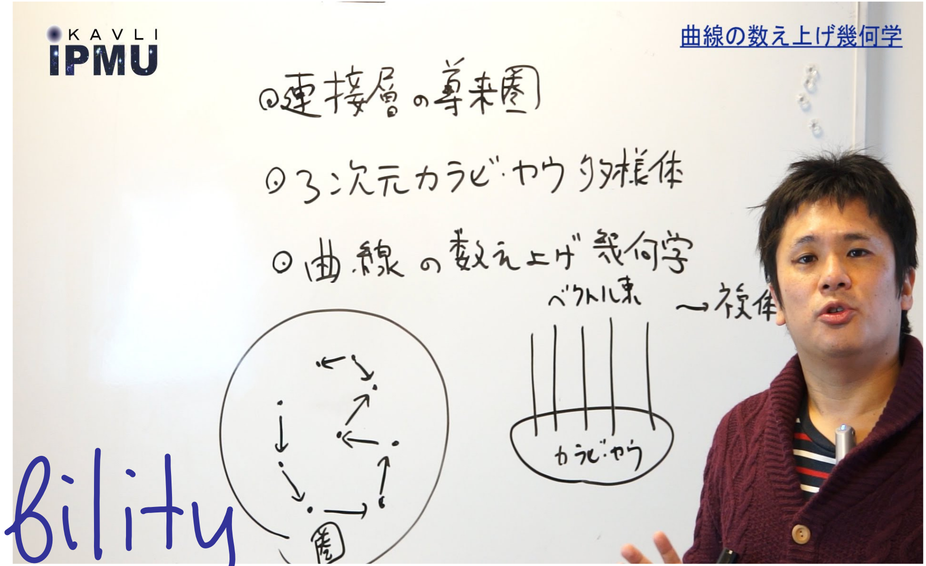
THE WORLD WITHOUT WALLS

KAVLI IPMU

Andrei Okounkov & Mina Aganagic

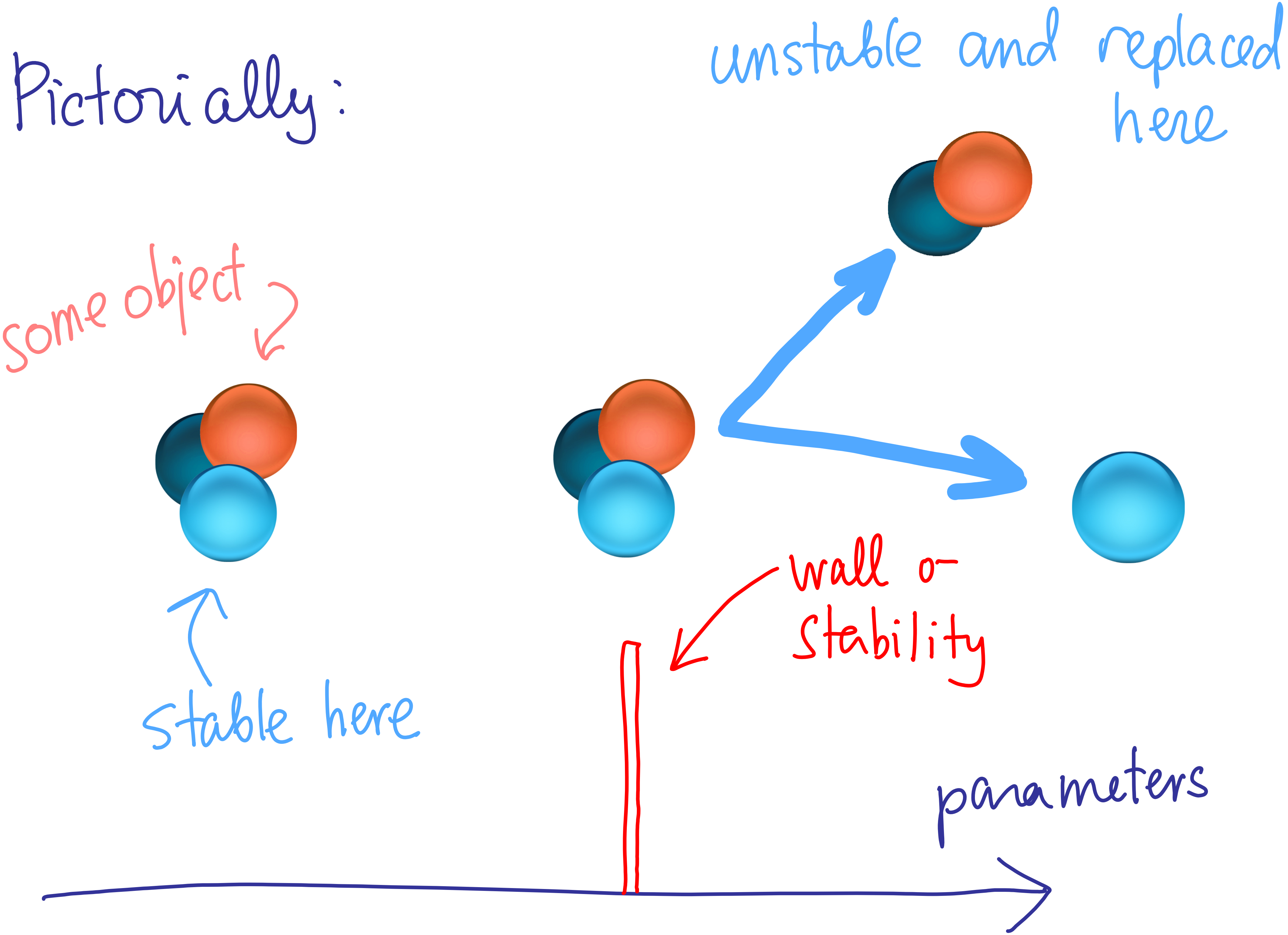


The ideas of

- stability
- variation of stability
- walls of stability
- wall-crossing etc

have been playing a very important role in today's mathematics and mathematical physics

Pictorially:



Basic example: quotients by a group action (e.g. by gauge transformations)

$$Y/G = \text{set of all } G\text{-orbits on } Y$$

Basic example: quotients by a group action (e.g. by gauge transformations)

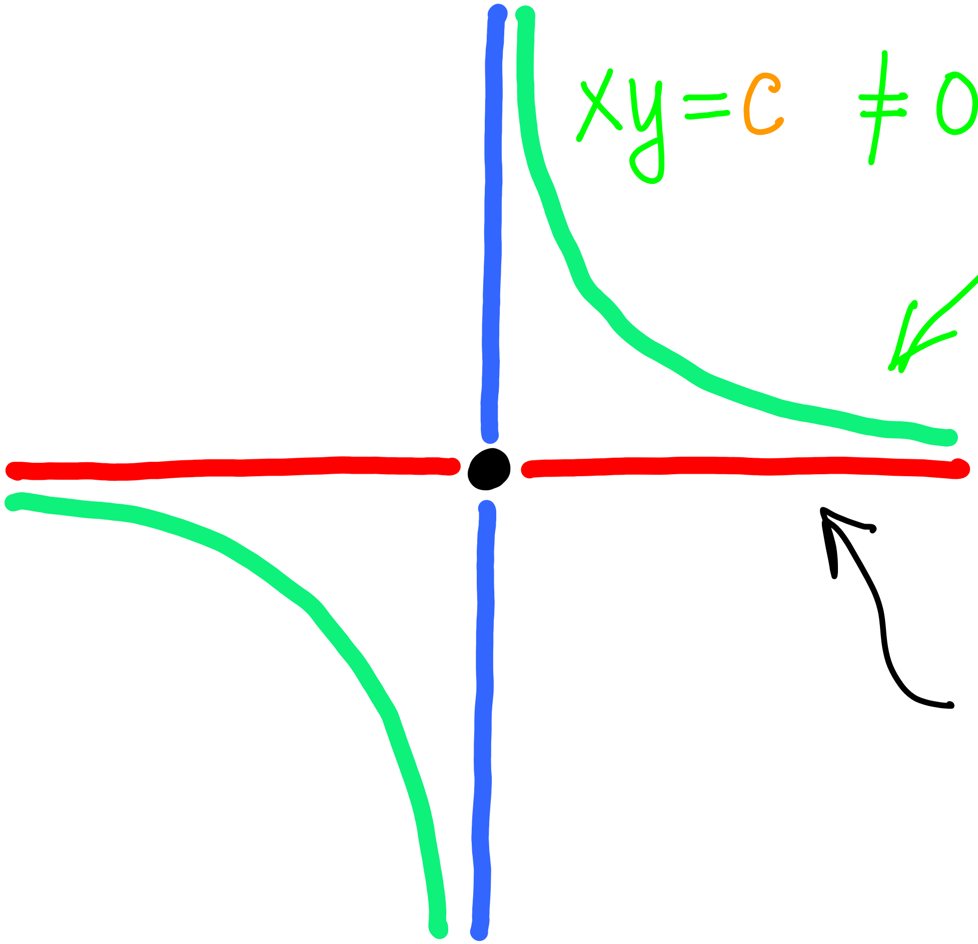
Y/G = set of ~~all~~ G-orbits on Y

= set of stable G-orbits on Y

Basic case $\quad Y = \mathbb{R}^2$ or $\mathbb{C}^2$

$$G = \left\{ \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \right\}$$

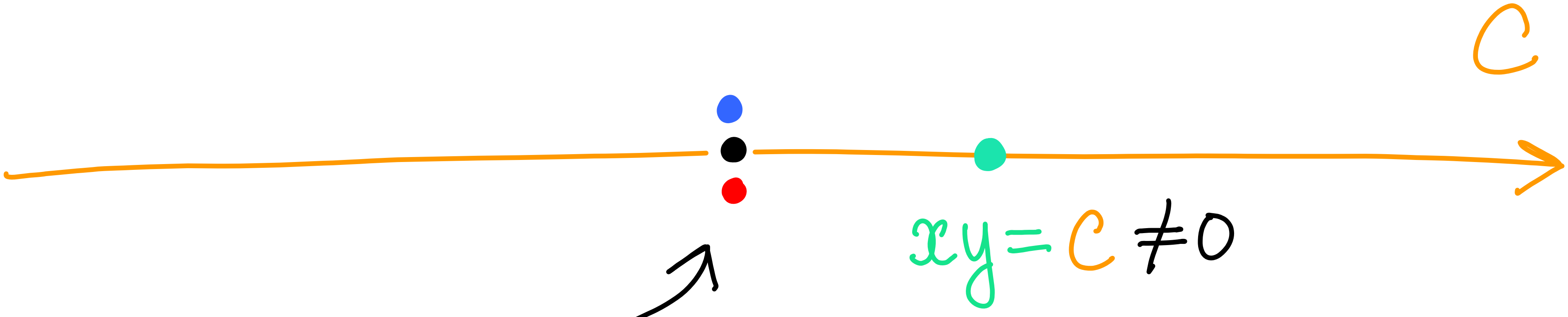
z real or complex



most orbits look like this

OK

problematic orbits come infinitely close to each other the set of all orbits looks like this



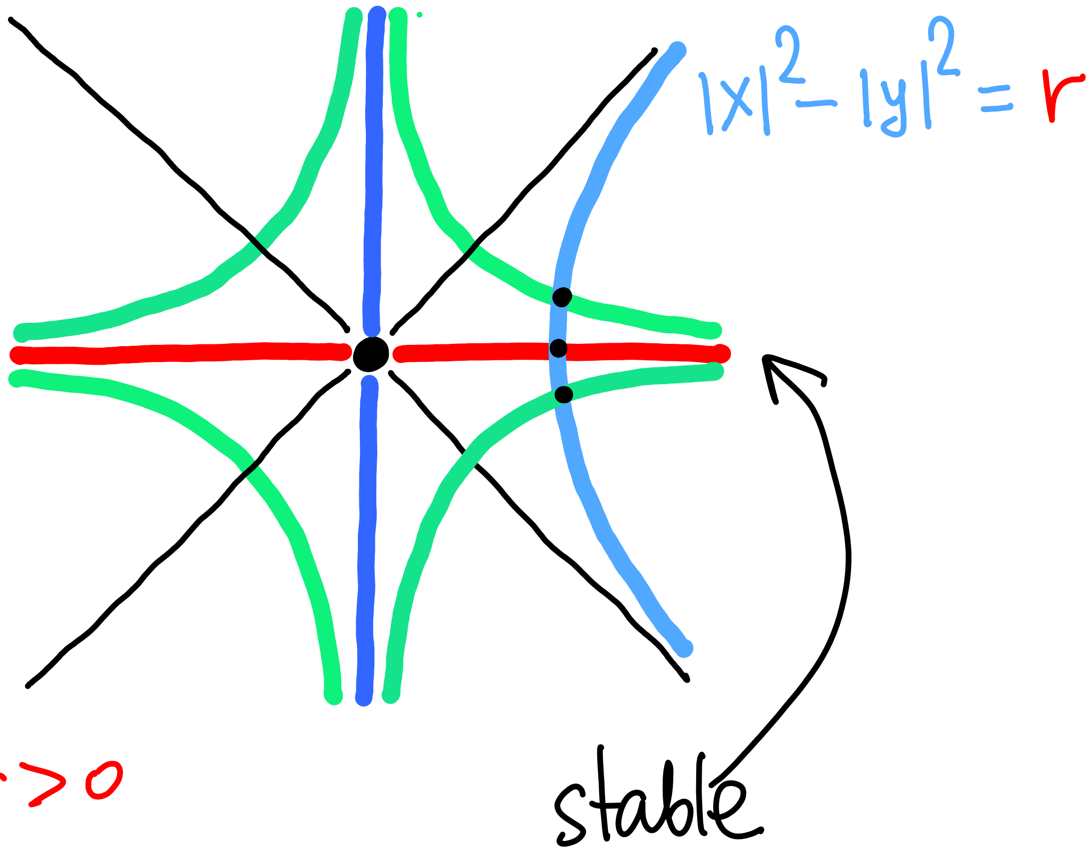
these three sit in the same chair

solution: declare only one of these stable, others unstable

can be implemented by

$Y/G \stackrel{\text{def}}{=}$ extra equation ~~$|z| = 1$~~

Hamiltonian for $z = e^{it}$



$r > 0$

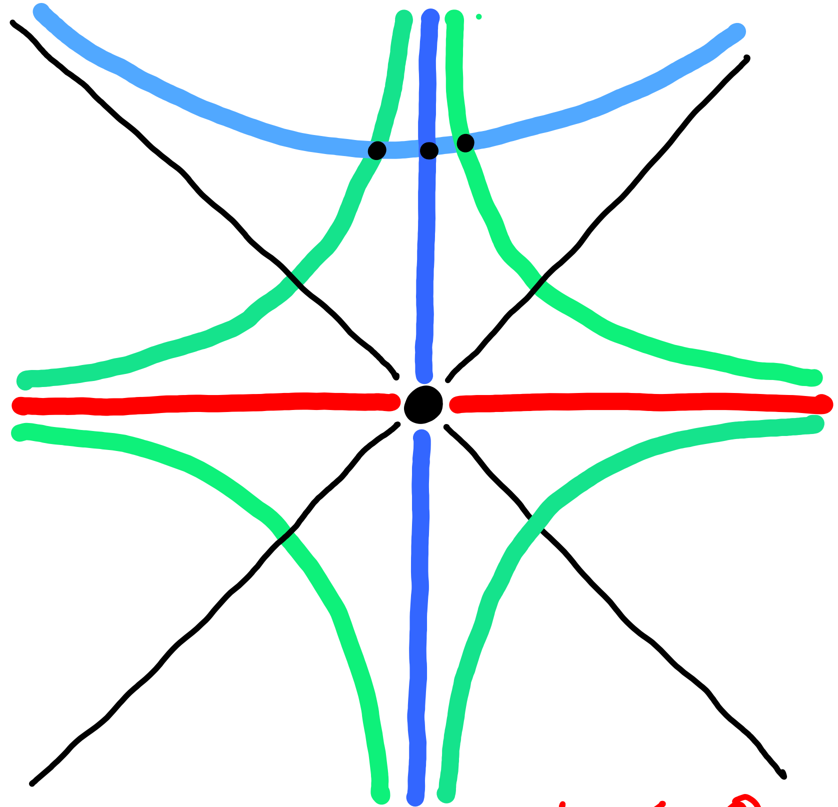
Can be implemented by

$Y/G \overset{\text{def}}{=}$ extra equation / $|z| = 1$

$\boxed{\text{Hamiltonian for } z = e^{it}}$

stable

$|x|^2 - |y|^2 = r$



$r < 0$

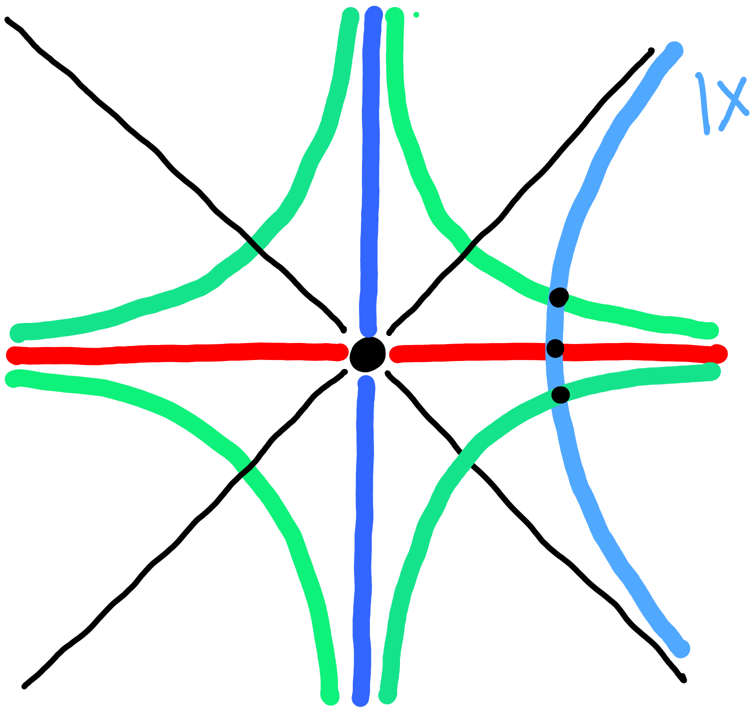
can be implemented by

$Y/G \stackrel{\text{def}}{=}$ extra equation

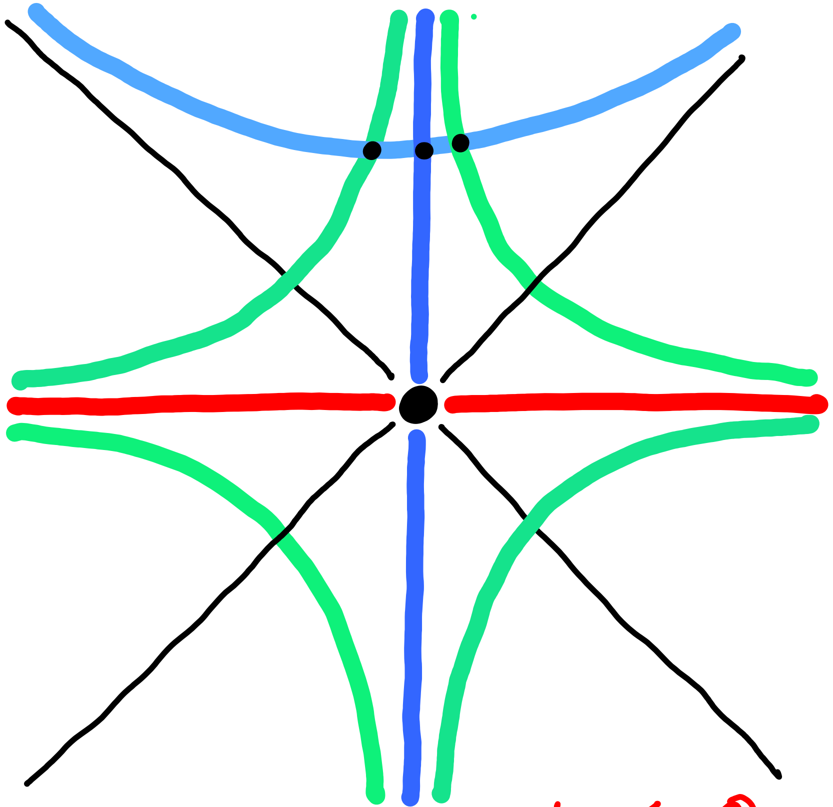
$|z| = 1$

Hamiltonian for $z = e^{it}$

stable



$|x|^2 - |y|^2 = r$



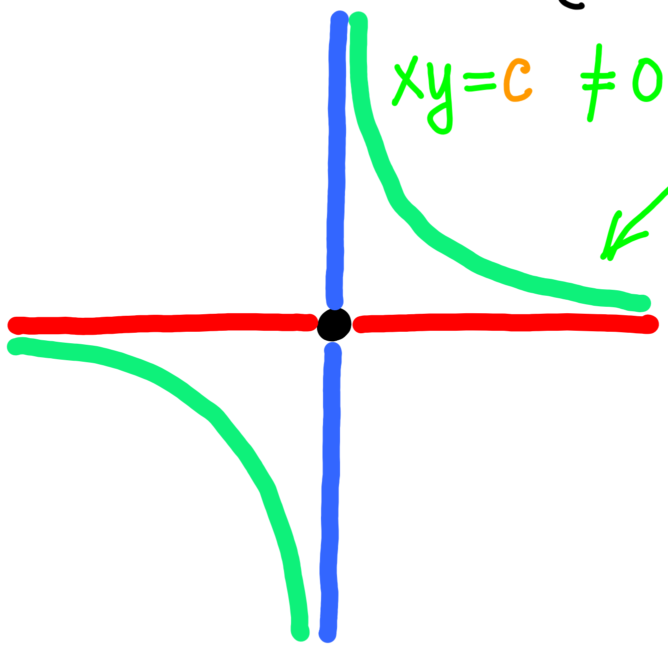
$r > 0$

stable

$r < 0$

Basic case $Y = \mathbb{R}^2$ or $\mathbb{C}^2$

$$G = \left\{ \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \right\}$$ z real or complex



most orbits look like this

<u>OK</u>

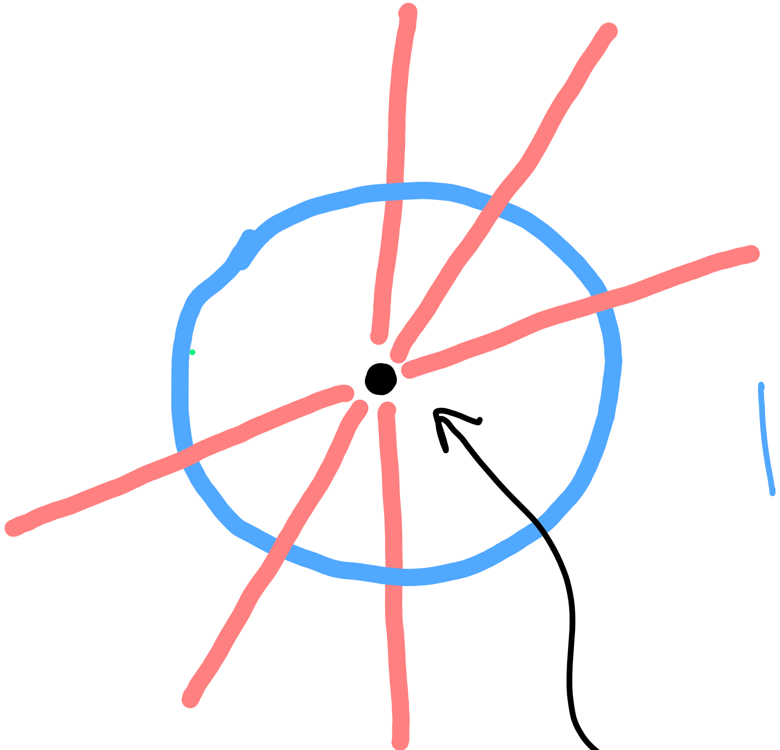
it was important here that G preserved the volume

$$\det \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} = 1$$

otherwise

Same groups acts <u>differently</u> on the same space

$$G = \left\{ \begin{pmatrix} z & \\ & z^{+1} \end{pmatrix} \right\}$$

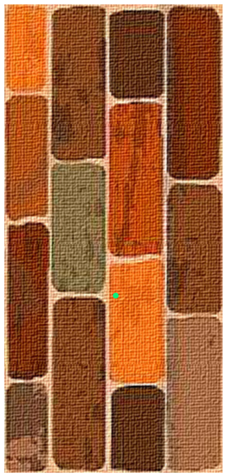


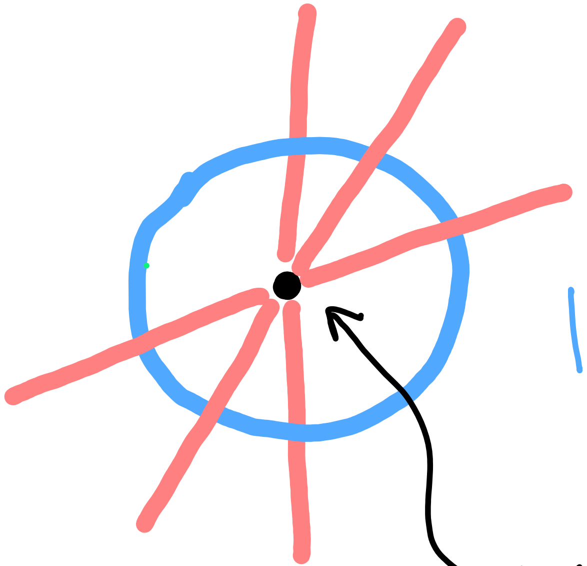
$$|x|^2 + |y|^2 = r > 0$$

unstable orbit

nothing, empty set

$r < 0$





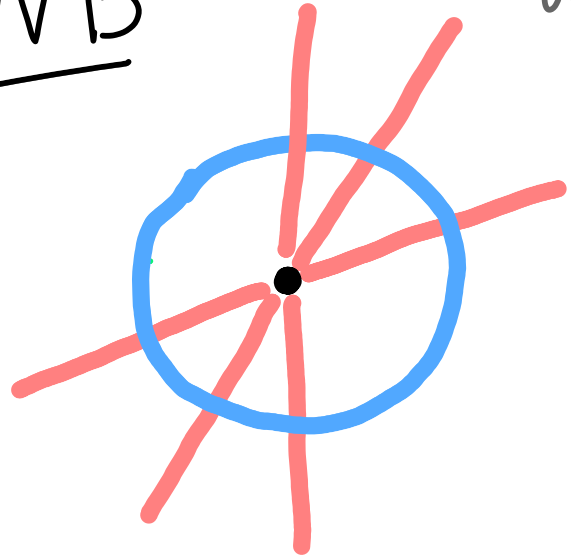
$|x|^2 + |y|^2 = r > 0$

unstable orbit

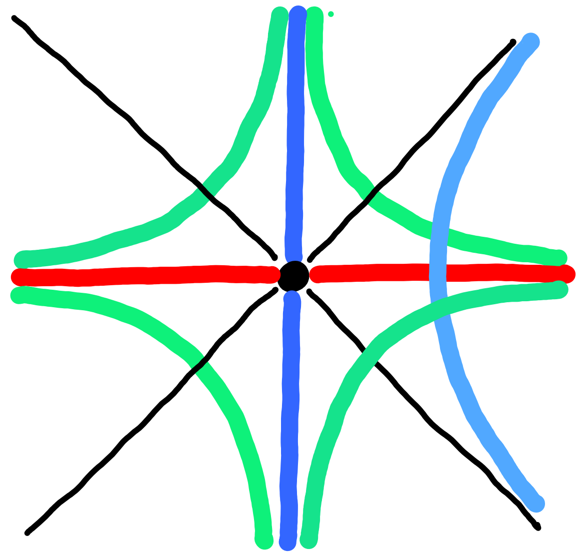
we don't want this, and in fact, we prefer when G preserves a symplectic form (like in classical mech.)

NB

in any case



$$|x|^2 \pm |y|^2 = r$$



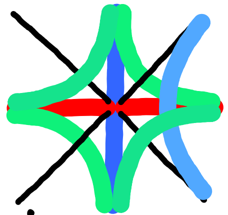
"Kähler moduli"

① the variable r corresponds to the "size" of the quotient

② in general, this is a vector that describes the volumes of different holomorphic / minimal surfaces $\subset Y/G$

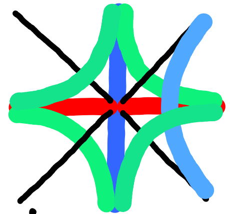
moduli of

$\mathcal{M} :=$ Higgs vacua = bottom of the potential / global gauge

in SUSY+ gauge theories are of the  kind
and they wallcross nicely ("flop")
as the parameters of the theory change

moduli of

$\mathcal{M}$:= Higgs vacua = bottom of the potential / global gauge

in SUSY+ gauge theories are of the  kind
and they wallcross nicely ("flop") as the parameters of the theory change

sophisticated geometries, in particular
Nakajima quiver varieties

the general philosophy of Bondal-Orlov & Kawamata predicts that while different $\mathcal{M}$ may not be the same, they as close as algebraic varieties can be, they have the same

complexes of coherent sheaves

boundary conditions in 2d

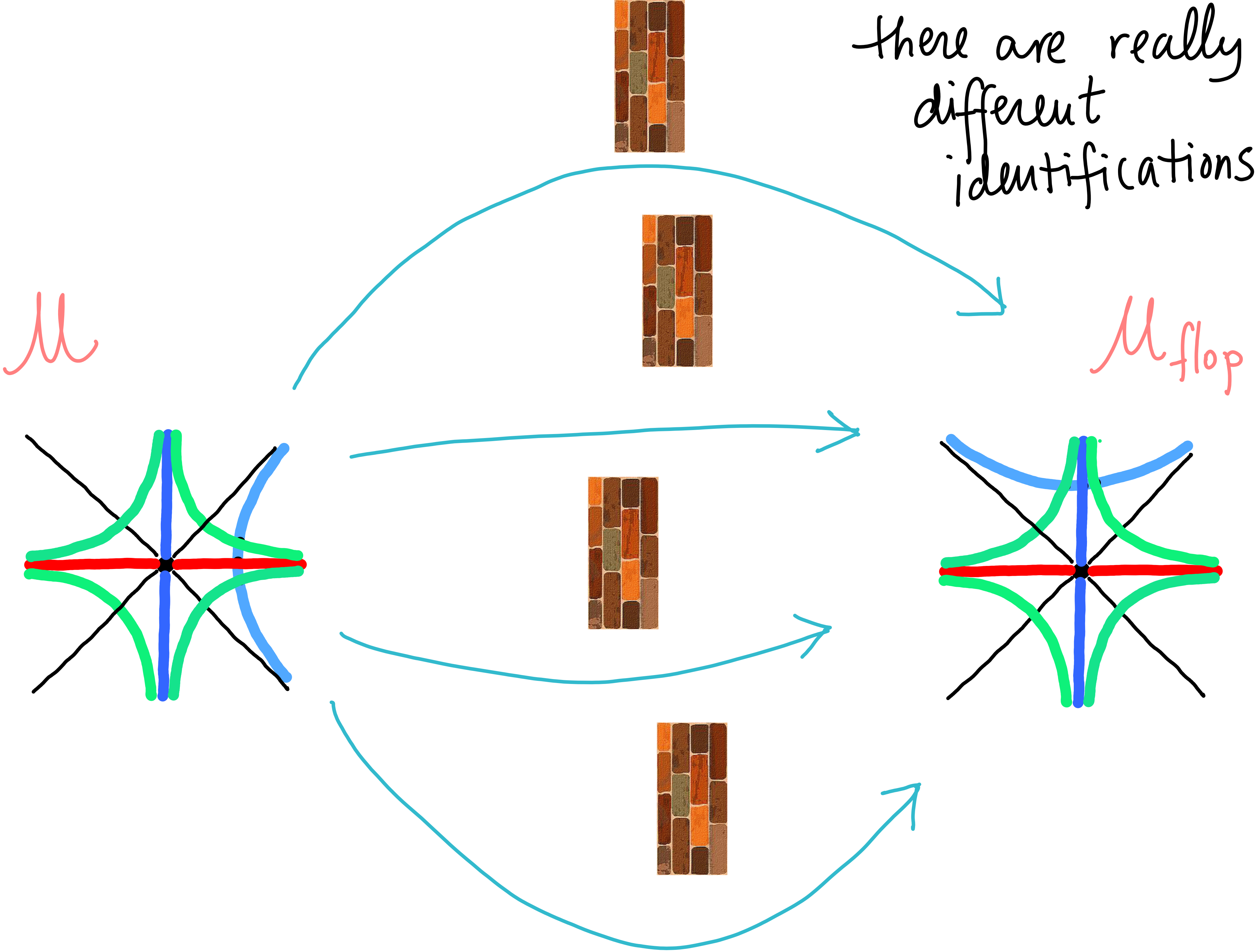
line operators in 3d

the general philosophy of Bondal-Orlov & Kawamata predicts that while different $\mathcal{M}$ may not be the same, they as close as algebraic varieties can be, they have the same complexes of coherent sheaves

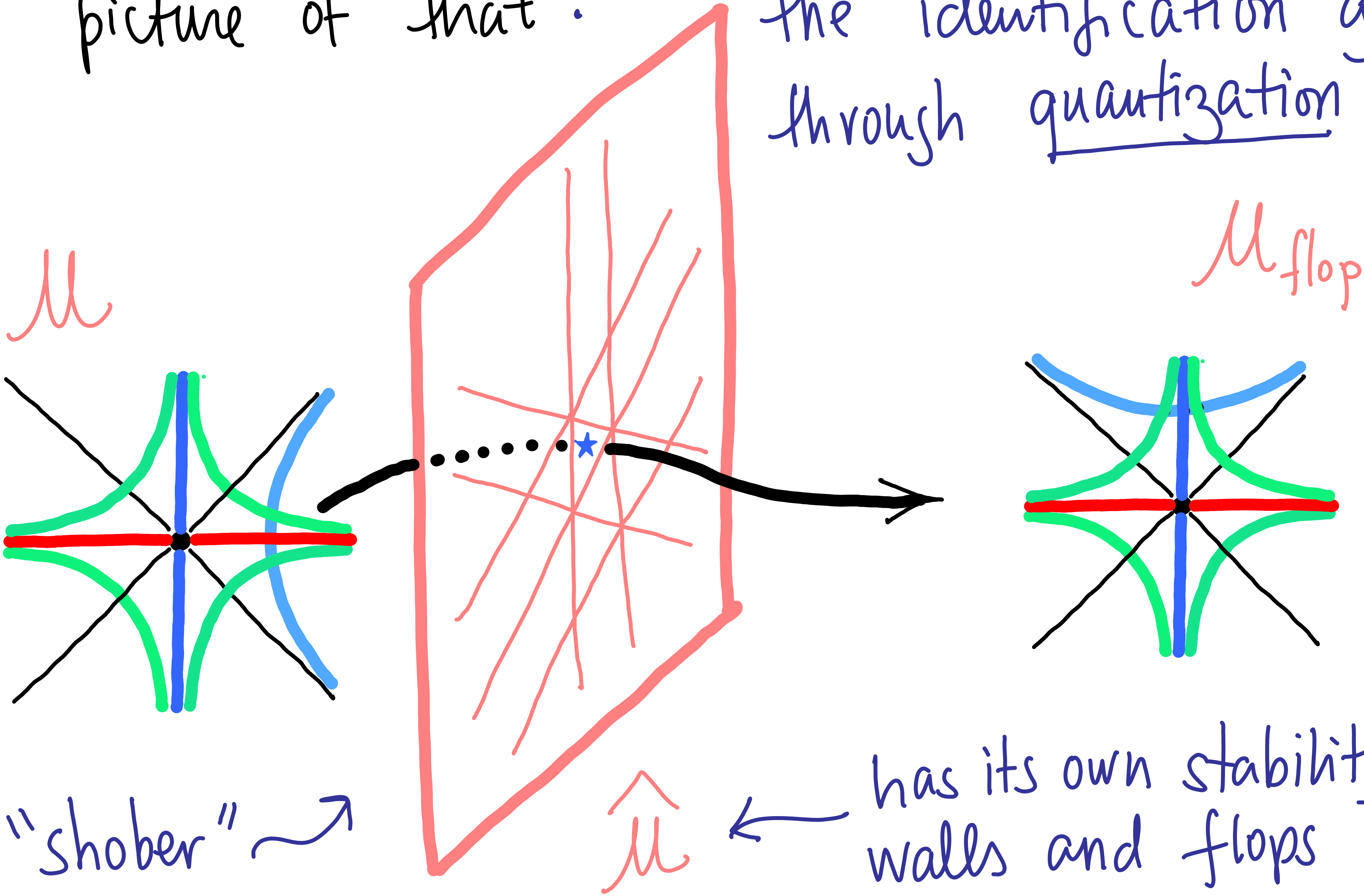
boundary conditions in 2d

line operators in 3d

sounds like we did not need to bother, but there is wallcrossing to wallcrossing!


there are really different identifications
$\mathcal{M}$
$\mathcal{M}_{flop}$

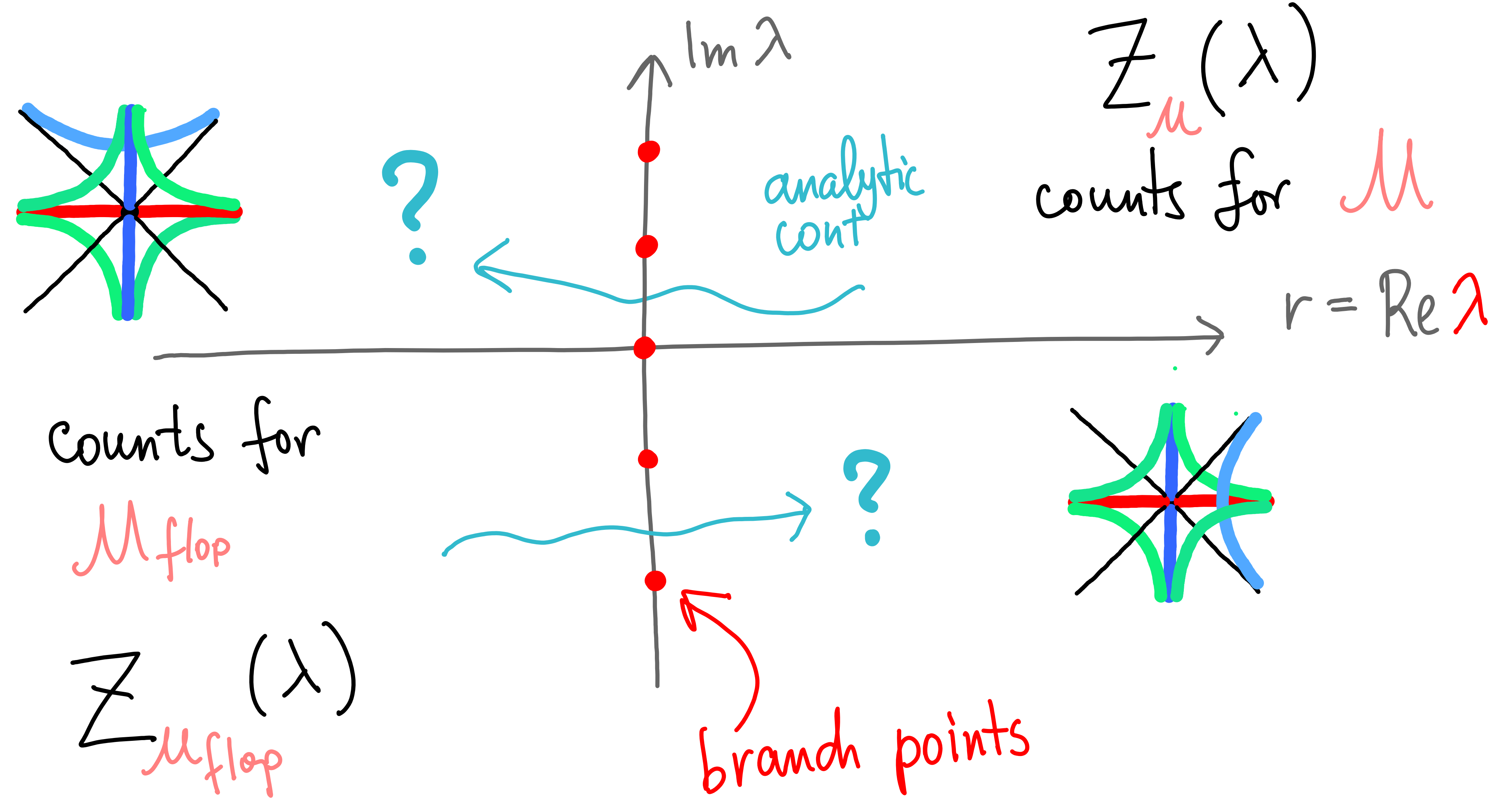
Bezrukavnikov & Kaledin have a really nice picture of that: the identification goes through quantization $\hat{\mathcal{M}}$

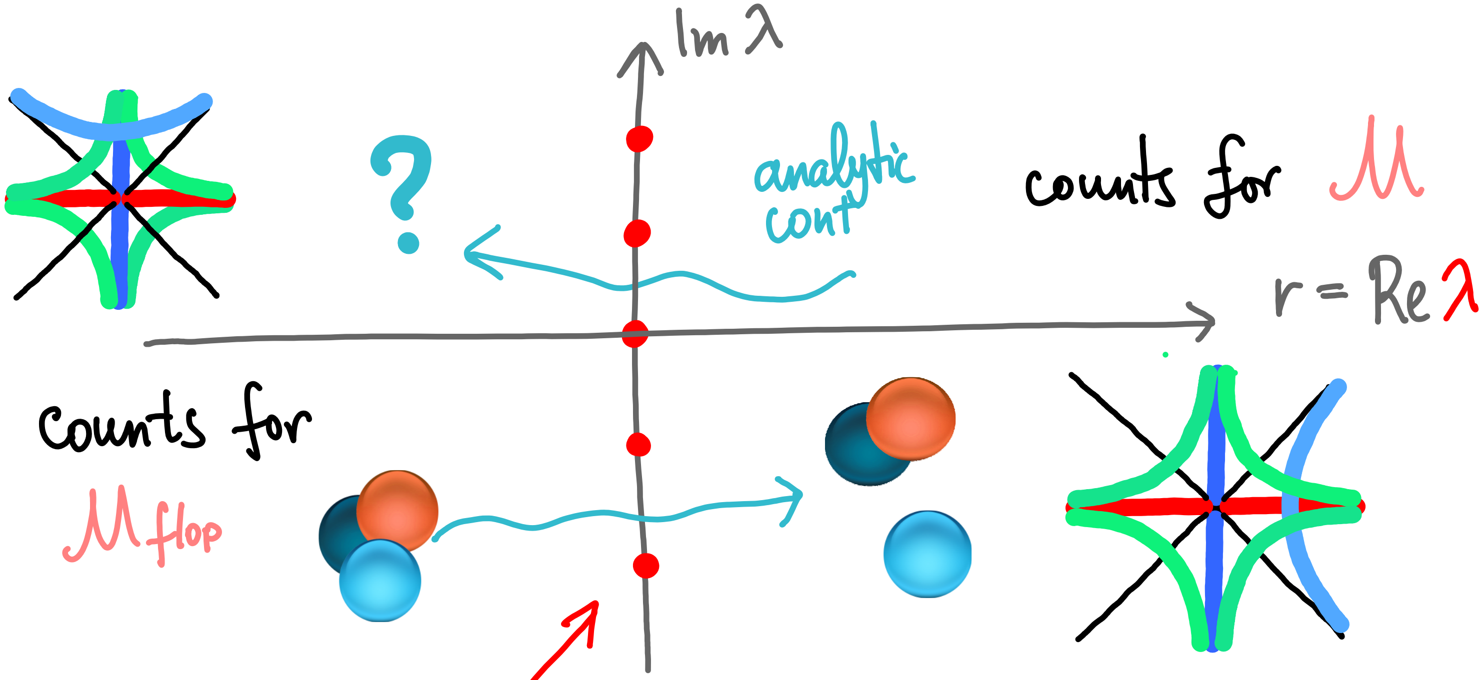


"shober"

has its own stability walls and flops

A quantifyable manifestation: susy indices / instanton counts in 2D have analytic continuation in Kähler moduli λ





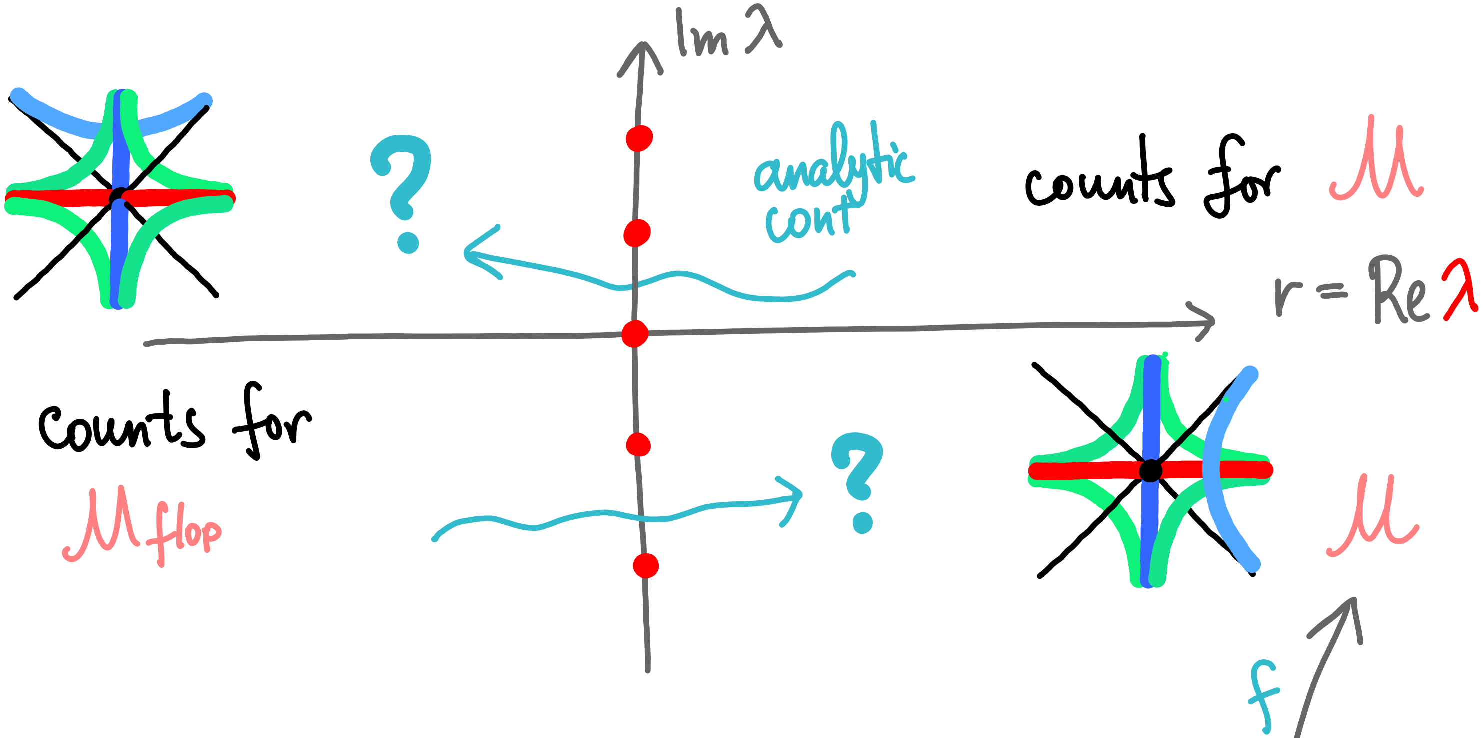
the vector

Theorem same vector-valued functions of λ up to different BK identifications of boundary cond.

in particular for the quantization $\hat{\mathcal{M}}$ these branch points are exactly the BK walls

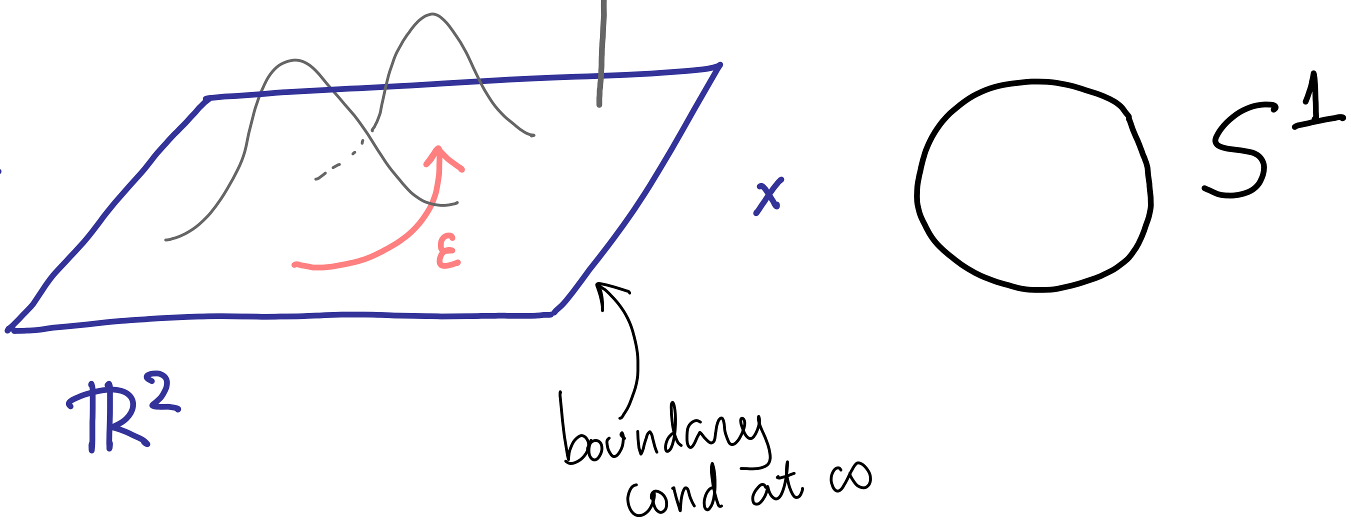
Actually, a rather delicate result to prove; took Bezrukavnikov & I almost 10 years of work....

An important techical ingredient came, in particular from a paper of [Aganagic-O.] where we solve the same analytic continuation problem but now ...



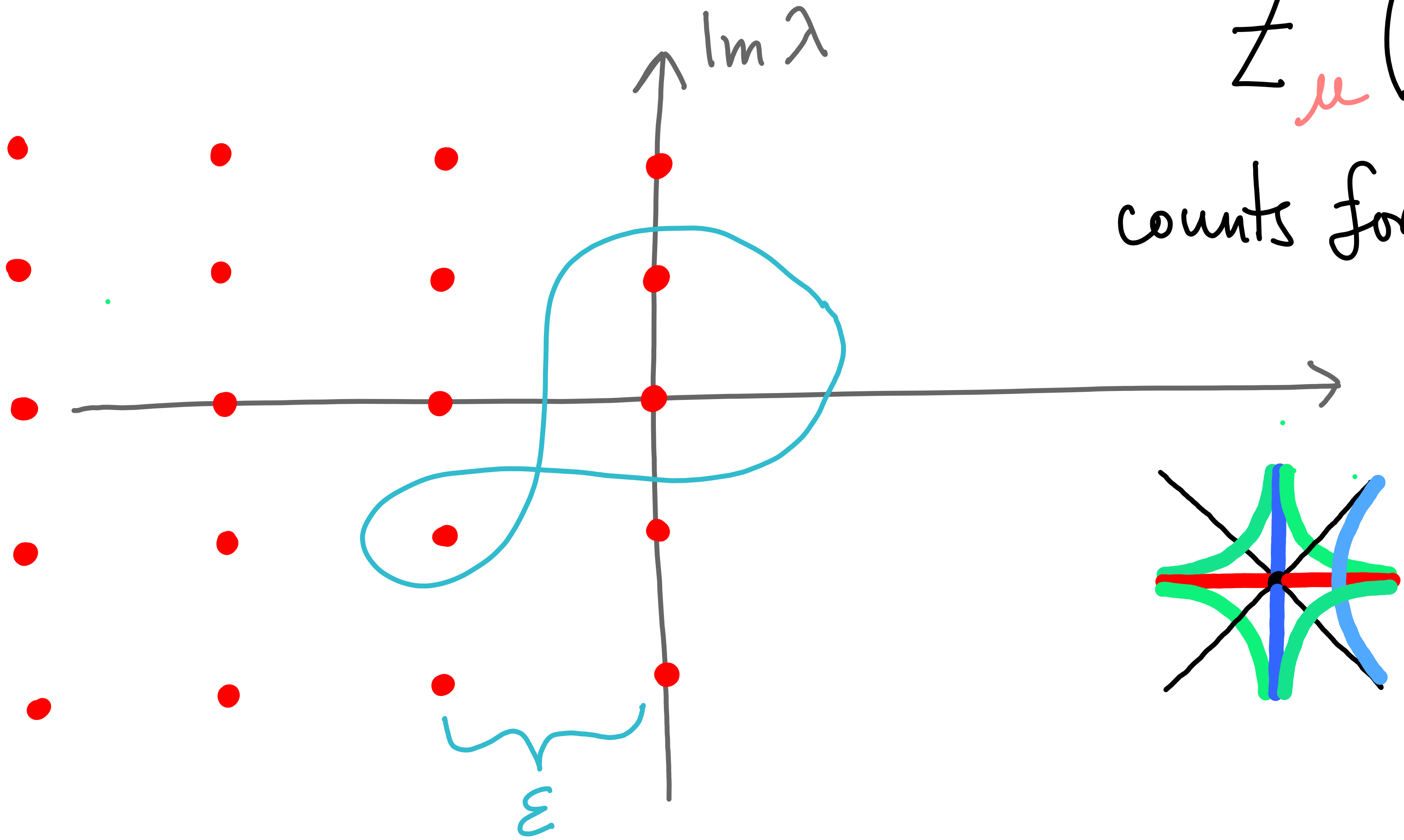
but now for 3-dimensional gauge theories on

space-time =



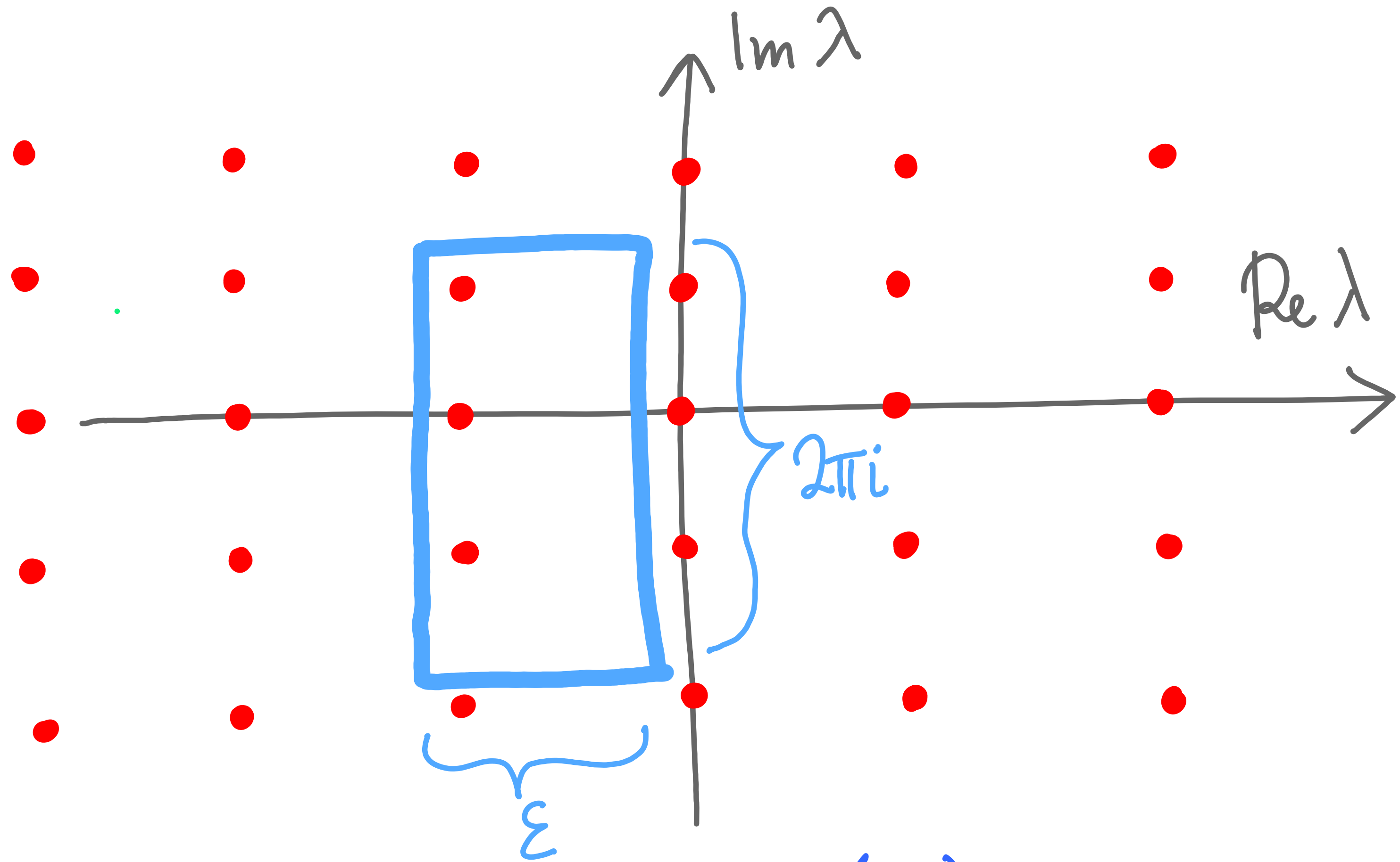
$Z_{\mathcal{M}}(\lambda)$

counts for $\mathcal{M}$



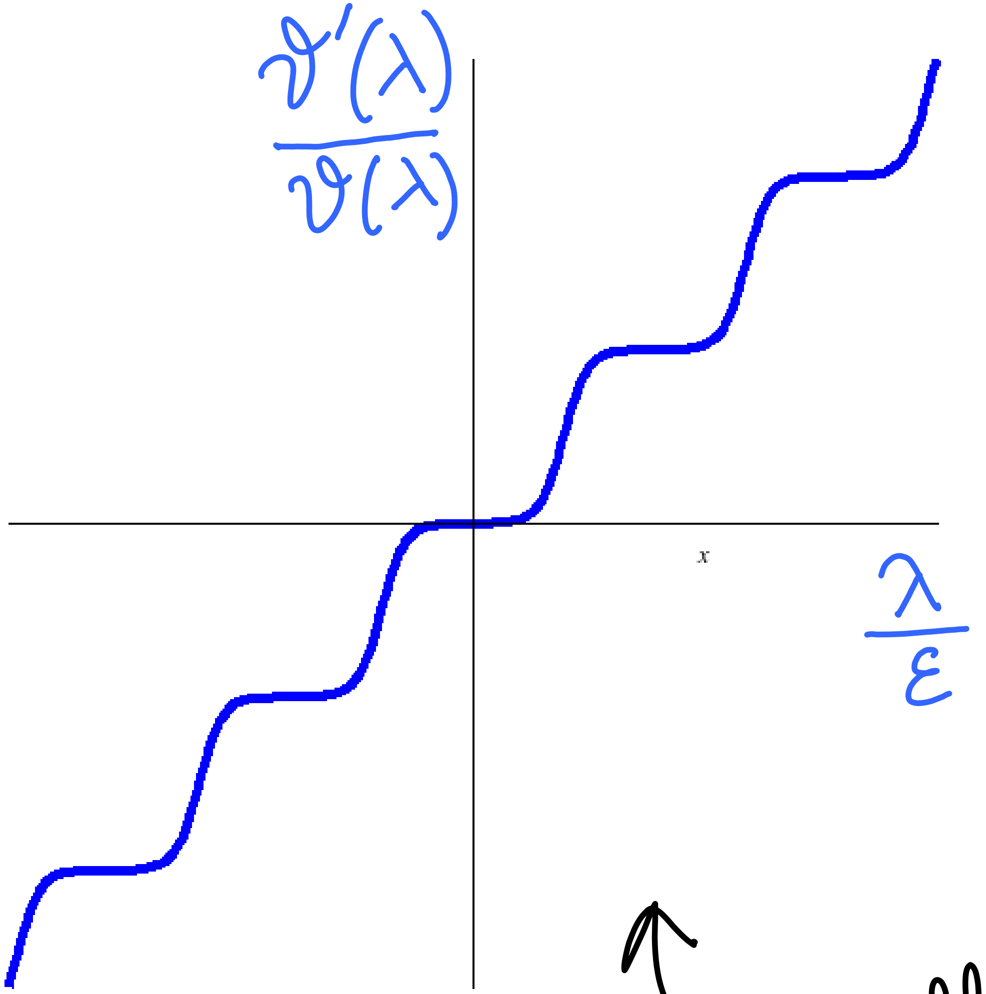
the counts are now <u>single-valued</u> with progressions of poles and if we compare the two

$$\frac{Z_{\mu}(\lambda)}{Z_{\mu_{flop}}(\lambda)} =$$



we get one elliptic matrix $W(\lambda)$ that contains <u>all</u> BK matrices as special cases depending on $\operatorname{Im}\lambda$

$\varepsilon \to 0$



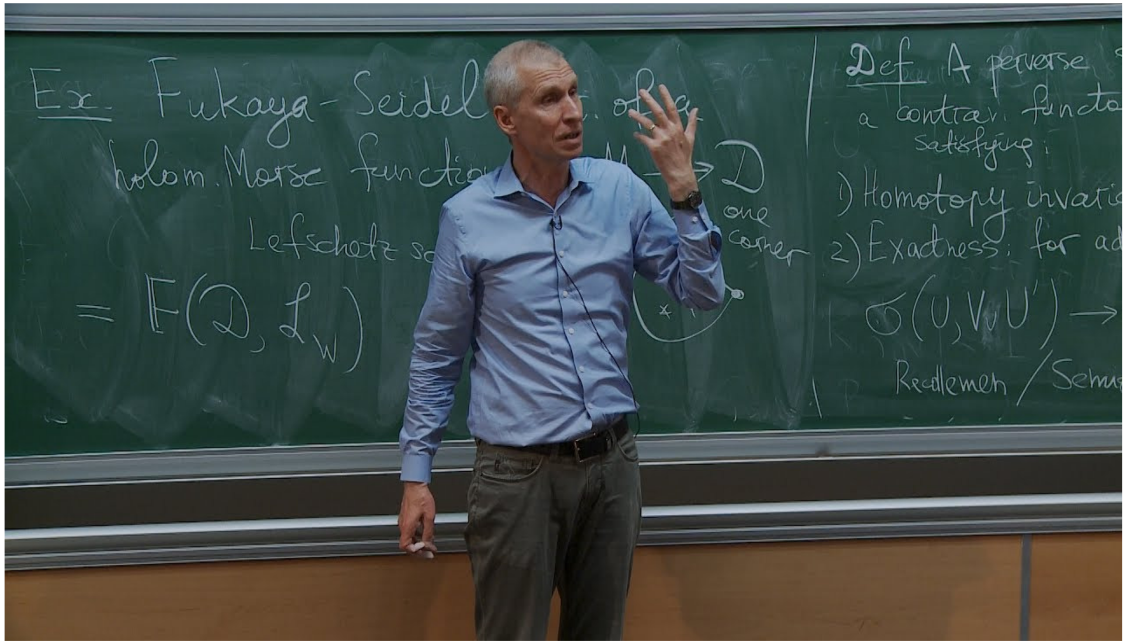
elliptic functions have a piecewise asymptotics as one of the periods grows/shrinks

walls develop

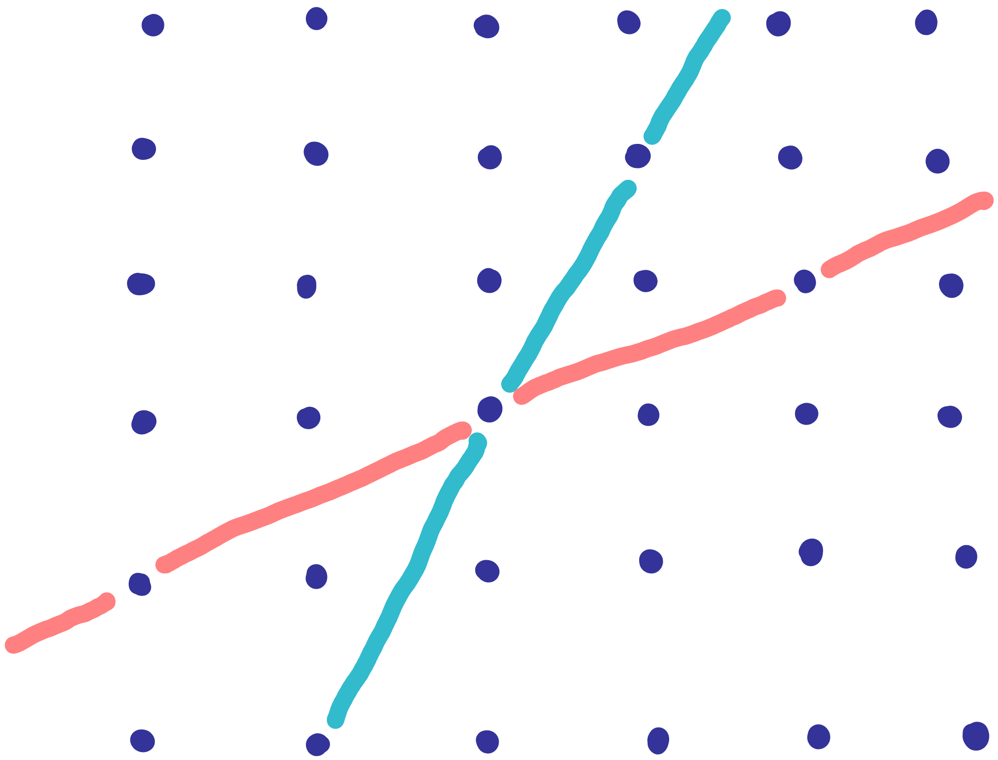
it identifies

$$\text{Elliptic cohomology}(\mathcal{M}) \xleftrightarrow{\ \mathcal{W}(\lambda)\ } \text{Elliptic cohomology}(\mathcal{M}_{\text{flop}})$$

and may be described as a "Weyl group" element in a certain elliptic quantum group



Apropos the idea that algebras should have generators is also about walls



different rank 1 subalgebras

$\Big\downarrow$ Weyl

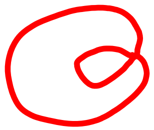
different BK monodromies

$\Uparrow$

one elliptic object

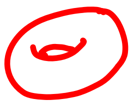
in particular, the quantum group relevant in 2D has many generators $\Leftarrow$ one elliptic object

Next step: categorify



$$D^b\,\mathrm{Coh}\,\mathcal{M} \xrightarrow[\text{Bezrukavnikov - Kaledin}]{\text{Bondal-Orlov, Kawamata, ...}} D^b\,\mathrm{Coh}\,\mathcal{M}_{flop}$$

$\Uparrow$ $q \to 0$



$\mathbb{C}^*/q^{\mathbb{Z}}$

D^b modules over certain operator algebras over $\mathcal{M}$ $\longrightarrow$ same for $\mathcal{M}_{flop}$

graded by q^{L_0}

boundary conditions in 3D, viewed categorically


THERE IS A WORLD WHERE
EVERYTHING IS ELLIPTIC
AND THERE ARE NO WALLS
KAVLI IPMU