# Mass and angular momentum in general relativity

Shing-Tung Yau

Harvard University

IPMU 10th Anniversary, Oct. 2017

This talk is based on joint work with Po-Ning Chen and Mu-Tao Wang.

More than 100 years ago, Einstein accomplished one of the most spectacular work in physics and radically changed the view of space and time in the history of mankind. The foundation laid by Isaac Newton on the theory of gravity was completely changed by the theory of general relativity.

In the very successful theory of Newton, space is static and time is independent of space. By 1905, when Einstein established special relativity along with Poincaré and others, it was realized that space and time are linked and that the very foundation of special relativity, and that information cannot travel faster than light, is in contradiction with Newtonian gravity where action at a distance was used.

Einstein learnt from his teacher in 1908 that special relativity is best described as the geometry of the Minkowski spacetime. He realized gravitational potential cannot be described by a scalar function. It should be described by a tensor. After tremendous helps from his two friends in mathematics : Grossmann and Hilbert, Einstein finally wrote down the famous Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}.$$

Note that Hilbert was the first one that write down the action principle of gravity, which plays the most important role in any attempts to quantize general relativity. The action is given by the total scalar curvature of the metric tensor which is considered to be the gravitational potential. If gravity is coupled with other matter, we simply add the matter Lagrangian.

The field equation was used by Einstein to calculate the perihelion of mercury and the light bending predicted by the Schwarzschild solution of the Einstein equation, which was found shortly. This was of course a great triumph of Einstein's theory of general relativity. However, since the theory is highly nonlinear and the geometry of space time is dynamical, the actual understanding of Einstein equation was very difficult: even to Einstein himself.

Einstein thought that the equation determined gravity completely. But that is actually not true as we cannot tell what is the initial condition for the field equation and we have difficulty to find the boundary condition.

There are many important problems in GR which are not solved. Many of them need deep understanding of geometry and analysis. The first major question is the question of the structure of singularities. Black hole appeared in Schwarzschild and Kerr solutions .

Penrose proposed the famous question of cosmic censorship. He claimed that for a generic space time, every singularity is hidden behind some membrane similar to black hole appeared in the above solutions.

The dynamical problem of the Einstein equation is still not solved yet. It is only understood in the very weak field limit case by the work of Christodoulou-Klainerman and Christodoulou.

There are many important physical quantities and questions that were understood in Newtonian mechanics. However, their counterparts are not easy to formulate, let alone to understand! These are largely due to the problem of gauge choice in general relativity. This problem started even before the field equation was written down, when Einstein attempted to use divergence free coordinate choice.

Einstein succeeded to define total mass for an isolated physical system, which was made precise by the famous work of Arnowitt-Deser-Misner. It was an important quantity to measure the whole physical system. Already Einstein found it difficult to know properties of such total mass. It needs to be positive for the system to be physically stable. This was finally proved by Schoen and I in 1979.

Physical quantities are gauge independent and their relation to geometry becomes very interesting. And we shall discuss them in this talk.

As is well known, it is not possible to find mass density of gravity in general relativity. The mass density would have to be first derivative of the metric tensor which is zero in suitable chosen coordinate at a point.

But we still desire to measure the total mass in a space like region bounded by a closed surface.

The mass due to gravity should be computable from the intrinsic and the extrinsic geometry of the surface. It has been an important question to find the right definition.

Penrose gave a talk on this question in my seminar at the Institute for Advanced Study in 1979. The quantity is called quasilocal mass.

Penrose listed it as the first major problem in his list of open problems.

Many people, including Penrose, Hawking-Horowitz, Brown-York, Bartnik, and others worked on this problem and various definitions were given.

About 15 years ago, I was interested in how to formulate a criterion for existence of black hole, that Kip Thorne called hoop conjecture. The statement says that if the quasi-local mass of a closed surface is greater than certain multiple of the diameter of the surface, then the closed surface will collapse to a black hole. (perhaps the length of shortest closed geodesic is a better quantity than diameter )

Hence a good definition of quasi-local mass is needed.

I list properties that the definition should satisfy :

1. It should be nonnegative and zero for any closed surfaces in flat Minkowski spacetime.

2. It should converge to the familiar ADM mass for asymptotically flat spacetime if we have a sequence of coordinate spheres that approaches the spatial infinity of an asymptotic flat slice.

3. It should converge to the Bondi mass when the spheres diverge to a cut at null infinity

4. It should be equivalent to the standard Komar mass in a stationary spacetime .

For a closed surface $S$ enclosing a spacelike region $D$, the Lagrangian formulation gave the definition of a surface Hamilton which can be described as follows:

$$\int_S N\,{}^2K + N^\mu p_{\mu\nu} r^\nu .$$

Here $w^\nu$ is a future time-like unit vector field along $S$ that corresponds to unit translation. We write $w^\mu = N\,n^\mu + N^\mu$ alone the surface $S$, where $n^\mu$ is the timelike unit normal of $D$ restricting to $S$. $N$ is the lapse and $N^\mu$ is the shift. $r^\mu$ is the space-like unit normal orthogonal to $n^\mu$ along $S$, and $p$ is the second fundamental form of $D$, and ${}^2K$ is the mean curvature of $S$ with respect to $r^\mu$.

Here the geometric quantities are determined by the spacelike region $D$ and the gauge choice of timelike vector fields along $D$ .

In time symmetric case, Brown and York set $N = 1$ and $N^\nu = 0$ in the definition of the surface Hamiltonian and subtract the same expression obtained by isometrically embedding the surface into the Euclidean 3-space.

Despite that this definition depends on the choice of the gauge along the boundary, Shi and Tam managed to prove the positivity if the local energy condition holds. In the time symmetric case, the local energy condition simply means the scalar curvature is non-negative.

Immediately after the work of Shi and Tam, Liu and Yau redefine the Brown-York mass to make it gauge independent and prove the positivity.

The Liu-Yau definition is

$$\frac{1}{8\pi}\int(H_0 - |H|)d\Sigma$$

where $H_0$ is the mean curvature of the isometric embedding into the Euclidean 3-space and $|H|$ is the norm of the mean curvature vector of the physical surface.

The positivity proof requires non-trivial estimate of Dirac spinors across a non-smooth surface.

The definition of quasi-local mass is finally achieved in 2009 by Wang and Yau. In the process, several important contributions are used: the quasi-spherical construction of Bartnik (which was also used by Shi-Tam), the positive mass theorem of Schoen and Yau which depends on solving the Jang equation, Witten's proof of positive mass theorem using spinors, and Shi-Tam's gluing construction.

To be precise, the Wang-Yau quasi local mass is defined in the following way:

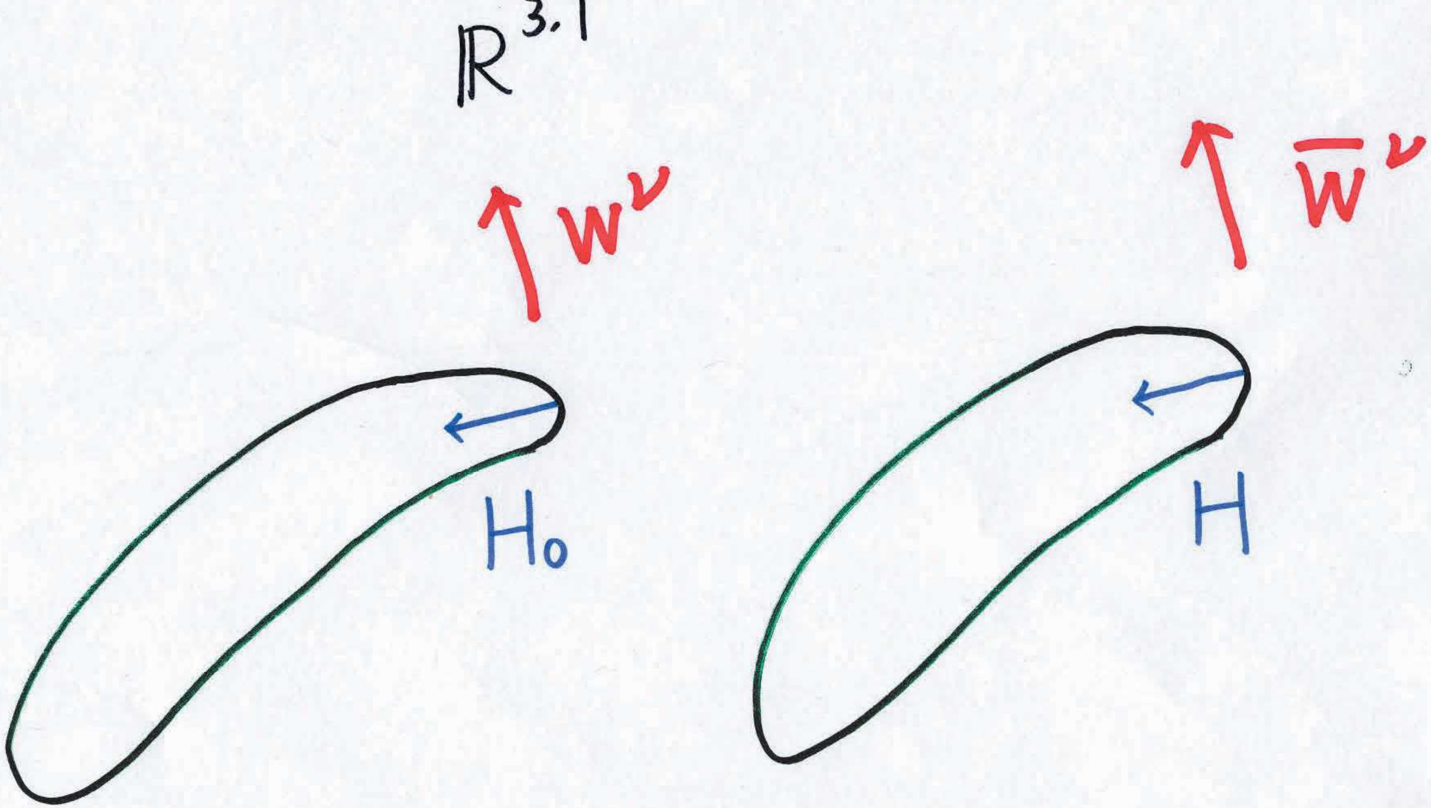
Given a surface $S$ in a physical spacetime, we assume that its mean curvature vector is spacelike. We embed $S$ isometrically into $\mathbb{R}^{3,1}$.

Given any constant unit future time-like vector $w$ (observer) in $\mathbb{R}^{3,1}$, we can define a future directed time-like vector field $\overline{w}$ along $S$ by requiring

$$\langle H_0, w \rangle \; = \; \langle H, \overline{w} \rangle$$

where $H_0$ is the mean curvature vector of $S$ in $\mathbb{R}^{3,1}$ (by the isometric embedding)

and $H$ is the mean curvature vector of $S$ in spacetime.




$$\langle H_0, w \rangle = \langle H, \bar{w} \rangle$$

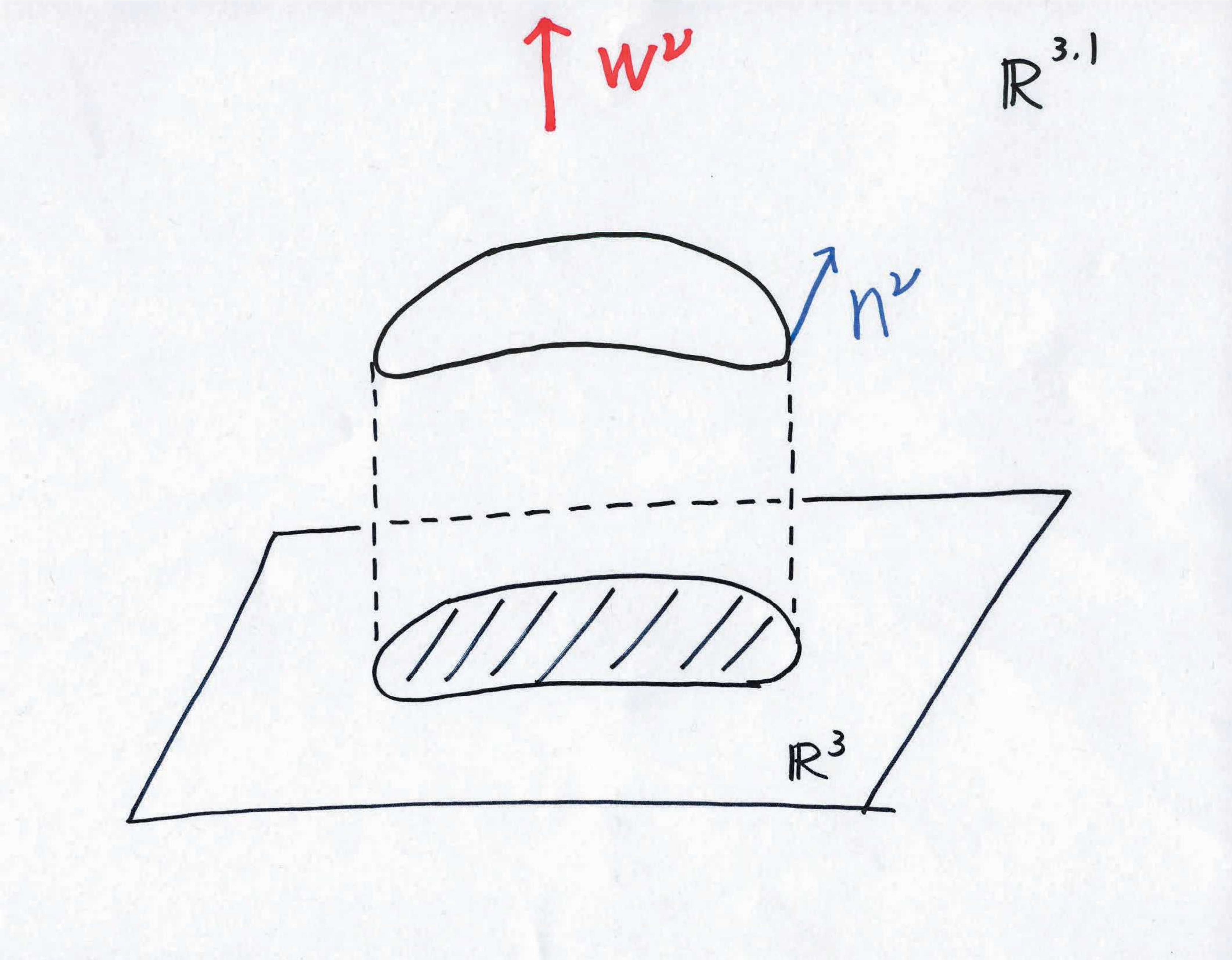
$$w^\nu = N n^\nu + N^\nu \qquad \bar{w}^\nu = N \bar{n}^\nu + N^\nu$$

Note that given any surface $S$ in $\mathbb{R}^{3,1}$ and a constant future time-like unit vector $w^\nu$, there exists a canonical gauge $n^\mu$ (future time-like unit normal along $S$) such that

$$\int_S N^2 K_0 + N^\mu (p_0)_{\mu\nu} r^\nu$$

is equal to the total mean curvature of $\hat{S}$, the projection of $S$ onto the orthogonal complement of $w^\mu$.

In the expression, we write $w^\mu = N\, n^\mu + N^\mu$ alone the surface $S$. $r^\mu$ is the space-like unit normal orthogonal to $n^\mu$, and $p_0$ is the second fundamental form calculated by the three surface defined by $S$ and $r^\mu$.



$w^\nu$
$\mathbb{R}^{3,1}$
$n^\nu$
$\mathbb{R}^3$

From the matching condition and the correspondence $(w^\mu, n^\mu) \to (\overline{w}^\mu, \overline{n}^\mu)$, we can define a similar quantity from the data in spacetime

$$\int_S N\, {}^2\overline{K} + N^\mu (\overline{p})_{\mu\nu} \overline{r}^\nu \ .$$

We write $E$ to be

$$8\pi E = \int_S N\, {}^2\overline{K} + N^\mu (\overline{p})_{\mu\nu} \overline{r}^\nu - \int_S N\, {}^2K_0 + N^\mu (p_0)_{\mu\nu} r^\nu$$

and define the quasi-local mass to be

$$\inf E$$

where the infimum is taken among all isometric embeddings into $\mathbb{R}^{3,1}$ and timelike unit constant vector $w \in \mathbb{R}^{3,1}$.

The Euler-Lagrange equation (called the optimal embedding equation) for minimizing $E$ is

$$
\begin{aligned}
& div_S\left(\frac{\nabla\tau}{\sqrt{1+|\nabla\tau|^2}}\cosh\theta|H| - \nabla\theta - \alpha_H\right) \\
& -(\widehat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}\hat{h}_{cd})\frac{\nabla_b\nabla_a\tau}{\sqrt{1+|\nabla\tau|^2}} = 0
\end{aligned}
$$

where $\sinh\theta = \frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}$, $\alpha_H$ is the connection one-form $\langle\nabla^N_{(\cdot)}\frac{J}{|H|}, \frac{H}{|H|}\rangle$ and $\hat{\sigma}$, $\widehat{H}$ and $\hat{h}$ are the induced metric, mean curvature and second fundamental form of $\hat{S}$ in $\mathbb{R}^3$.

In general, the above equation should have an unique solution $\tau$. We prove that $E$ is non-negative among admissible isometric embedding into Minkowski space.

In summary, given a closed space-like 2-surface in spacetime whose mean curvature vector is space-like, we associate an energy-momentum four-vector to it that depends only on the first fundamental form and the mean curvature vector.

In addition to satisfying the properties mentioned earlier, when we take the limit approaching a point along null geodesics, we recover the energy-momentum tensor of matter density when matter is present, and the Bel-Robinson tensor in vacuum.

Po-Ning Chen joined in the research about seven years ago and we can now defined quasilocal angular momentum and center of gravity.

We define quasi-local conserved quantities in general relativity by using the optimal isometric embedding to transplant Killing fields in the Minkowski spacetime back to the 2-surface a physical spacetime.

To each optimal isometric embedding, a dual element of the Lie algebra of the Lorentz group is assigned. Quasi-local angular momentum and quasi-local center of mass correspond to pairing this element with rotation Killing fields and boost Killing fields, respectively.

Consider the following quasi-local energy density $\rho$

$$\rho = \frac{\sqrt{|H_0|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}} - \sqrt{|H|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}}}{\sqrt{1+|\nabla\tau|^2}}$$

and momentum density $j$

$$j = \rho\nabla\tau - \nabla[\sinh^{-1}(\frac{\rho\Delta\tau}{|H_0||H|})] - \alpha_{H_0} + \alpha_H.$$

The optimal embedding equation takes a simple form:

$$div(j) = 0.$$

The quasi-local conserved quantity of $\Sigma$ with respect to an optimal isometric embedding $(X, T_0)$ and a Killing field $K$ is

$$E(\Sigma, X, T_0, K) = \frac{(-1)}{8\pi} \int_\Sigma \left[ \langle K, T_0 \rangle \rho + j(K^\top) \right] d\Sigma.$$

Suppose $T_0 = A(\frac{\partial}{\partial X^0})$ for a Lorentz transformation $A$.

The quasi-local conserved quantities corresponding to $A(X^i \frac{\partial}{\partial X^j} - X^j \frac{\partial}{\partial X^i})$ are called the quasi-local angular momentum and the ones corresponding to $A(X^i \frac{\partial}{\partial X^0} + X^0 \frac{\partial}{\partial X^i})$ are called the quasi-local center of mass integrals.

The quasi-local angular momentum and center of mass satisfy the following important properties:

[1] The quasi-local angular momentum and center of mass vanish for any surfaces in the Minkowski space

[2] They obey classical transformation laws under the action of the Poincaré group.

We further justify these definitions by considering their limits as the total angular momentum $\tilde{J}^i$ and the total center of mass $\tilde{C}^i$ of an isolated system. They satisfy the following important properties:

[1] All total conserved quantities vanish on any spacelike hypersurface in the Minkowski spacetime, regardless of the asymptotic behavior.

[2] The new total angular momentum and total center of mass are always finite on any vacuum asymptotically flat initial data set of order one.

[3] Under the vacuum Einstein evolution of initial data sets, the total center of mass obeys the dynamical formula $\partial_t C^i(t) = \frac{p^i}{p^0}$ where $p^\nu$ is the ADM energy-momentum four vector.

The last formula is the relativistic version of $p = mv$.

Let me say a few more about the angular momentum as this is perhaps of more current interest.

There are difficulties even for the definition of total angular momentum (and also the total center of mass) of an asymptotically flat initial data set.

Recall that $(M, g, k)$ is an asymptotically flat initial data set if outside a compact subset, there exists an asymptotically flat coordinate system $(x^1, x^2, x^3)$ on each end, such that $g = \delta + O_2(r^{-q})$ and $k = O_1(r^{-p})$, $r = \sqrt{\sum_{i=1}^{3}(x^i)^2}$ for $q > \frac{1}{2}$ and $p > \frac{3}{2}$.

The decay order $(q > \frac{1}{2}, p > \frac{3}{2})$ guarantees that the ADM mass is a valid definition and the positive mass theorem holds.

In addition to the ADM mass (energy-momentum), there is also a companion definition of angular momentum that is attributed to ADM (Ashtekar-Hansen, Christodoulou, Chrusciel, etc.)

$$J_{ADM} = \frac{1}{8\pi} \int_{S_\infty} \pi(x^i \partial_j - x^j \partial_i, \nu), i < j, \text{ where } \pi = k - (tr_g k) g.$$

$x^i \partial_j - x^j \partial_i$ is considered to be an asymptotically rotation Killing field.
Note that, however, the calculation of angular momentum is more subtle, as the expression of $J_{ADM}$ diverges apparently.

There are proposals (Regge-Teitelboim) of parity condition on $(g, k)$ to assure finiteness of the improper integral $J_{ADM}$.

On the other hand, we found the following examples:

(Chen-Huang-Wang-Y.) There exist asymptotically flat spacelike hypersurfaces in the Minkowski spacetime or the Schwarzschild spacetime with finite, nonzero ADM angular momentum $J_{ADM}$ such that $g = \delta + O(r^{-\frac{4}{3}})$ and $k = O(r^{-\frac{5}{3}})$.

As the spacetime is static, the above examples show that $J_{ADM}$ can be unphysical even when the decay order is within the range with which the ADM mass is well-defined.

(Chrusciel) If $p + q > 3$, then the ADM angular momentum $J_{ADM}$ is finite.

Suppose the ADM mass of $(M, g, k)$ is positive, then there is a unique, locally energy-minimizing, optimal isometric embedding of $S_r$ whose image approaches a large round sphere in $\mathbb{R}^3$.

Take the limit as $r \to \infty$ of the quasi-local conserved quantities on $S_r$, we obtain $(E, P_i, \tilde{J}_i, \tilde{C}_i)$ where $(E, P_i)$ is the same as the ADM energy-momentum.

$\tilde{J}_i$ is the new total angular momentum we defined and it may differ from the ADM angular momentum.

We prove a finiteness theorem and an invariance theorem for the newly defined total angular momentum.

The finiteness theorem of the new total angular momentum we proved does not assume any parity condition.

In evaluating the new total angular momentum, the optimal isometric embeddings provides the necessary correction to cancel any unphysical terms.

In addition, the definition also satisfies

$$\partial_t \tilde{J} = 0$$

along the vacuum Einstein equation.

Note that we can take the limit along a family of large spheres to define total angular momentum at null infinity as well (a special case was studied by Rizzi previously).

We have been using the Minkowski spacetime as the reference spacetime in defining the quasi-local energy. The critical points of the quasilocal energy are optimal isometric embeddings into $\mathbb{R}^{3,1}$.

Recently we are able to take into account of cosmological constants and define quasilocal energy and optimal isometric embeddings in reference to the de-Sitter (dS) or the Anti-de-Sitter (AdS) spacetime.

The *AdS* spacetime admits 10 dimensional Killing fields. Hence, we obtain 10 quasi-local conserved quantities.

Furthermore, for asymptotically AdS spacetimes, the limits of the quasi-local conserved quantities recover the total conserved quantities (global charges) $E$, $P^i$, $C^i$ and $J^i$. Here $E$ and $J$ are the total energy and total angular momentum of the asymptotically AdS spacetime, respectively.

Under the Einstein equation, we obtain the following evolution equation for the total conserved quantities:

$$\begin{aligned}\partial_t E =& 0\\ \partial_t P^i =& -C^i\\ \partial_t C^i =& P^i\\ \partial_t J^i =& 0\end{aligned}$$

We recently also compute the quasi-local mass of “spheres of unit size” at null infinity to capture the information of gravitational radiation.

For a gravitational perturbation of the Schwarzschild, the quasi-local gives a quantity of order $\frac{1}{d^2}$ where $d$ is the distance to the source. In addition, the vanishing of the $\frac{1}{d}$ gives a limiting integrand that integrates to zero on the limiting 2-sphere at null infinity. To each closed loop on the limiting 2-sphere at null infinity, we thus associate a non-vanishing arc integral that is of the order of $\frac{1}{d}$.

We expect the freedom in varying the shape of the loop can increase the detectability of gravitational waves.