Contractibility in algebraic geometry

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Complex algebraic variety

The space of solutions to polynomial equations in complex numbers.

Contractions

A fundamental way to relate varieties.

Used in constructions, and classification.

Definition

For subvariety $Y \subset X$, say algebraic map

$$f\colon X \to Z$$

is a contraction of Y if

- $f|_Y$ has image a point,
- $f|_{X-Y}$ is an isomorphism.

Simple example

Let $Z_1 = \mathbb{C}^2$ with coordinates (a, b). Take

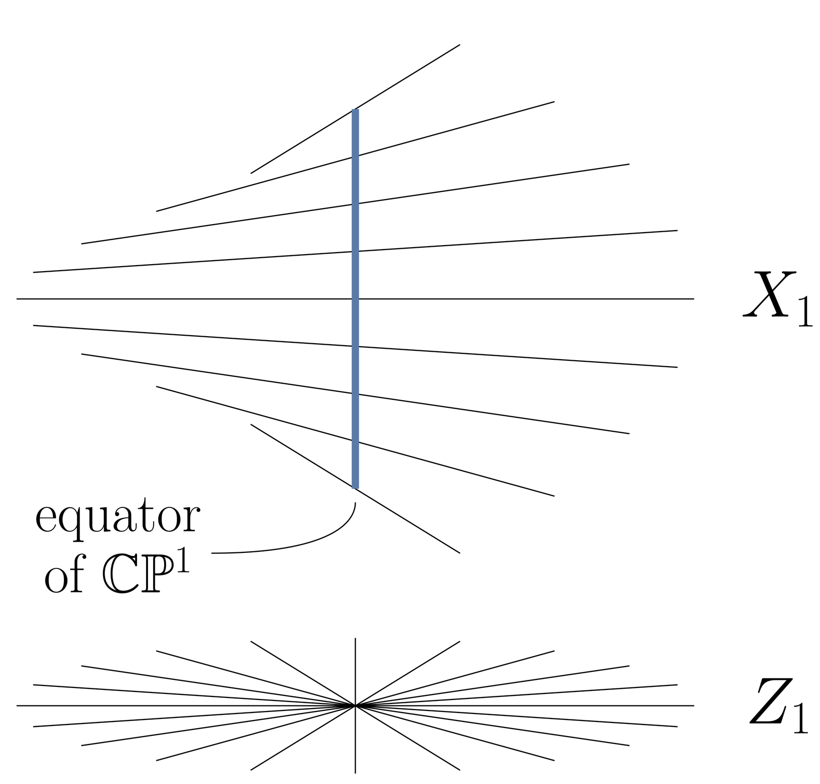
$$X_1 \subset \mathbb{C}^2 \times \mathbb{CP}^1$$

defined by equation

$$aB = Ab$$

where $(A\colon B)$ are coordinates on $\mathbb{CP}^1$.

Natural map $X_1 \to Z_1$ is contraction of $\mathbb{CP}^1 \subset X_1$ defined by $a = b = 0$:



Singular example

Often image Z of contraction is singular.

For example,

$$Z_2 \subset \mathbb{C}^3$$

defined by equation

$$rs = t^2$$

has a resolution $X_2 \to Z_2$, which is a contraction of $\mathbb{CP}^1 \subset X_2$.

Question

For $C \subset X$ where $C = \mathbb{CP}^1$, when is C contractible?

Answer in dimension 2

Assume $\dim X = 2$.

Let $d =$ self-intersection number of C.

$$C \text{ contractible} \iff d < 0.$$

For X_1 and X_2, have $d = -1, -2$ resp.

Higher dimensions

By Jiménez [5], for X of any dimension,

C contractible $\iff$

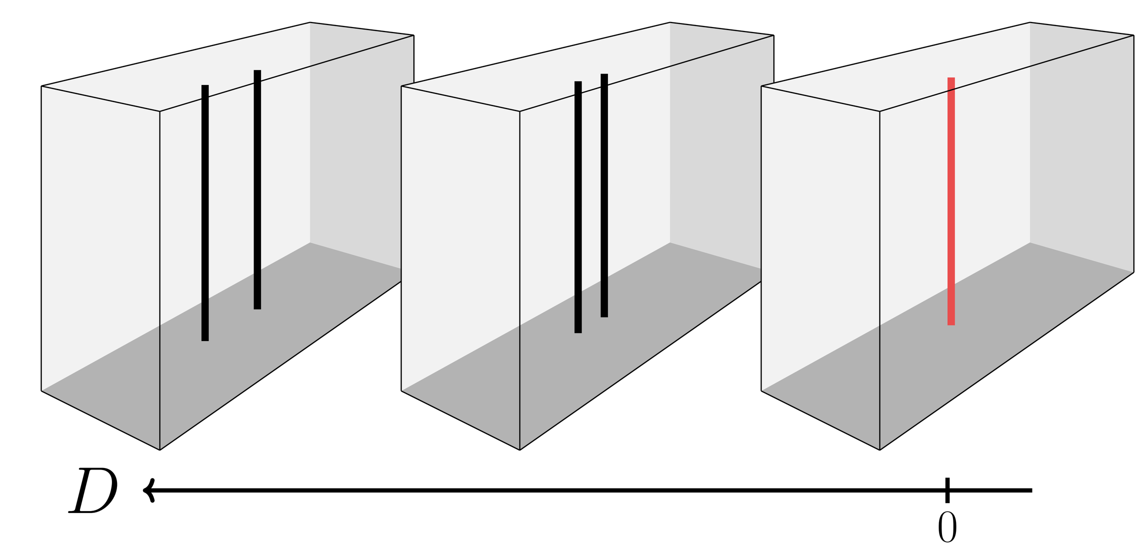
exists line bundle on neighbourhood of C with certain properties.

Challenge

Find a simple criterion for contractibility.

Issue in dimension 3

Necessary to study degenerate curves:



Resolutions of $rs = t^2 + (u^2 + D^2)^2$

Red curve above has infinitesimal first-order deformation; Reid [7] used such deformations to study contractibility in dimension 3.

Contraction algebra

Noncommutative deformations of C controlled by contraction algebra A from [2, 3].

Computable in examples. In dimension 3 example above, $A = \mathbb{C}[\epsilon]/\epsilon^2$.

Theorem: Donovan–Wemyss [3]

$$C \text{ contractible} \iff A \text{ finite-dimensional.}$$

Classical deformations

Classical deformations of C controlled by abelianization A^{ab}. We show that

$$C \text{ contractible} \not\Longleftarrow A^{\mathrm{ab}} \text{ finite-dimensional.}$$

Noncontractible example

Let $X_3 = X_2 \times \mathbb{C}$. Then for $C = \mathbb{CP}^1 \times 0 \subset X_3$, find $A = \mathbb{C}[[T]]$, so C *not* contractible.

Related work

Generalizations of A used to study enumerative geometry and derived category of X, by Donovan–Wemyss [4], and Bodzenta–Bondal [1], Kawamata [6], Toda [8], and others.

[1] Bodzenta, Bondal. arXiv:1511.00665.
[2] Donovan, Wemyss. *Duke Math J* 2016.
[3] Donovan, Wemyss. arXiv:1511.00406.
[4] Donovan, Wemyss. *To appear JEMS.*
[5] Jiménez. *Duke Math J* 1992.
[6] Kawamata. arXiv:1512.06170.
[7] Reid. *Proceedings, Tokyo* 1981.
[8] Toda. *Compositio Math* 2017.

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