopes $\square$

Theorem B Assume that the family Y of hypersurfaces in $(\mathbb{C}^*)^n$ or $\mathbb{C}^n$ is schön. Then for $\lambda \notin R_f$, we have

$$\mathrm{Gr}^W_\ell H^j_c(Y_t)_\lambda = 0 \quad (\ell \neq j)$$

$H^j(\psi_{t,\lambda}(j_! R\pi_! \mathbb{C}_Y)) \simeq H^j_c(Y_t)$

(This means that "The complex $\psi_{t,\lambda}(j_! R\pi_! \mathbb{C}_Y)$ has a pure weight 0")

Recall

$$\begin{tikzcd} \mathbb{C}^* \times (\mathbb{C}^*)^n \arrow[dr, "\pi"'] & & Y \arrow[ll, hook'] \arrow[dl, "\pi_Y"] \\ & \mathbb{C}^* \arrow[r, hook, "j"] & \mathbb{C} \end{tikzcd}$$

proof (In the case of $Y_t \subset (\mathbb{C}^*)^n$)

We use the idea of Sabbah 2006.

Suppose for $\lambda \notin R_f$, $j \in \mathbb{Z}$,

$$H^j_c(Y_t;\mathbb{C})_\lambda \xrightarrow{\sim} H^j(Y_t;\mathbb{C})_\lambda \quad \cdots (*)$$

$\mathrm{Gr}^W_\ell H^j_c(Y_t)_\lambda = 0$ $(\ell > j)$

$\mathrm{Gr}^W_\ell H^j(Y_t)_\lambda = 0$ $(\ell < j)$

Then we have $\mathrm{Gr}^W_\ell H^j_c(Y_t)_\lambda = 0 \quad (\ell \neq j)$

$\rightsquigarrow$ It suffices to show $(*)$!!

The proof of $H^j_c(Y_t;\mathbb{C})_\lambda \xrightarrow{\sim} H^j(Y_t;\mathbb{C})_\lambda$

Recall the diagram :

$$\begin{tikzcd}
 & & X_\Sigma & \\
\mathbb{C}^* \times (\mathbb{C}^*)^n \arrow[urr, hook] \arrow[dr, "\pi"'] & Y \arrow[l, phantom, "\supset"] \arrow[r, "j_Y"'] \arrow[ur, hook] \arrow[d, "\pi_Y"] & \overline{Y} \arrow[u, hook] \arrow[d, "\pi_{\overline{Y}}"] & D := \overline{Y} \setminus Y \arrow[l, hook', "i_D"] \arrow[ul, hook] \\
 & \mathbb{C}^* \arrow[r, hook, "j"] & \mathbb{C} &
\end{tikzcd}$$

$\exists$ Distinguished triangle in $D^b(\mathrm{Mod}(\mathbb{C}))$:

$$\psi_{t,\lambda}(j_! R\pi_! \mathbb{C}_Y) \longrightarrow \psi_{t,\lambda}(j_! R\pi_* \mathbb{C}_Y) \longrightarrow \psi_{t,\lambda}(R\pi_{\overline{Y}*} i_{D*} i_D^{-1} Rj_{Y*} \mathbb{C}_Y) \xrightarrow{+1}$$

Taking j-th cohomology

$$H^j_c(Y_t;\mathbb{C})_\lambda \longrightarrow H^j(Y_t;\mathbb{C})_\lambda \qquad R\Gamma(\pi_{\overline{Y}}^{-1}(0); \psi_{\pi_{\overline{Y}},\lambda}(i_{D*} i_D^{-1} Rj_{Y*} \mathbb{C}_Y)) \simeq 0$$

$\lambda \notin R_f$

by the precise description of the "primitive decomposition" of the nearby cycle sheaves associated to some normal crossing divisors.

$\square$

Step 3 computation of the Jordan normal form of Φ_{n-1}

$$J_{\lambda,k} := \# \left\{ k \left\{ \overbrace{\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}^{k} \right. \in \Phi_{n-1} \circlearrowright H_c^{n-1}(Y_t) \right\}$$

Theorem (S-T) Assume that the family Y of hypersurfaces in $(\mathbb{C}^*)^n$ or $\mathbb{C}^n$ is schön. Then for $\lambda \notin R_f$, we have

$$\sum_{m=0}^{n-1} J_{\lambda,n-m}\, s^{m+2} = \sum_{F \in \mathcal{S}} s^{\dim F}\, l^*_\lambda(F, \nu_F; 1) \cdot \tilde{l}_P(\mathcal{S}, F; s^2)$$

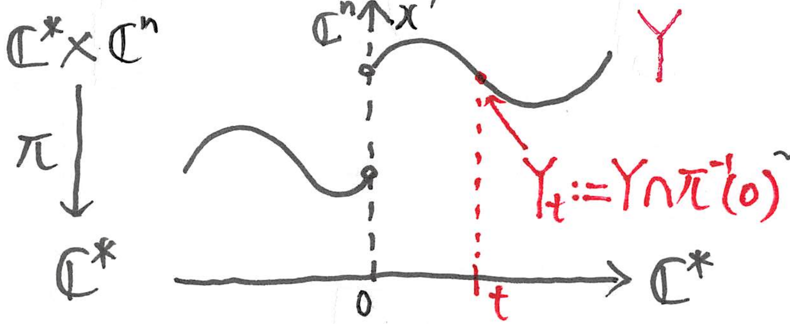
(l^*_λ and $\tilde{l}_P$ are polynomials with coefficients in $\mathbb{Z}$ defined by P, $\mathcal{S}$, ν_F)

Family of complete intersection varieties

$$f_1(t,x), \cdots, f_k(t,x) \in \mathbb{C}(t)[x_1, \cdots, x_n]$$

$$\rightsquigarrow Y := f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0) \subset \mathbb{C}^* \times \mathbb{C}^n$$

$\rightsquigarrow$ We get a family of subvarieties in $\mathbb{C}^n$. (In certain cases, this is a family of complete intersection var in $\mathbb{C}^n$.)



$$\rightsquigarrow H_c^j(Y_t) \circlearrowleft \Phi_j$$

$$J_{\lambda,k} := \#\left\{ k\left\{ \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \right. \in \Phi_{n-k} \right\}$$

<u>Thm</u> Assume that Y is schön. Then for $\lambda \notin R_f$, we have $H_c^j(Y_t)_\lambda = 0$ $(j \neq n-k)$, and $J_{\lambda,k}$ can be described in terms of polynomials defined by $UH_{f_1}, \cdots, UH_{f_k}$.