A mathematical definition of Gopakumar-Vafa invariants

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arXiv:1610.07303 (with Davesh Maulik)
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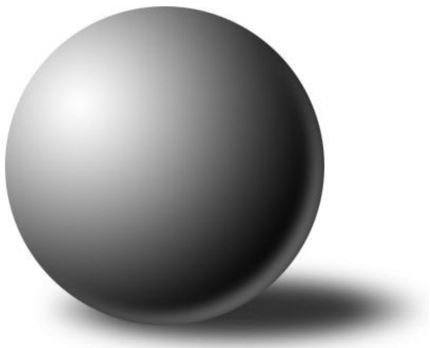
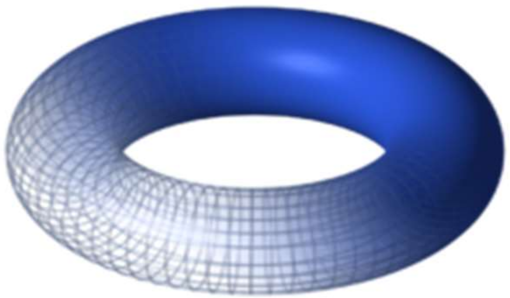
- Algebraic curves

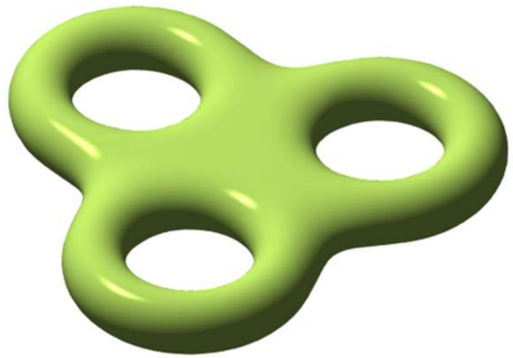
= one dimensional complex algebraic varieties, classified by the number of handles (called genus)



Rational curve



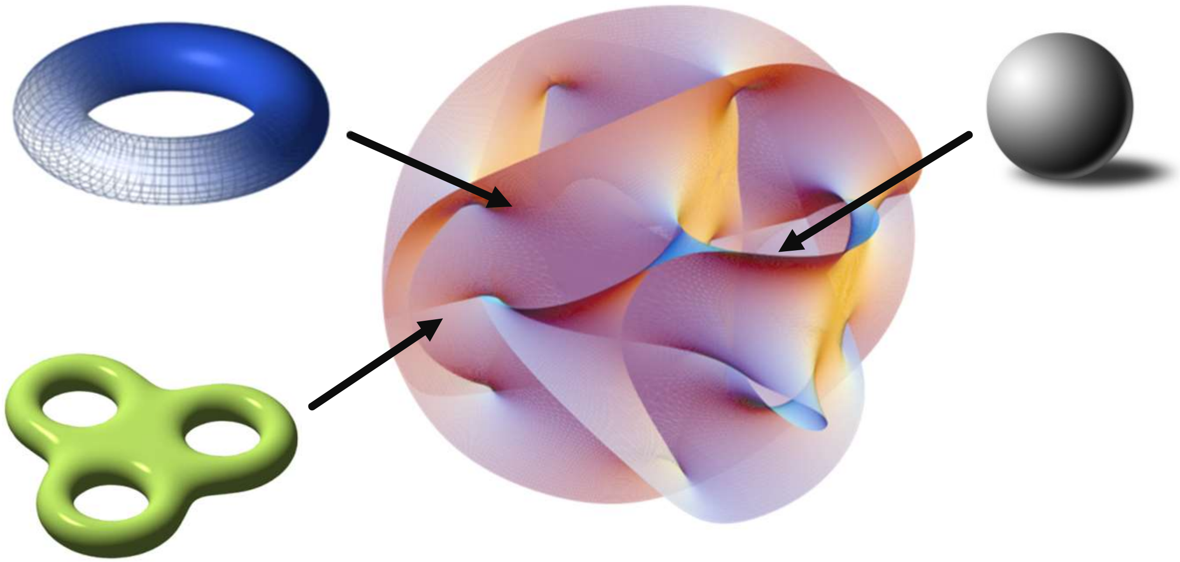
Elliptic curve



General type

- It is an important problem to count algebraic curves on algebraic varieties, both in mathematics and physics.

- The most interesting case is when the target is a Calabi-Yau 3-fold, i.e. complex 3-folds with trivial canonical bundle.



- Example: rational curves on quntic 3-folds

$$X = \{f(x_1, x_2, x_3, x_4, x_5) = 0\} \subset \mathbb{P}^4_{\mathbb{C}}$$

is a Calabi-Yau 3-fold, where f is a general homogeneous polynomial of degree five.

# of degree one rational curves= 2875

# of degree two rational curves= 609250

# of degree three rational curves= 317206375

In general, there exist infinite number of curves on a CY 3-fold with given genus and degree. So we need their virtual counting via compact moduli space of curves.

There exist at least 3 curve counting theories on CY 3-folds, which are conjecturally equivalent:

- Gromov-Witten (GW) invariants
- Pandharipande-Thomas (PT) invariants
- Gopakumar-Vafa (GV) invariants

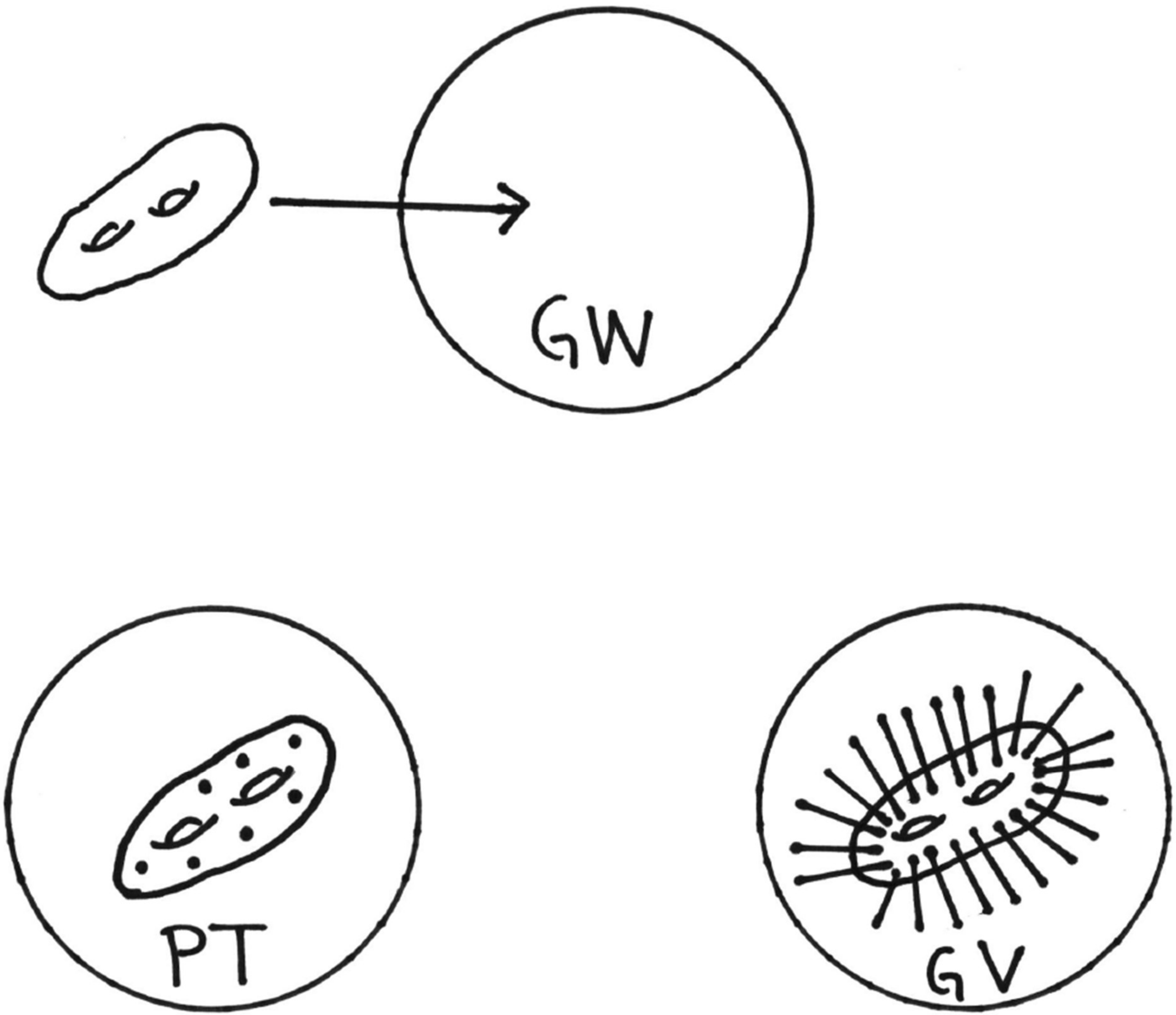
GW invariants $\mathrm{GW}_{g,\beta} \in \mathbb{Q}$
= # of curves + maps to CY3
(stable maps, worldsheet)

Maulik-Nekrasov-Okounkov-Pandharipande conjecture

Gopakumar-Vafa conjecture

PT invariants $P_{n,\beta} \in \mathbb{Z}$
= # of curves on CY3 and points on them
(stable pairs, D0-D2-D6 bound states)

GV invariants $n_{g,\beta} \in \mathbb{Z}$
= # curves on CY3 and vector bundles on them
(one dimensional stable sheaves, D2 branes)


GW
PT
GV

Issue:

Contrary to GW and PT invariants, which are mathematically well-established, the definition of GV invariants are not yet mathematically satisfactory.

There have been several proposal on mathematical definition on GV invariants

$$n_{g,\beta} \in \mathbb{Z}, \;\; g \in \mathbb{Z}_{\geq 0}, \;\; \beta \in H_2(X, \mathbb{Z})$$

But when the relevant moduli space has singularities the proposed definitions (before our paper with D. Maulik) do not match with GW and PT invariants, except the g=0 case.

• PT/GV correspondence

How to define ?

PT invariants $P_{n,\beta}$? $\Longleftrightarrow$ GV invariants $n_{g,\beta}$

This should be analogy of the well-known fact for smooth curve C of genus g:

$$\sum_{n\geq 0} \chi(\mathrm{Sym}^n(C))q^n = (q+1)^{2g-2}$$

$$= \frac{1}{(q+1)^2} P_q(\mathrm{Pic}^0(C))$$

- In genus zero g=0, the PT/GV correspondence is explained by wall-crossing phenomena of Bridgeland stability conditions in derived categories. (T, 2008)

- The above idea led to the proof of the rationality conjecture of the generating series of PT invariants, and many other properties of Donaldson-Thomas type invariants.

- Again in higher genus g>0, even a mathematical definition of GV invariants has not been clear. We need some refined version of cohomology theory for singular moduli spaces.

History toward math definition of GV invariants

- Gopakumar-Vafa (1998), physics proposal of integer valued invariants equivalent to GW invariants.

- Hosono-Saito-Takahashi (2001), proposal of math definition of GV invariants via $sl_2 \times sl_2$ action on intersection cohomology of moduli spaces of one dimensional stable sheaves.

- Katz (2006), definition of $n_{0,\beta}$ via virtual class.

- Kiem-Li (2012), modification of HST approach via perverse sheaves of vanishing sheaves and their gr of weight filtration.

Theorem (Maulik-T, 2016)

(i) Kiem-Li’s definition is not well-defined, and depends on a choice of `orientation data’ of moduli spaces of one dimensional stable sheaves.

(ii) There is an example where Kiem-Li’s definition does not match with GW or PT invariants for any choice of orientation data.

In a joint work with D. Maulik, by modifying the Kiem-Li's definition, we defined the GV invariant from the moduli space M of one dimensional stable sheaves on a CY 3-fold X with homology class β . The key ingredients are

- Use the character formula of $sl_2 \times sl_2$ action on the hyper-cohomology of perverse sheaves of vanishing cycles.

- We take a special type of orientation data of M, which we call Calabi-Yau orientation data.

Based on the work of Töen et al on (-1)-shifted symplectic derived algebraic geometry, Joyce's group showed that M has a d-critical structure. This means that M is covered by open subsets U written as critical locus in smooth schemes V (called d-critical chart)

$$U = \{df = 0\} \subset V \xrightarrow{f} \mathbb{C}$$

Moreover these data remember global information of (-1)-shifted symplectic form so that they form a global section of some sheaf on M canonically attached to it.

By Joyce et al and Kiem-Li, the perverse sheaves of vanishing cycles on each d-critical chart

$$U = \{df = 0\} \subset V \xrightarrow{f} \mathbb{C}$$

can be glued to give a global perverse sheaf

$$\phi_M \in \mathrm{Perv}(M)$$

once we choose an orientation data, which is a square root line bundle on M of the canonical gluing of line bundles $K_V|_U^{\otimes 2}$ (called virtual canonical line bundle).

Definition of GV invariants

The Hilbert-Chow map

$$\pi \colon M \to \mathrm{Chow}(X)$$

is defined by sending a one dimensional sheaf to its support. Here Chow(X) is the moduli space of effective one cycles on X.

In [Maulik-T], we define the GV invariant $n_{g,\beta} \in \mathbb{Z}$ by

$$\sum_{i \in \mathbb{Z}} \chi({}^p\mathcal{H}^i(R\pi_* \phi_M)) y^i = \sum_{g \geq 0} n_{g,\beta} (y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{2g}.$$

Here ${}^p\mathcal{H}^i(-)$ is the i-th perverse cohomology.

A subtlety of this definition is a choice of a correct orientation data. We impose the condition on it to be pulled-back from the Chow variety under the Hilbert-Chow map

$$\pi : M \to \mathrm{Chow}(X)$$

We call such an orientation data as Calabi-Yau orientation data, and conjecture that such a choice always exists. Our definition of GV invariant is independent of orientation data as long as it is Calabi-Yau.

PT/GV Conjecture

- Let $P_{n,\beta} \in \mathbb{Z}$ be the PT invariant which (roughly speaking) virtually counts genus g curves with homology class β and $(n+g-1)$ points. Then we have the identity:

$$\log\left(1+\sum_{\beta>0, n\in\mathbb{Z}} P_{n,\beta} q^n t^\beta\right)$$

$$=\sum_{\beta>0}\sum_{g\in\mathbb{Z}, k\geq 1} \frac{n_{g,\beta}}{k}(-1)^{g-1}((-q)^{\frac{k}{2}}-(-q)^{-\frac{k}{2}})^{2g-2}t^{k\beta}.$$

Theorem (Maulik-T, 2016)

- PT/GV conjecture is true for local surfaces, i.e.

$$X = \mathrm{Tot}_S(K_S)$$

where S is a smooth projective surface, and the curve class β is irreducible, i.e. β is not written as $\beta_1 + \beta_2$ for effective curve classes β_1, β_2

Dependence of stability conditions

A priori, the definition of GV invariant may depend on a choice of a stability condition. Given an element

$$\sigma = B + i\omega \in H^2(X, \mathbb{C})$$

such that ω is ample, we have the notion of σ-semistable objects and the associated GV invariants

$$n_{g,\beta}(\sigma) \in \mathbb{Z}$$

Theorem (T, 2017)

The invariant $n_{g,\beta}(\sigma) \in \mathbb{Z}$ is independent of a choice of a stability condition σ. So we can define GV invariant $n_{g,\beta} \in \mathbb{Z}$ from any choice of a stability condition.

Corollary (Flop invariance)

Let $\phi\colon X \to Y \leftarrow X^{\dagger}$ be a flop between Calabi-Yau 3-folds. Then we have the identity

$$n_{g,\beta} = n_{g,\phi_* \beta}$$

In the proof, we show that under wall-crossing diagram of moduli spaces under change stability conditions

$$M_+ \stackrel{f_+}{\to} M_0 \stackrel{f_-}{\leftarrow} M_-$$

for each open neighborhood $p \in U \subset M_0$
there is a diagram

$$\begin{array}{ccccc} Y_+ & \longrightarrow & Y_0 & \longleftarrow & Y_- \\ & {\scriptstyle W_+}\searrow & \downarrow {\scriptstyle g} & \swarrow {\scriptstyle W_-} & \\ & & \mathbb{C} & & \end{array}$$

where $Y_+ \dashrightarrow Y_-$ is a flop between smooth varieties and isomorphisms of d-critical schemes

$$f_{\pm}^{-1}(U) \cong \{dW_{\pm} = 0\}$$

Future perspective

I call a diagram like $M_+ \to M_0 \leftarrow M_-$ as a d-critical flop. Such a notion can be generalized to other birational transformations like d-critical flips, d-critical divisorial contractions, etc.

This indicates that d-critical schemes may fit into new type of birational geometry, which I would call d-critical birational geometry.

Developing such a geometry would lead to geometric understanding of wall-crossing for CY 3-folds, generalization of Bondal-Orlov, Kawamata's D-K equivalence conjecture for d-critical schemes, etc.

Thank you very much !