Homological representations of framed braid groups

Akishi Ikeda
akishi.ikeda@ipmu.jp
Kavli IPMU, University of Tokyo

1 Goal

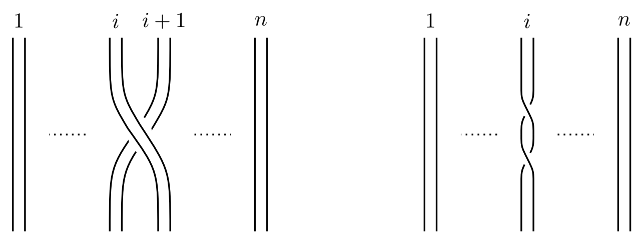
The goal is to construct representations of the framed braid groups by using the geometry of surfaces.

2 Framed braid groups

Definition (Framed braid groups)**.** The *framed braid group* FB_n is a group generated by $\sigma_1, \dots, \sigma_{n-1}$ and $\tau_1, \dots, \tau_n$ with the relations

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \quad \text{if } |i-j| \geq 2 \\ \tau_i\tau_j &= \tau_j\tau_i \\ \sigma_i\tau_j &= \begin{cases} \tau_{i+1}\sigma_i & \text{if } j = i \\ \tau_i\sigma_i & \text{if } j = i+1 \\ \tau_j\sigma_i & \text{if } j \neq i, i+1. \end{cases} \end{aligned}$$

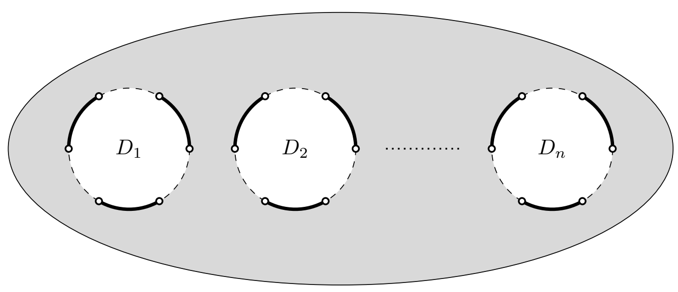
Graphically generators of the framed braid group FB_n can be described by using n ribbons as in the following Figure.



3 Description as the mapping class group

Let $\mathbb{D}$ be a closed disk and take disjoint n closed disks $D_1, \dots, D_n \subset \mathbb{D}$. Take disjoint r open intervals on each boundary ∂D_k and denote by $\mathbb{A}^{(r)}$ the set of such rn open intervals. Define the surface $\mathbb{S}_n^{(r)}$ (as in the following) by

$$\mathbb{S}_n^{(r)} = \mathbb{D} \setminus (D_1 \cup \cdots \cup D_n) \cup \bigcup_{A \in \mathbb{A}^{(r)}} A.$$



Let $\mathfrak{M}(\mathbb{S}_n^{(r)})$ be the mapping class group of $\mathbb{S}_n^{(r)}$.

Proposition. There is an isomorphism of groups

$$FB_n \xrightarrow{\sim} \mathfrak{M}(\mathbb{S}_n^{(r)}).$$

4 Relative homology groups

Let $\mathcal{C}_{n,m}^{(r)}$ be the configuration space of unordered m distinct points on $\mathbb{S}_n^{(r)}$:

$$\mathcal{C}_{n,m}^{(r)} := \{(t_1, \dots, t_m) \in (\mathbb{S}_n^{(r)})^m \mid t_i \neq t_j \text{ if } i \neq j\}/\mathfrak{S}_m.$$

There is a group homomorphism from the fundamental group $\pi_1(\mathcal{C}_{n,m}^{(r)}, *)$ to a free abelian group of rank two

$$\alpha \colon \pi_1(\mathcal{C}_{n,m}^{(r)}, *) \to \langle q \rangle \oplus \langle t \rangle$$

where the generator q corresponds to the loop around the cylinders $\{t_1 \in D_k\}$ and the generator t corresponds to the loop around the hyperplanes $\{t_i = t_j\}$. Let

$$\pi \colon \widetilde{\mathcal{C}}_{n,m}^{(r)} \to \mathcal{C}_{n,m}^{(r)}$$

the covering space corresponding to α. Introduce the subset $\mathcal{A}^{(r)} \subset \mathcal{C}_{n,m}^{(r)}$ by

$$\mathcal{A}^{(r)} := \{\, \{t_1, \dots, t_m\} \in \mathcal{C}_{n,m}^{(r)} \mid t_1 \in A \text{ for some } A \in \mathbb{A}^{(r)} \,\}$$

and its inverse image $\widetilde{\mathcal{A}}^{(r)} = \pi^{-1}(\mathcal{A}^{(r)}) \subset \widetilde{\mathcal{C}}_{n,m}^{(r)}$.

Definition (Relative homology group)**.** Define the relative homology group

$$\mathcal{H}_{n,m}^{(r)} := H_m(\widetilde{\mathcal{C}}_{n,m}^{(r)}, \widetilde{\mathcal{A}}^{(r)}; \mathbb{Z}).$$

$\mathcal{H}_{n,m}^{(r)}$ has a $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$-module structure coming from the action of the deck transformation group $\langle q \rangle \oplus \langle t \rangle$.

5 Main result

Theorem.
For positive integers $m, r > 0$, there is a representation of the framed braid group

$$\rho_{n,m}^{(r)} \colon FB_n \to \mathrm{GL}(\mathcal{H}_{n,m}^{(r)}, \mathbb{Z}[q^{\pm 1}, t^{\pm 1}])$$

which is induced from the natural action of $FB_n \cong \mathfrak{M}(\mathbb{S}_n^{(r)})$ on $\mathcal{H}_{n,m}^{(r)}$ with the following properties:

(1) The rank of $\mathcal{H}_{n,m}^{(r)}$ over $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ is given by

$$\operatorname{rank}_{\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]} \mathcal{H}_{n,m}^{(r)} = \binom{rn + n + m - 2}{m}.$$

(2) The representation $\rho_{n,m}^{(r)}$ is faithful for $m \geq 2$.

(3) There is a subrepresentation $\mathcal{N}_{n,m}^{(r)} \subset \mathcal{H}_{n,m}^{(r)}$ such that the quotient $\mathcal{H}_{n,m}^{(r)}/\mathcal{N}_{n,m}^{(r)}$ coincides with the homological representation of the usual braid group constructed by Lawrence (containing Lawrence-Krammer-Bigelow representation and Burau representation).