Extension Theorems for Differential Forms on Singular Spaces

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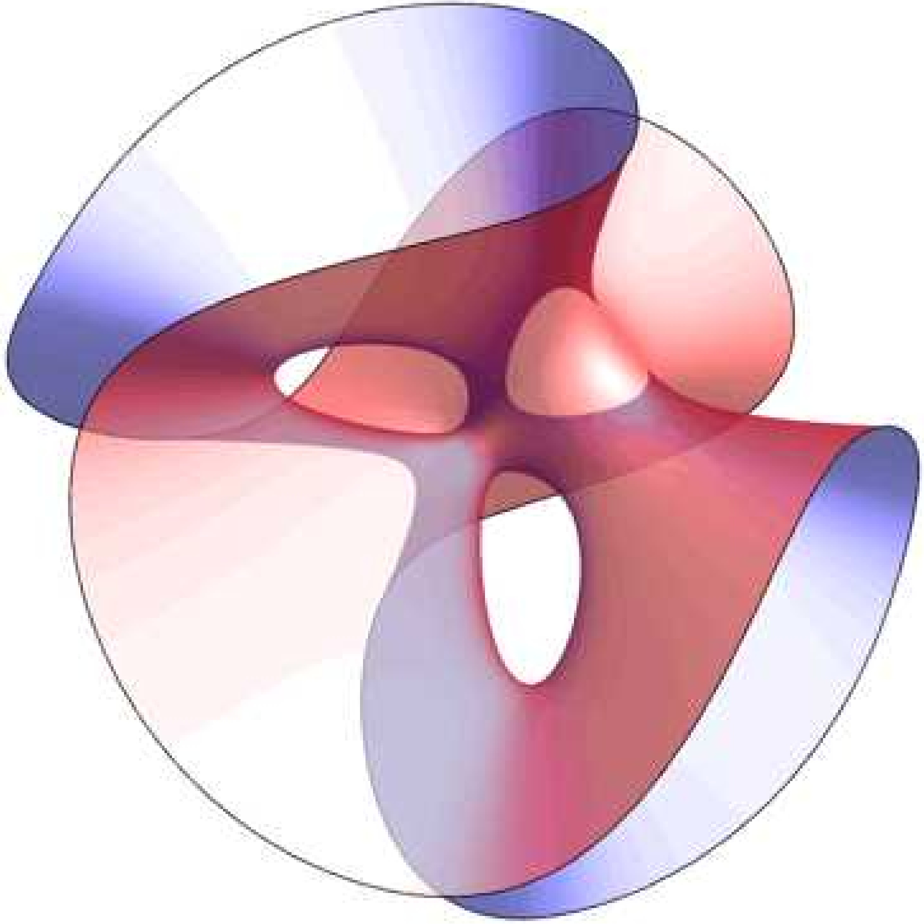
Part I

Complex spaces

Complex spaces

Near the origin in $\mathbb{C}^d$, consider the common zero set of finitely many holomorphic functions. This is called a complex space.



Clebsch cubic surface: $1 + z_1^3 + z_2^3 + z_3^3 = (1 + z_1 + z_2 + z_3)^3$

Example: Hypersurfaces

A typical example is a hypersurface X, defined by

$$f(z_1, z_2, \ldots, z_d) = 0.$$

Here f is a convergent power series in $z_1, z_2, \ldots, z_d$.

Near points where $\partial f / \partial z_j \neq 0$, one can solve for z_j as a convergent power series in the other variables, and therefore parametrize the set X using $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_d$.

At such points, X is a complex manifold of dimension $d - 1$.

Example: Hypersurfaces

A typical example is a hypersurface X, defined by

$$f(z_1, z_2, \ldots, z_d) = 0.$$

Here f is a convergent power series in $z_1, z_2, \ldots, z_d$.

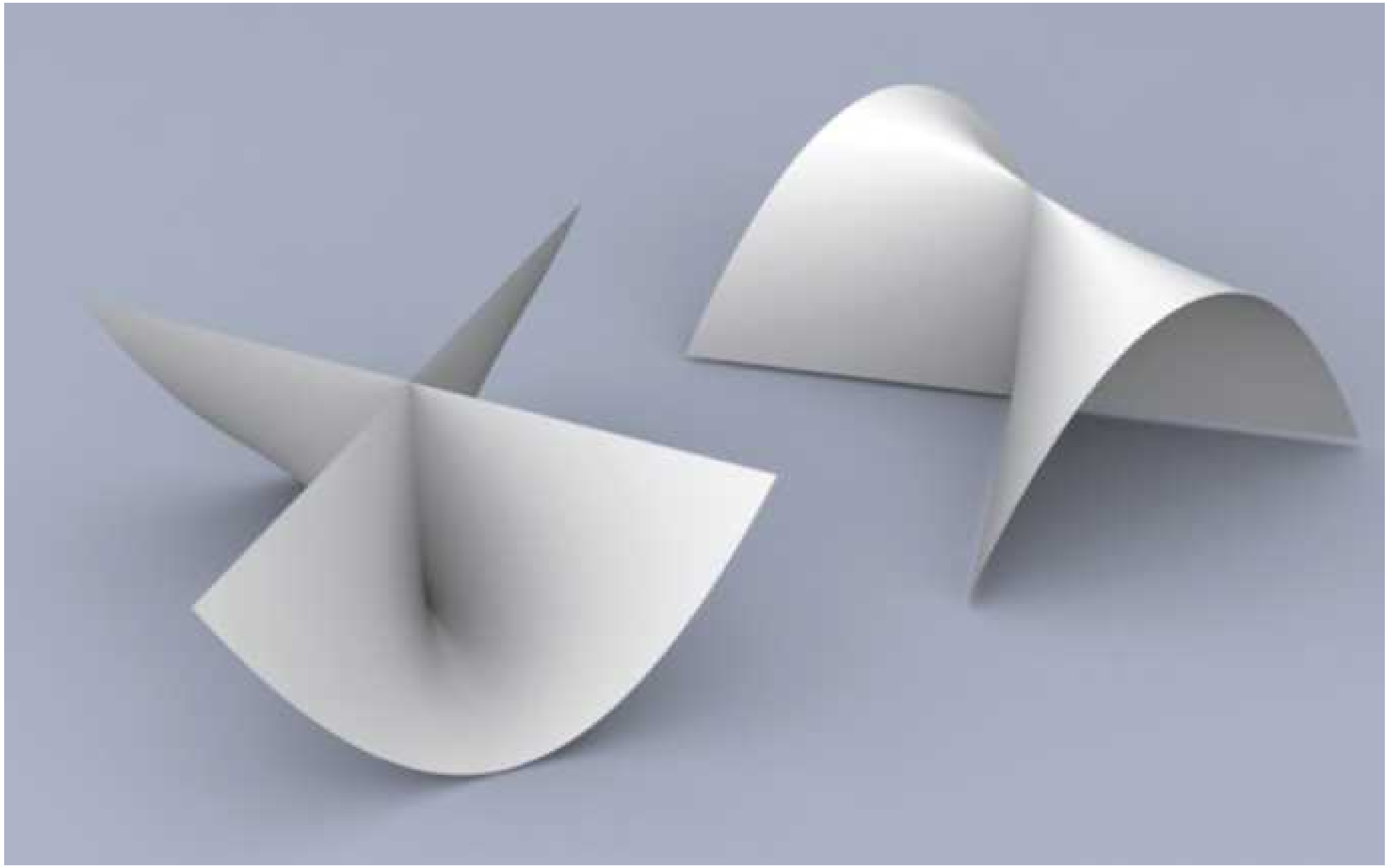
The points of the hypersurface where

$$f = \frac{\partial f}{\partial z_1} = \cdots = \frac{\partial f}{\partial z_d} = 0,$$

are called singular; they again form a complex space.

Regular and singular points

Near most points, a complex space X can be parametrized by holomorphic functions; such points are called regular.



Points where there is no parametrization are called singular.

Regular and singular points

Let X be a complex space.

The set of regular points is denoted by X_{reg}.

X_{reg} is a complex manifold that is open and dense in X.
Its dimension is called the dimension of the complex space.

The set of singular points is denoted by X_{sing}.

X_{sing} is again a complex space, but of smaller dimension.

Resolution of singularities

According to a very famous theorem by Hironaka (from 1964), one can always “resolve” the singularities of a complex space.

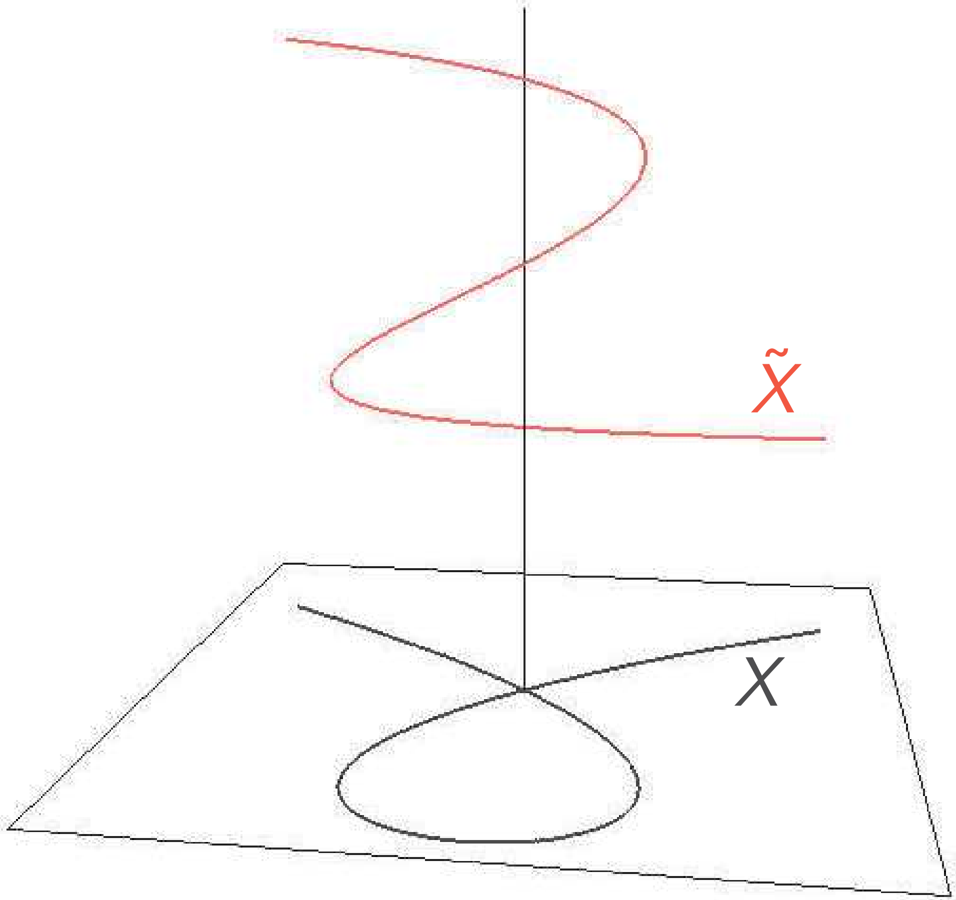
This process turns a complex space into a complex manifold.

Resolution of singularities

A resolution of singularities is a proper holomorphic mapping

$$r : \tilde{X} \to X,$$

from a complex manifold $\tilde{X}$, that is an isomorphism over the set of regular points X_{reg}.

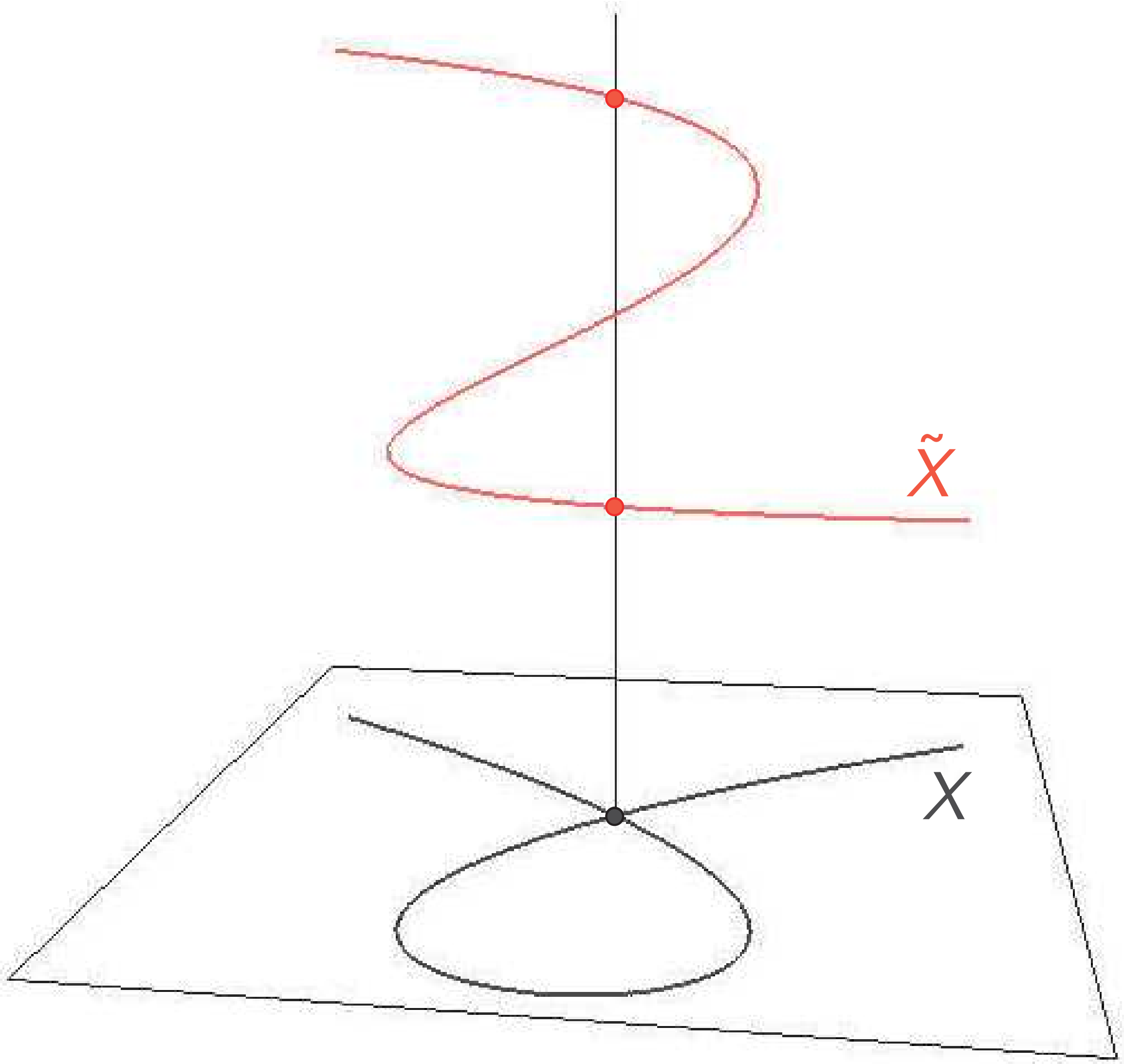


Part II

The extension problem

The extension problem

X_{reg} and its preimage in $\tilde{X}$ are isomorphic.



This allows us to transplant "objects" from one to the other.

The extension problem

My talk is about the following problem:

Let X be a complex space, and let $r\colon \tilde{X} \to X$ be a resolution of singularities that is an isomorphism over X_{reg}.

Given some “object” on the complex manifold X_{reg}, under what conditions does it extend to the complex manifold $\tilde{X}$?

Examples of such “objects” are:

- holomorphic functions
- holomorphic differential forms

Extending holomorphic functions

One example are holomorphic functions (= functions that, in local coordinates, can be written as convergent power series).

A complex space X is called normal if every holomorphic function on X_{reg} extends to a holomorphic function on $\tilde{X}$.

If X is normal, then

$$\dim X_{sing} \leq \dim X - 2;$$

for hypersurfaces, this condition is equivalent to normality.

Holomorphic differential forms

Another example are holomorphic differential forms.

On a complex manifold, choose local coordinates $z_1, \ldots, z_n$.

A holomorphic 1-form is an expression

$$f_1 dz_1 + f_2 dz_2 + \cdots + f_n dz_n$$

where $f_1, \ldots, f_n$ are holomorphic functions.

A holomorphic 2-form is an expression

$$f_{1,2} dz_1 \wedge dz_2 + f_{1,3} dz_1 \wedge dz_3 + \cdots + f_{n-1,n} dz_{n-1} \wedge dz_n$$

where $f_{1,2}, f_{1,3}, \ldots, f_{n-1,n}$ are holomorphic functions. Etc.

Holomorphic differential forms

A holomorphic *n*-form is an expression

$$f\,dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n,$$

where f is a holomorphic function.

If we change coordinates from $z_1, \ldots, z_n$ to, say, $w_1, \ldots, w_n$, the differentials transform by the rule

$$dz_j = \sum_{k=1}^{n} \frac{\partial z_j}{\partial w_k} dw_k.$$

The local expressions for a holomorphic p-form are required to transform accordingly.

Extending holomorphic differential forms

More precisely, my talk is about the following problem:

Let X be a normal complex space, and let $r\colon \tilde{X} \to X$ be a resolution of singularities that is an isomorphism over X_{reg}.

Given a holomorphic p-form on X_{reg}, under what conditions does it extend to a holomorphic p-form on $\tilde{X}$?

Recall that $\dim X_{sing} \leq \dim X - 2$ when X is normal.

Part III

Two Results

Greb, Kebekus, Kovacs, Peternell

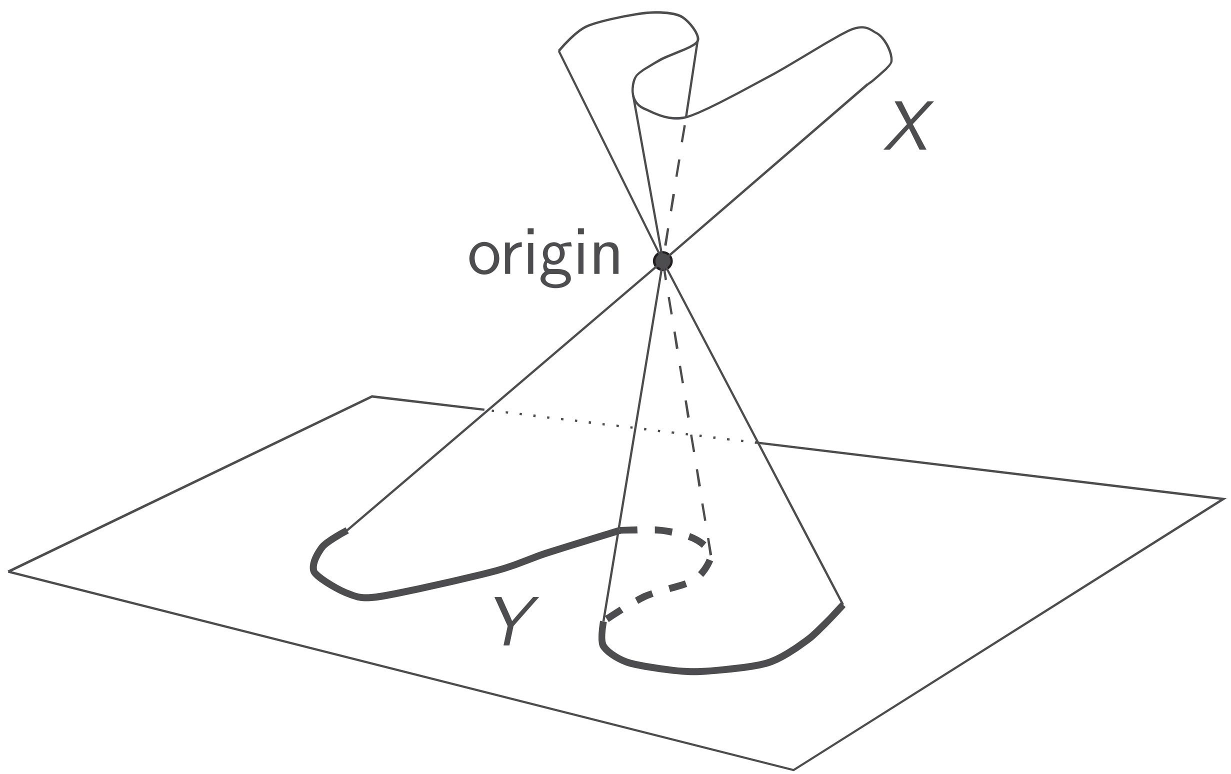
In 2011, Greb, Kebekus, Kovacs, and Peternell published what was then considered the last word on the extension problem:

Let X be a normal algebraic variety (defined by polynomials), and let $r\colon \tilde{X} \to X$ be a resolution of singularities that is an isomorphism over X_{reg}.

If the singularities of X are sufficiently nice (klt), then all holomorphic p-forms on X_{reg} extend to $\tilde{X}$ (for $1 \leq p \leq \dim X$).

Example: Cones

Cones over projective complex manifolds are a good test case.

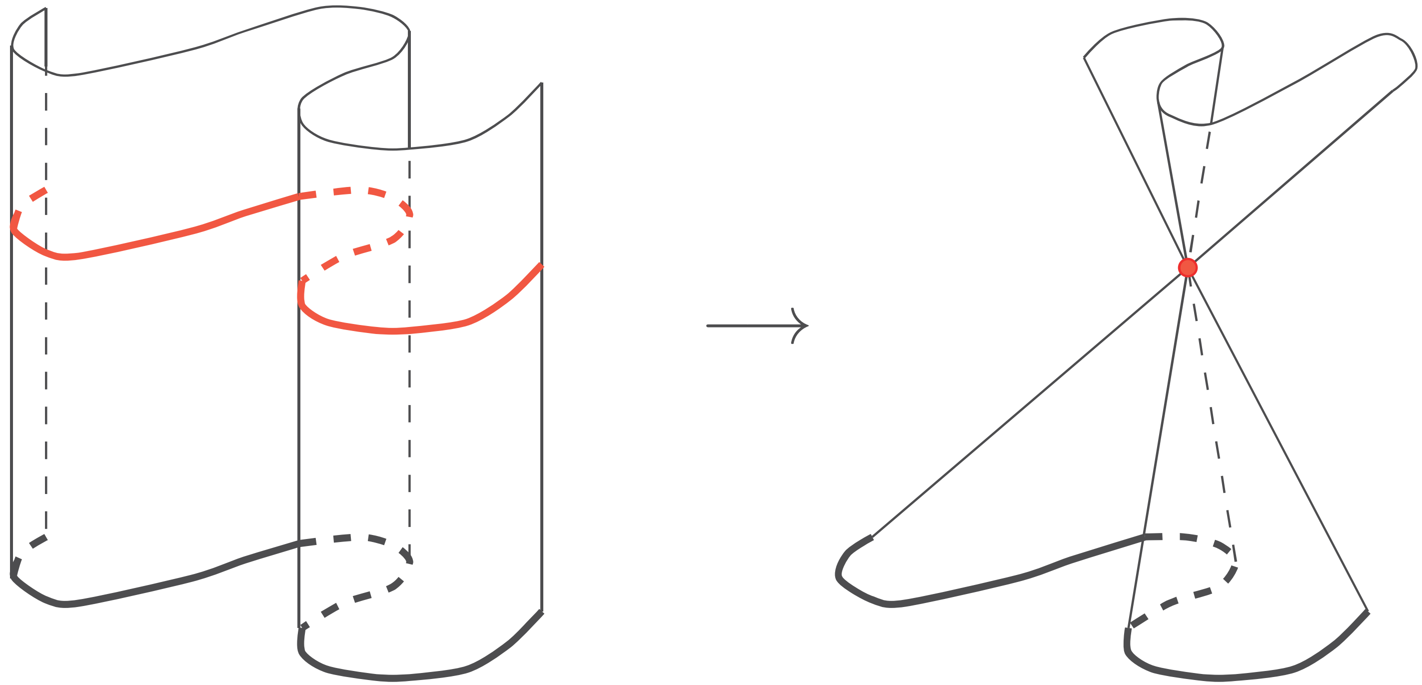


Let Y be a complex submanifold of dimension $n-1$ in $\mathbb{P}^{d-1}$.

Each point of Y is a line in $\mathbb{C}^d$, and the union of all those lines is an algebraic variety X, called the cone over Y.

Example: Cones

The vertex of the cone is the only singular point: $X_{sing} = \{0\}$.



The singularities of X are resolved by separating the lines that meet at the vertex. The resulting complex manifold $\tilde{X}$ is a line bundle over Y; the zero section gets contracted to the vertex.

Example: Cones

When X is a cone, one can calculate everything explicitly:

- When is X normal?
- When is X klt?
- When do all holomorphic p-forms on X_{reg} extend to $\tilde{X}$?

The conclusion is surprising:

If the cone X is normal and if all holomorphic n-forms extend, then all holomorphic p-forms with $1 \leq p \leq n$ also extend.

No further assumptions about the singularities are necessary!

Main result

The example of cones suggested the following optimal result, which Kebekus and I proved this summer:

Let X be a normal complex space, and let $r\colon \tilde{X} \to X$ be a resolution of singularities that is an isomorphism over X_{reg}.

Suppose that all holomorphic n-forms on X_{reg} extend to $\tilde{X}$.

Then all holomorphic p-forms on X_{reg} also extend to $\tilde{X}$.

Recall that $n = \dim X$, and $1 \leq p \leq n$.

Thank you!