Geography of algebraic varieties

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Goal

One of the main goals of algebraic geometry is to classify algebraic varieties.

Method

To study invariants.

What is Geography?

The study of the relations between invariants.

Example: distribution of Calabi–Yau 3-folds

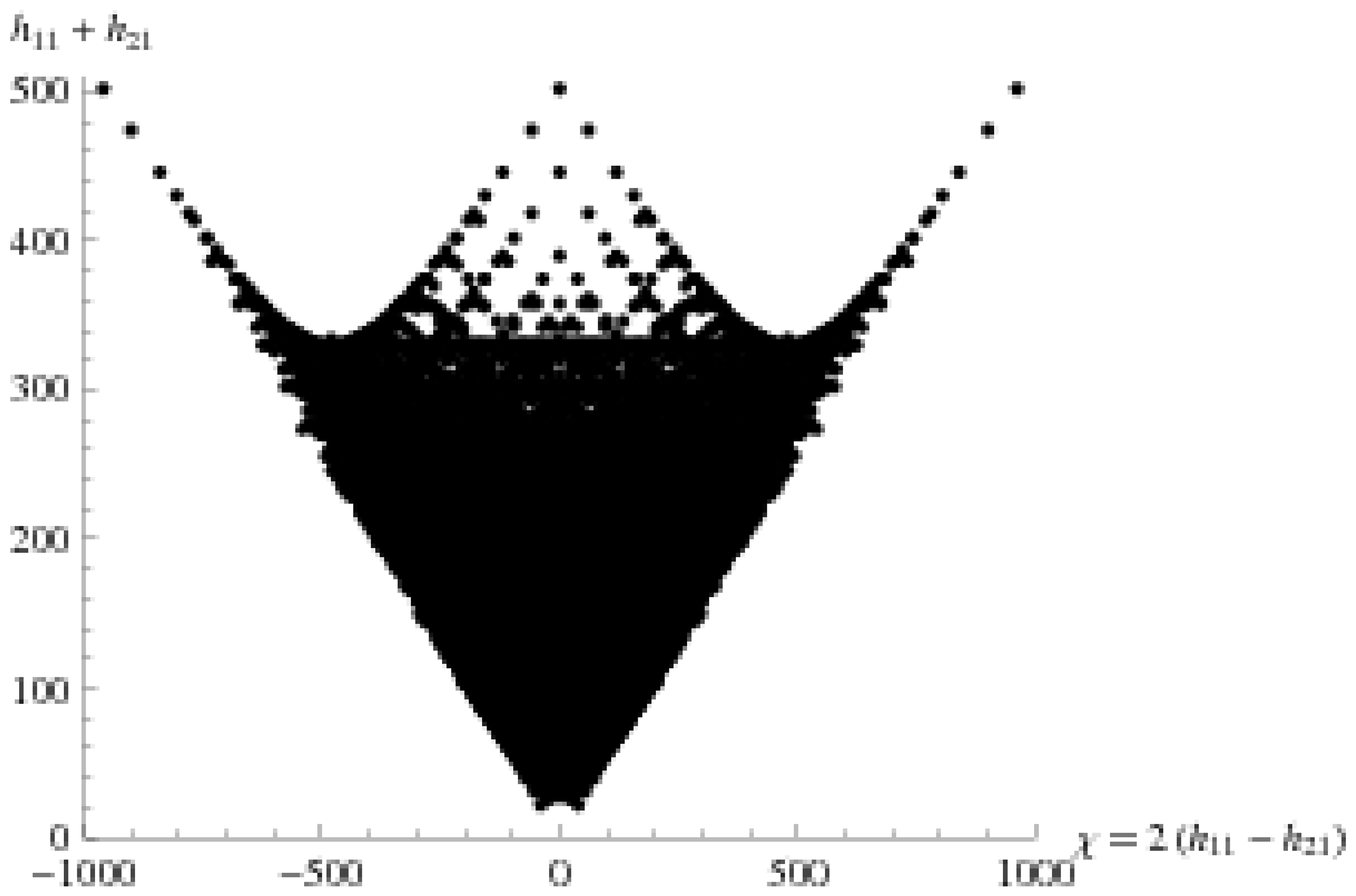


Figure from [4]. The 30,108 distinct Hodge numbers of the 473.8 million Calabi–Yau threefolds identified by Kreuzer and Skarke. Each point represent a pair $(2(h^{1,1}(X) - h^{2,1}(X)), h^{1,1}(X) + h^{2,1}(X))$ for some Calabi–Yau threefold X.

What invariants are we interested in?

We are working on *birational geometry*, i.e., to study algebraic varieties under birational equivalence. So we are interested in birational invariants, in particular, for a projective smooth variety X of dimension n, we are interested in the following two birational invariants.

- *Geometric genus*

$$p_g(X) := h^0(X, \omega_X),$$

- *Canonical volume*

$$\mathrm{vol}(X) := \lim_{m\to\infty} \frac{h^0(X, \omega_X^{\otimes m})}{m^n/n!}$$

In particular, we are interested in projective smooth varieties *of general type* (i.e. $\mathrm{vol}(X) > 0$).

Curve case

When $\dim X = 1$, the relation between $p_g(X)$ and $\mathrm{vol}(X)$ is clear, that is

$$\mathrm{vol}(X) = 2p_g(X) - 2.$$

Surface case

When $\dim X = 2$, the relation between $p_g(X)$ and $\mathrm{vol}(X)$ is not as simple as the curve case, but still it is well understood due to the surface theory.

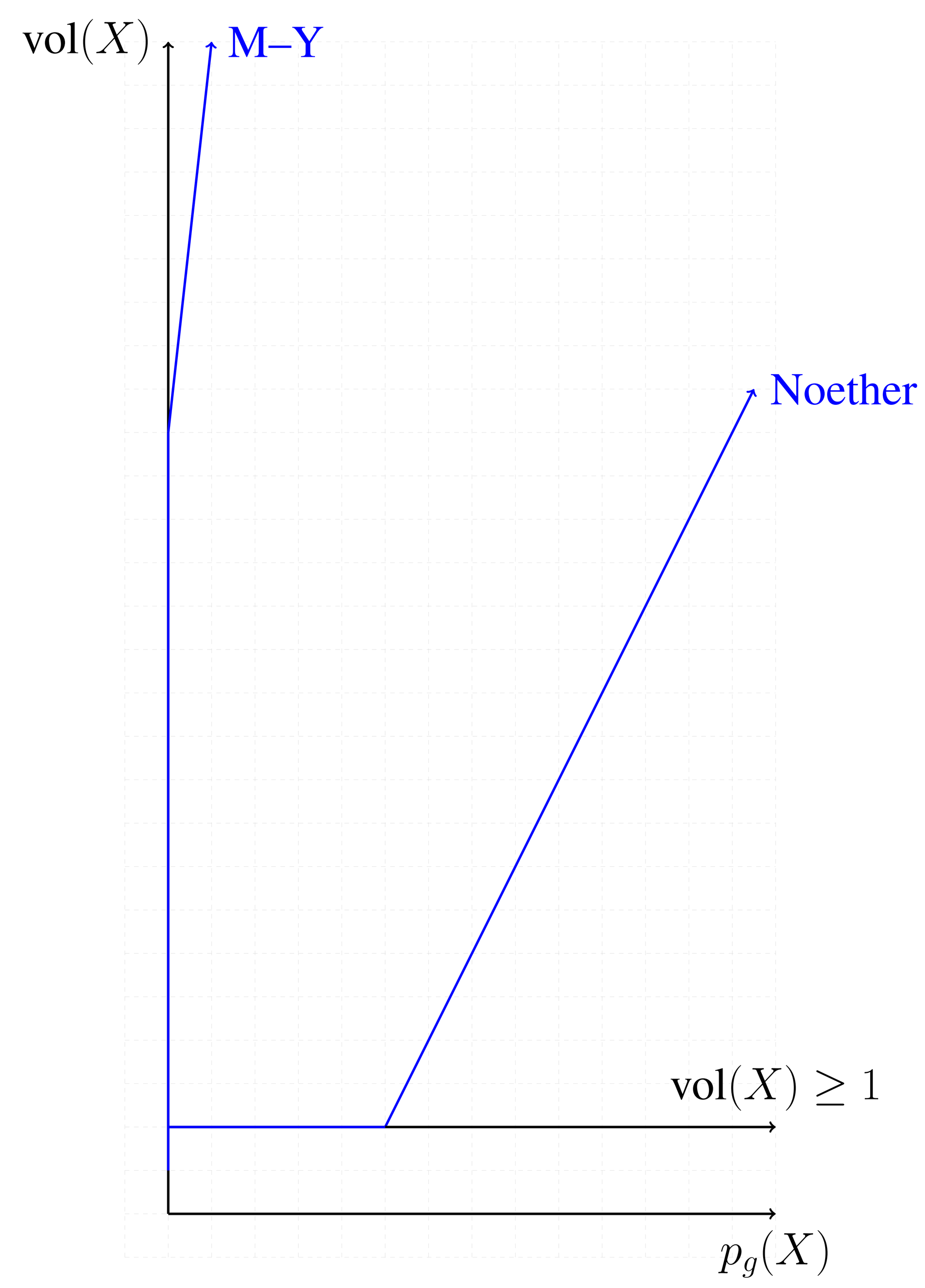
On one hand, we have the **Miyaoka–Yau inequality**

$$\mathrm{vol}(X) \leq 9(p_g(X) + 1).$$

On the other hand, we have the **Noether inequality** [3]

$$\mathrm{vol}(X) \geq 2p_g(X) - 4.$$

Hence $(p_g(X), \mathrm{vol}(X))$ can only appear as integer points inside the area surround by blue lines in the following figure.



3-fold case

When $\dim X = 3$, the relation between $p_g(X)$ and $\mathrm{vol}(X)$ is more complicated.

Theoretically, for any n, we know that there exists positive numbers a_n and b_n such that for any projective smooth variety X of dimension n of general type,

$$\mathrm{vol}(X) \geq a_n p_g(X) - b_n.$$

In 1992, Kobayashi [2] found infinitely many projective smooth 3-folds X of general type satisfying

$$\mathrm{vol}(X) = \frac{4}{3}p_g(X) - \frac{10}{3}.$$

People believe that this should be the right form of Neother inequality in dimension 3.

In 2015, J. A. Chen and M. Chen proved that for any projective smooth 3-fold X of general type with Gorenstein minimal model,

$$\mathrm{vol}(X) \geq \frac{4}{3}p_g(X) - \frac{10}{3}.$$

Most recently, in a joint work in progress with J. A. Chen and M. Chen, we show that for any projective smooth 3-fold X of general type with $p_g(X) \geq 28$,

$$\mathrm{vol}(X) \geq \frac{4}{3}p_g(X) - \frac{14}{3}.$$

Moreover, our proof suggests that there are certain candidates of infinitely many projective smooth 3-folds X of general type satisfying

$$\mathrm{vol}(X) = \frac{4}{3}p_g(X) - \frac{14}{3}.$$

Once those candidates are confirmed to exist, our theorem gives the right Noether inequality for "most" of projective smooth 3-folds of general type, since there are only finitely many deformation classes of projective smooth 3-folds of general type with $p_g(X) < 28$.

References

[1] J. A. Chen, M. Chen, *The Noether inequality for Gorenstein minimal 3-folds*. Comm. Anal. Geom. **23** (2015), no. 1, 1–9.

[2] M. Kobayashi, *On Noether's inequality for threefolds*, J. Math. Soc. Japan **44** (1992), no. 1, 145–156.

[3] M. Noether, *Zur Theorie der eindeutigen Entsprechungen algebraischer Gebilde*, Math. Ann. **8** (1875), 495–533.

[4] W. Taylor, *On the Hodge structure of elliptically fibered Calabi–Yau threefolds*, JHEP 1208 (2012) 032.