Solving QFT by a method inspired by Holography

Ami Katz

Boston University

w/ N. Anand, L. Fitzpatrick, J. Kaplan
Z. Khandker, L. Vitale, M. Walters.

Basic goal is to solve:

$$H_{QFT}|\psi\rangle = E|\psi\rangle$$

But what basis should we choose?

Traditionally: Fock Space Basis

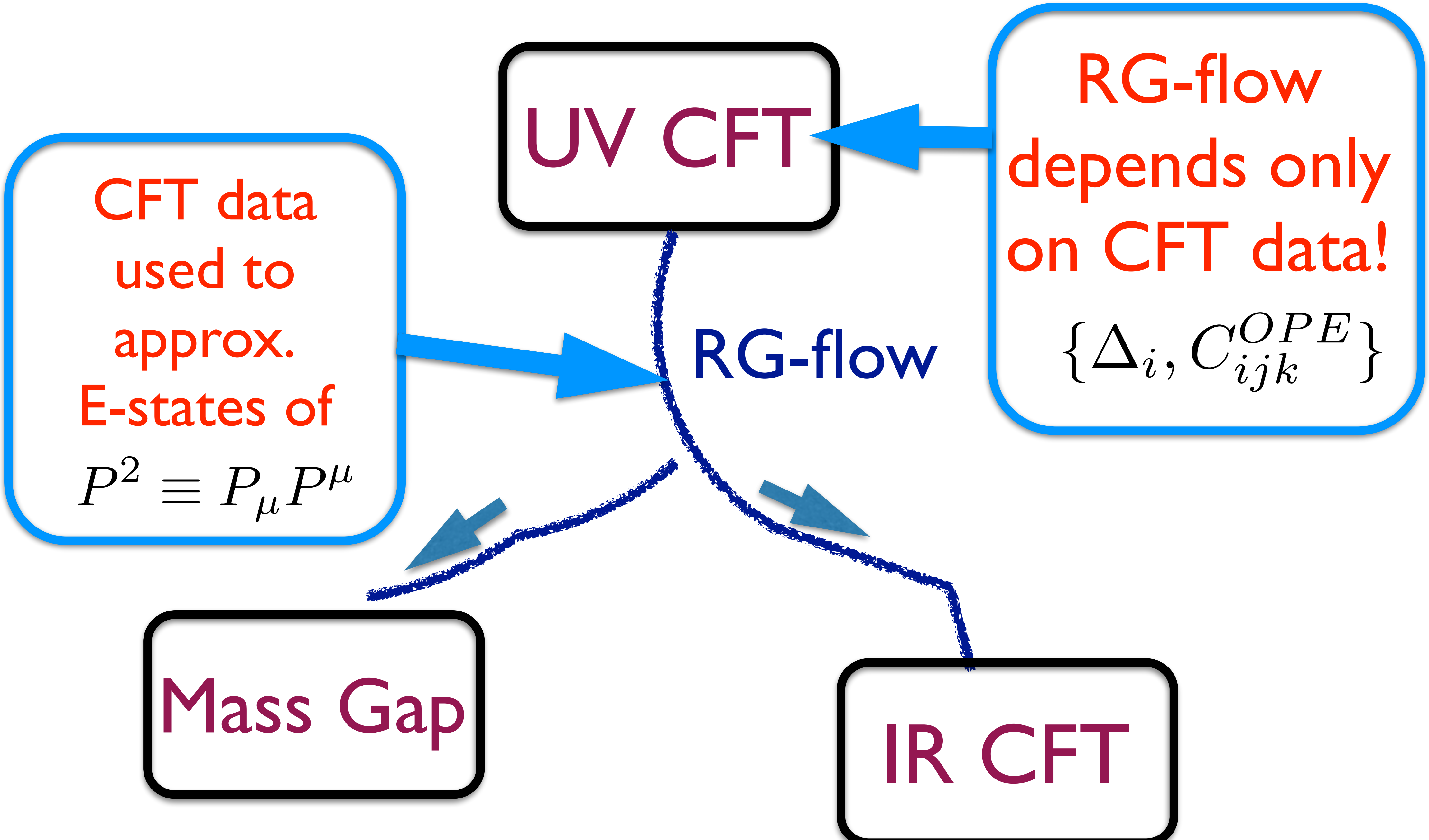
At strong coupling not a very useful basis

Is there a basis which approximates well low-energy dynamical observables in QFT even at strong coupling?

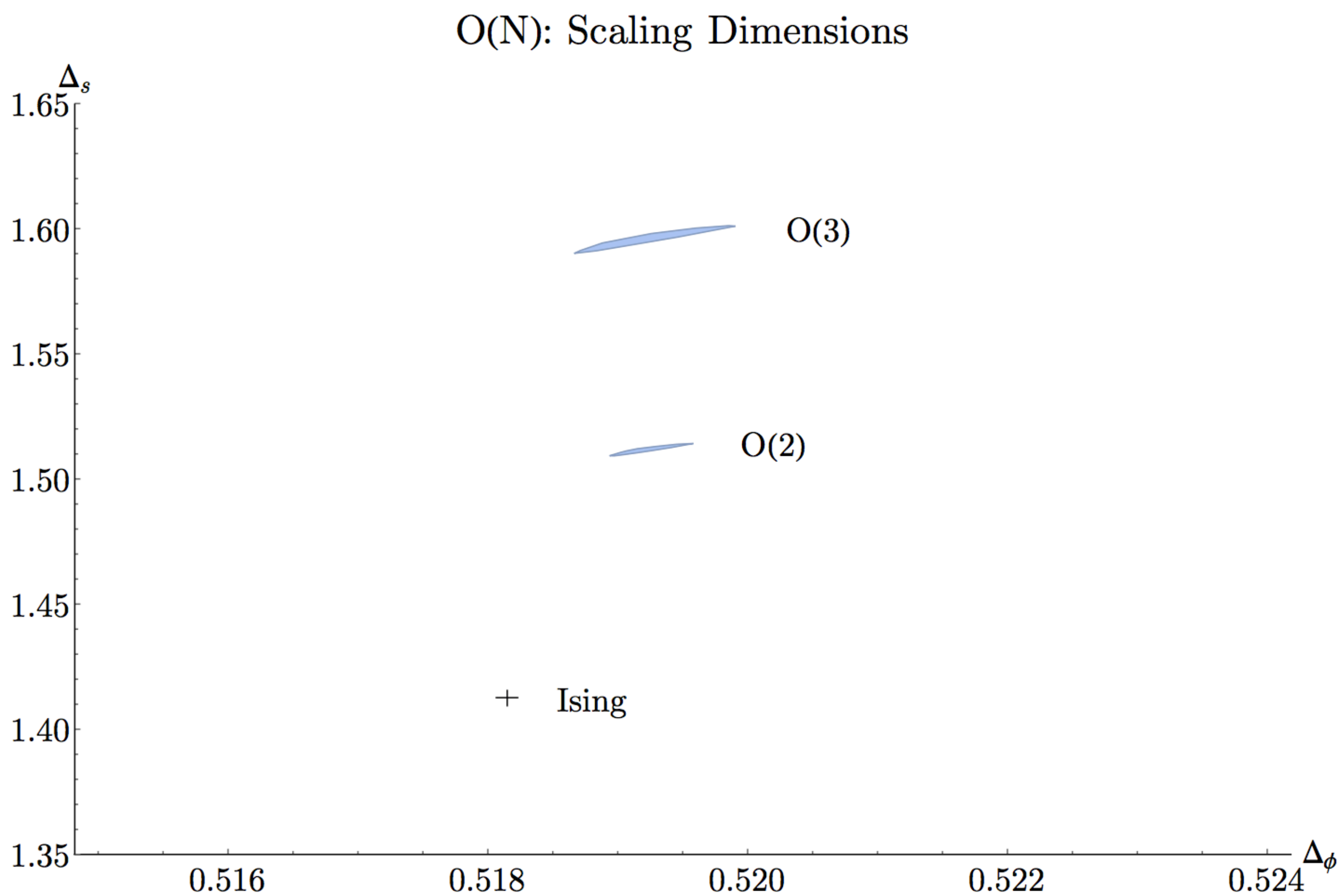
(Reasonable given Eigenstate Thermalization Hypothesis)

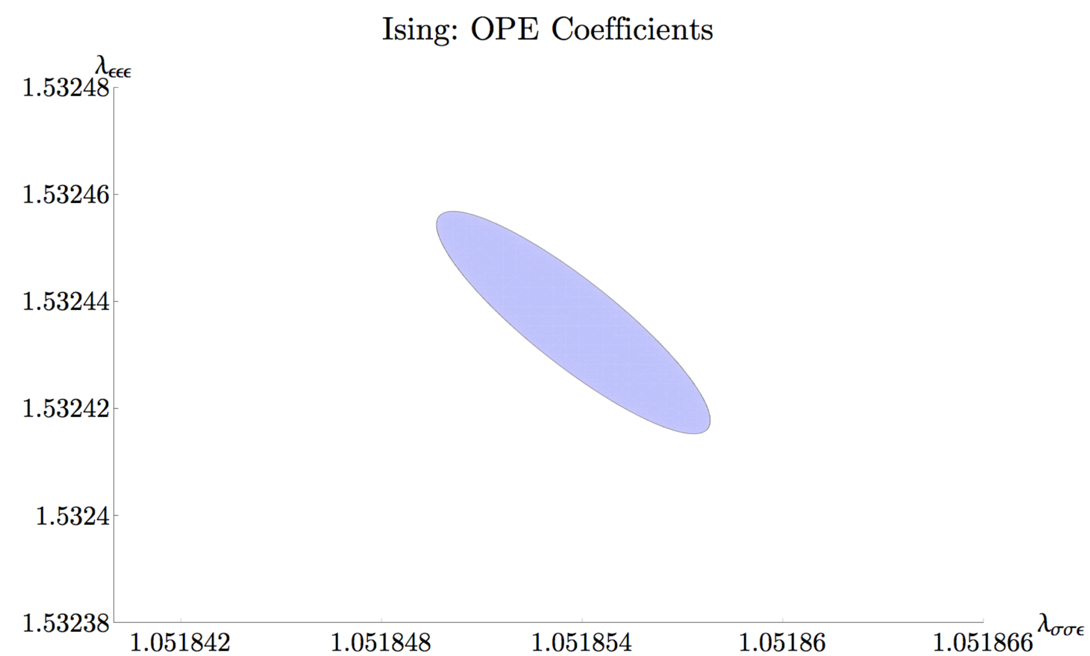
Holography says Yes:
A basis organized by conformal structure!

Any Lorentzian QFT:



Such a scheme is conceptually satisfying given recent conformal bootstrap work:





Unitary CFTs are very special and perhaps there's a hidden formulation which will categorize their intrinsic data: $\{\Delta_i, C_{ijk}^{OPE}\}$

Outline

1. Intro to “conformal truncation” on the light-cone: Conformal structure + Light-cone quantization.
2. Tests of the method in 2D - non-pert. RG-flows
3. Sketch of how to apply the method to non-abelian gauge theories (including chiral gauge theories in 4D).
4. Large-N RG-flows and the Light-cone: A tale of zero-modes.
5. Conclusions and hopes for the future.

Conformal Truncation on the LC

I. A conformal basis of the Hilbert space

CFT primaries:

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle \sim \delta_{ij}\frac{1}{x^{2\Delta_i}}\,(polarization)$$

Kallen-Lehmann states are monogamous!

$$|\mathcal{C}, l; \vec{P}, \mu\rangle \equiv \int d^dx\; e^{-iP\cdot x}\;\mathcal{O}(x)|0\rangle$$

w/ $\mathcal{O}$ some primary op. & $P^2 = \mu^2$

A Lorentzian op-state correspondence

CFT Basis: $|\mathcal{C}, l; \vec{P}, \mu\rangle \equiv \int d^d x\, e^{-iP\cdot x}\, \mathcal{O}(x)|0\rangle$

E-states of momentum, $\vec{P}$

and Conformal Casimir, $\mathcal{C} = \Delta(\Delta - d) + l(l + d - 2)$

$$P^2_{CFT}|\mathcal{C}, l; \vec{P}, \mu\rangle = \mu^2|\mathcal{C}, l; \vec{P}, \mu\rangle$$

2pt fcn induces a inner product on these states:

$$\langle \mathcal{C}', l'; \vec{P}', \mu'|\mathcal{C}, l; \vec{P}, \mu\rangle = \delta_{\mathcal{C}\mathcal{C}'}\delta_{ll'}\ \rho_{\mathcal{O}}(\mu^2)\delta(\mu^2 - \mu'^2)\delta^{d-1}(\vec{P}' - \vec{P})$$

w/ $\rho_{\mathcal{O}}(\mu^2)$ the KL spectral density of the op.

(Scalar op: $\rho_{\mathcal{O}}(\mu^2) \sim \mu^{2\Delta - d}$)

Above states are natural in understanding holographic RG-flows

Modes in the Poincare patch of AdS

$$ds^2 = \frac{1}{z^2}(dx^2 - dz^2)$$

Ex: Take a scalar AdS field: $\Phi_{AdS}(x^\mu, z)$

w/ $M^2 = \mathcal{C} = \Delta(\Delta - d)$

$$\langle \Phi_{AdS}(x^\mu, z) | \mathcal{C}, l; \vec{P}, \mu \rangle \sim z^{d/2} J_{\Delta - d/2}(\mu z) \; e^{-iP \cdot x}$$

We can now use the above states to describe any RG flow starting from our UV CFT

$$H = H_{CFT} + V \quad , \quad V = \int d^{d-1}x \; \lambda \mathcal{O}_R(\vec{x})$$

$$\langle \mathcal{C}', P' | V | \mathcal{C}, P \rangle =$$

$$\lambda \int d^d x' d^d x d^{d-1} y \; e^{i(x' \cdot P' - x \cdot P)} \langle \mathcal{O}'(x') \mathcal{O}_R(\vec{y}) \mathcal{O}(x) \rangle$$

Determined by CFT data alone!

Conformal Truncation on the LC

2. Quantization on the Light-cone

The problem with standard quantization is lack of manifest Lorentz-covariance

$$\langle \mathcal{C}', P_x, \mu' | V | \mathcal{C}, P_x, \mu \rangle \sim V_{\mathcal{O}'\mathcal{O}}(\mu, \mu', P_x)$$

$$P^2 = (H_0 + V)^2 - \vec{P}^2 = P_0^2 + H_0 V + V H_0 + V^2$$

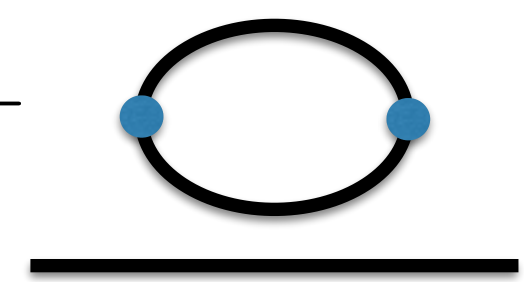
$$\langle \mathcal{C}', P_x, \mu' | P^2 | \mathcal{C}, P_x, \mu \rangle \sim M^2_{\mathcal{O}'\mathcal{O}}(\mu, \mu', P_x)$$

Simplest ex: $\delta\mathcal{L} = -\frac{1}{2}\delta m^2\phi^2$

$$\langle P_x|V|P_x\rangle = \frac{\delta m^2}{2\sqrt{P_x^2+m^2}} \; , \; \langle P_x|V|p_1,p_2,p_3\rangle \neq 0$$

V^2

$$\langle P_x|P^2|P_x\rangle = m^2 + \delta m^2 + \frac{\delta m^4}{2(P_x^2+m^2)} +$$



Weinberg's Infinite Momentum Limit: $P_x \to -\infty$

$$\langle P_x|P^2|P_x\rangle \to m^2 + \delta m^2$$

The Light-cone limit: $P_x \to -\infty$

$$\langle \mathcal{C}', P_x, \mu' | V | \mathcal{C}, P_x, \mu \rangle \to \langle \mathcal{C}', P_-, \mu' | \delta P_+ | \mathcal{C}, P_-, \mu \rangle \; , \; P_- = |P_x|$$

$$\delta P_+ = \int dx^- d^{d-2}x \; \lambda \mathcal{O}_R(x^-, \vec{x}^\perp)$$

(true for $\Delta_R > \frac{d}{2}$)

Why is LC manifestly Lorentz-covariant?

Root of the problem with regular quantization:

All boosts depend on interactions

$$[J_{0i}, P^2] = 0 \quad , \quad J_{0i} = \int d^{d-1}x \; x^i T_{00}(\vec{x})$$

Ex: $T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \lambda\phi^4$

Boosts mix unperturbed states in a complicated way!

Not so with LC: $J_{+-} = \int dx^- d^{d-2}x \; x^- T_{--}^{CFT} = J_{+-}^{CFT}$

Ex: $T_{--} = (\partial_-\phi)^2$

since relevant ops
only modify the trace:

$$T^\mu_\mu = 2T_{+-} - \sum_\perp T_{\perp\perp} \sim \lambda \mathcal{O}_R$$

States $|\mathcal{C}, P_-, \mu\rangle$ transform simply under J_{+-}

One can choose a convention where:

$$\langle \mathcal{C}', P_-, \mu' | \delta P_+ | \mathcal{C}, P_-, \mu\rangle = \lambda \int d^d x' d^d x d^{d-1} y \; e^{i(x'\cdot P' - x\cdot P)} \langle \mathcal{O}'(x') \mathcal{O}_R(y^-, \vec{y}^\perp) \mathcal{O}(x)\rangle$$

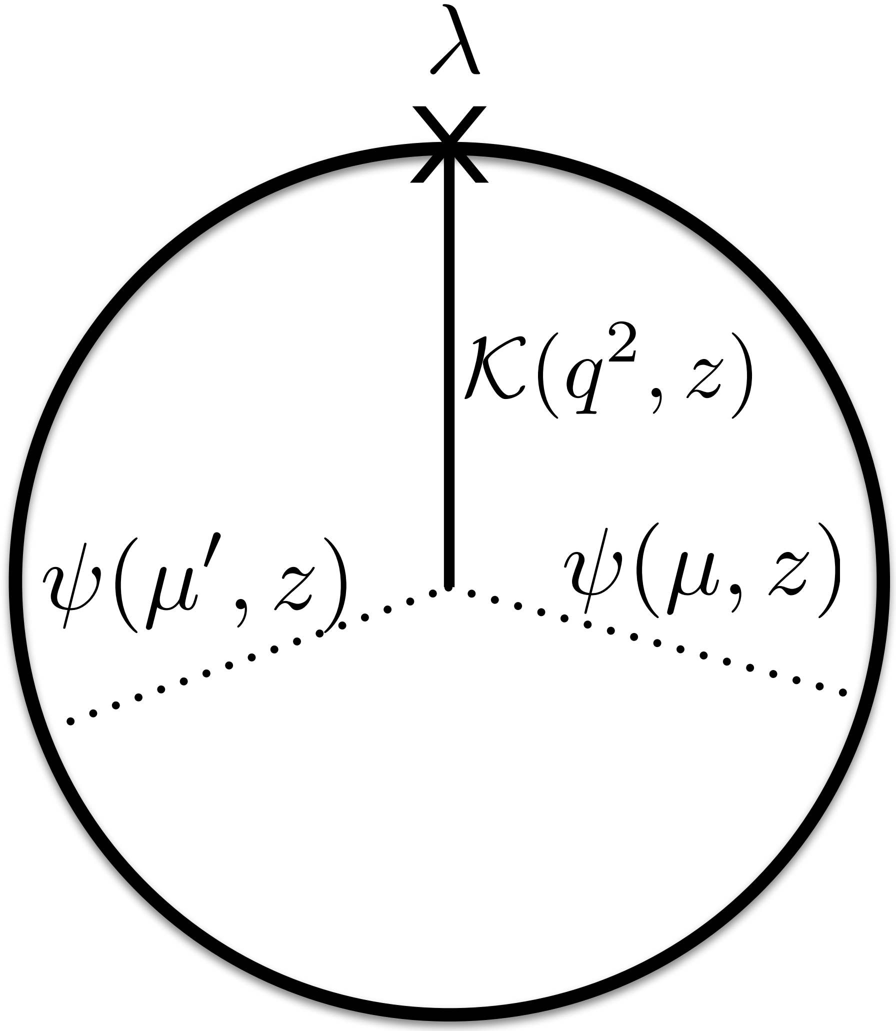
$$\equiv \lambda C_{R\mathcal{O}\mathcal{O}'} \boxed{\mathcal{M}^R_{\mathcal{O}\mathcal{O}'}(\mu, \mu')} \, \delta^{d-1}(\vec{P}' - \vec{P})$$

CFT Kinematic

Boost invariant/Lorentz covariant

How to see Lorentz-covariance of amplitude in LC limit?

$$\langle \mathcal{C}', P_x, \mu' | V | \mathcal{C}, P_x, \mu \rangle =$$



$$\to \mathcal{M}^R_{\mathcal{O}'\mathcal{O}}(\mu, \mu')$$

(L-inv. data consistent with holographic RG-flow intuition)

$$q_0 = \sqrt{\mu^2 + P_x^2} - \sqrt{\mu'^2 + P_x^2}$$

$$P_x \to -\infty:$$

$$q_0 \to \frac{\mu^2 - \mu'^2}{2P_x} \to 0$$

$$\mathcal{K}(q^2 \to 0, z) \sim z^{d-\Delta}$$

RG-flow as a Hamiltonian equation:

$$P^2|\psi\rangle = \mu_\psi^2|\psi\rangle$$

expressed in our basis as:

$$\mu^2\psi_{\mathcal{O}}(\mu) \ + \ \lambda \sum_{\mathcal{O}'} C_{R\mathcal{O}\mathcal{O}'} \int d\mu'^2 \ \mathcal{M}^R_{\mathcal{O}\mathcal{O}'}(\mu,\mu') \ \psi_{\mathcal{O}'}(\mu') = \mu_\psi^2 \psi_{\mathcal{O}}(\mu)$$

QFT

w/ $\langle \mathcal{C}, l; \mu|\psi\rangle \equiv \rho_{\mathcal{O}}(\mu)\psi_{\mathcal{O}}(\mu)$

Ok - but how do we implement this practically?

(interested in examples with "bad AdS duals")

1. Need to discretize the μ label:

$$|\mathcal{C}, l; \vec{P}, k\rangle \equiv \int_0^{\Lambda^2} d\mu^2 \; g_k(\mu)|\mathcal{C}, l; \vec{P}, \mu\rangle$$

w/ Λ a cutoff and polys of deg k, $g_k(\mu)$

obeying: $$\int_0^{\Lambda^2} d\mu^2 \rho_{\mathcal{O}}(\mu) \; g_k(\mu) g_{k'}(\mu) = \delta_{kk'}$$

2. Truncate the Hilbert space in order to calculate:

$$k \leq k_{max} \qquad\qquad \mathcal{C} \leq \mathcal{C}_{max}$$

$k \leq k_{max}$: bounds the IR resolution.

$$\frac{\Lambda}{k_{max}} \lesssim \mu < \Lambda$$

k-max ~ several 100

$\mathcal{C} \leq \mathcal{C}_{max}$: dials the complexity of the basis.

$$\mathcal{C}_{max} = \Delta^2_{max} + \cdots$$

Delta max ~ (few) x 10

Due to the complexity of computing the OPE coefficients (even for a free theory!)

Summary of the method in a heuristic holographic picture:

UV cutoff

$$z \sim \frac{1}{\Lambda}$$

Bulk fields with mass:

$$M_{AdS} \leq M_{max}$$

IR cutoff

$$z \sim \frac{k_{max}}{\Lambda}$$

Holographic picture suggests why we expect the truncation to converge!

We're integrating out bulk fields by restricting the Conformal Casimir.

Naive expectation is that low-energy quantities converge as: $\sim \dfrac{1}{(\Delta_{max})^p}$

Really this should be some property of the OPE coefficients:

$$\sum_{\Delta_i = \Delta_H} \frac{C^2_{\mathcal{O}_L \mathcal{O}_R \mathcal{O}_{H_i}}}{C_{\mathcal{O}_{H_i} \mathcal{O}_R \mathcal{O}_{H_i}}} \sim \frac{1}{(\Delta_H)^q}$$

2D Test: QCD at small-N

SU(N): $\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(F^2) + i\bar{\Psi}\gamma^{\mu}D_{\mu}\Psi$

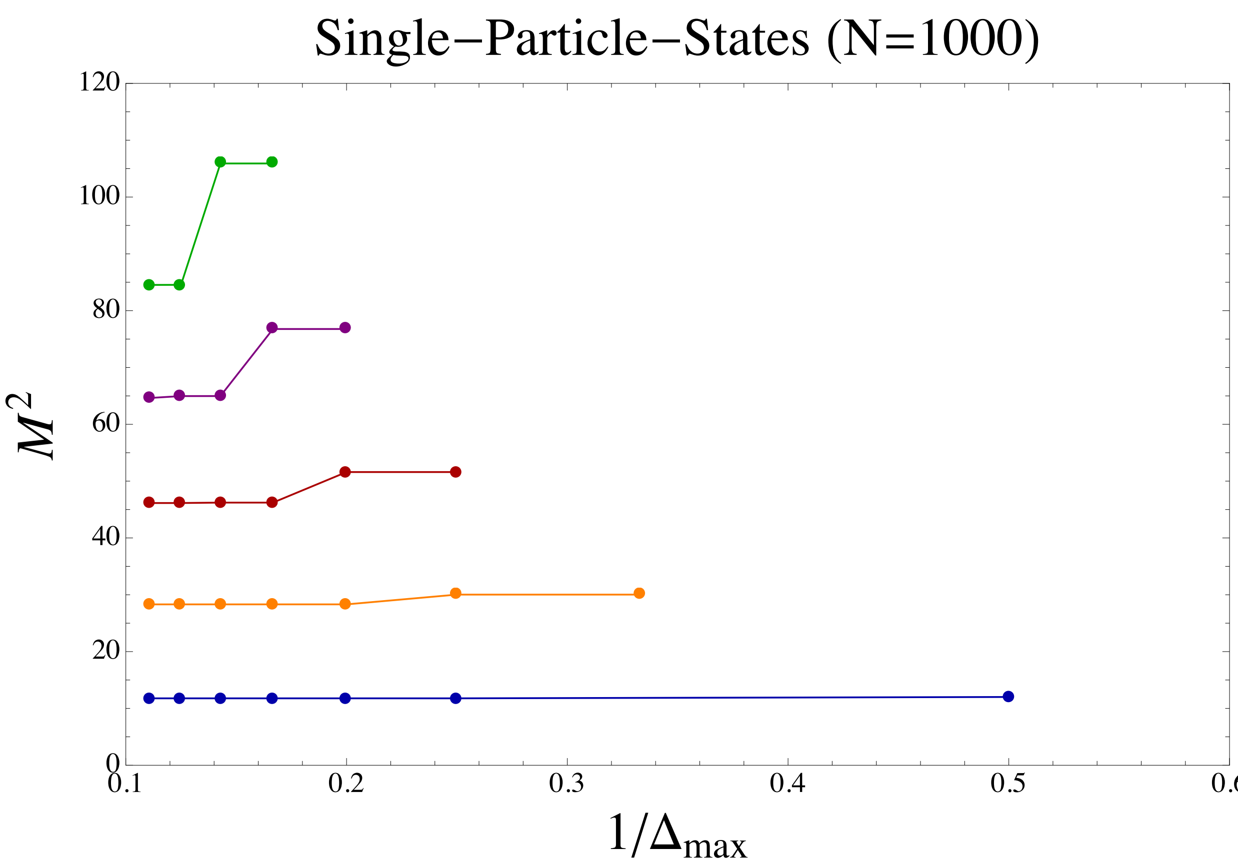
$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left((\partial_- A_+)^2\right) + i\psi^{\dagger}D_+\psi + i\chi^{\dagger}\partial_-\chi$$

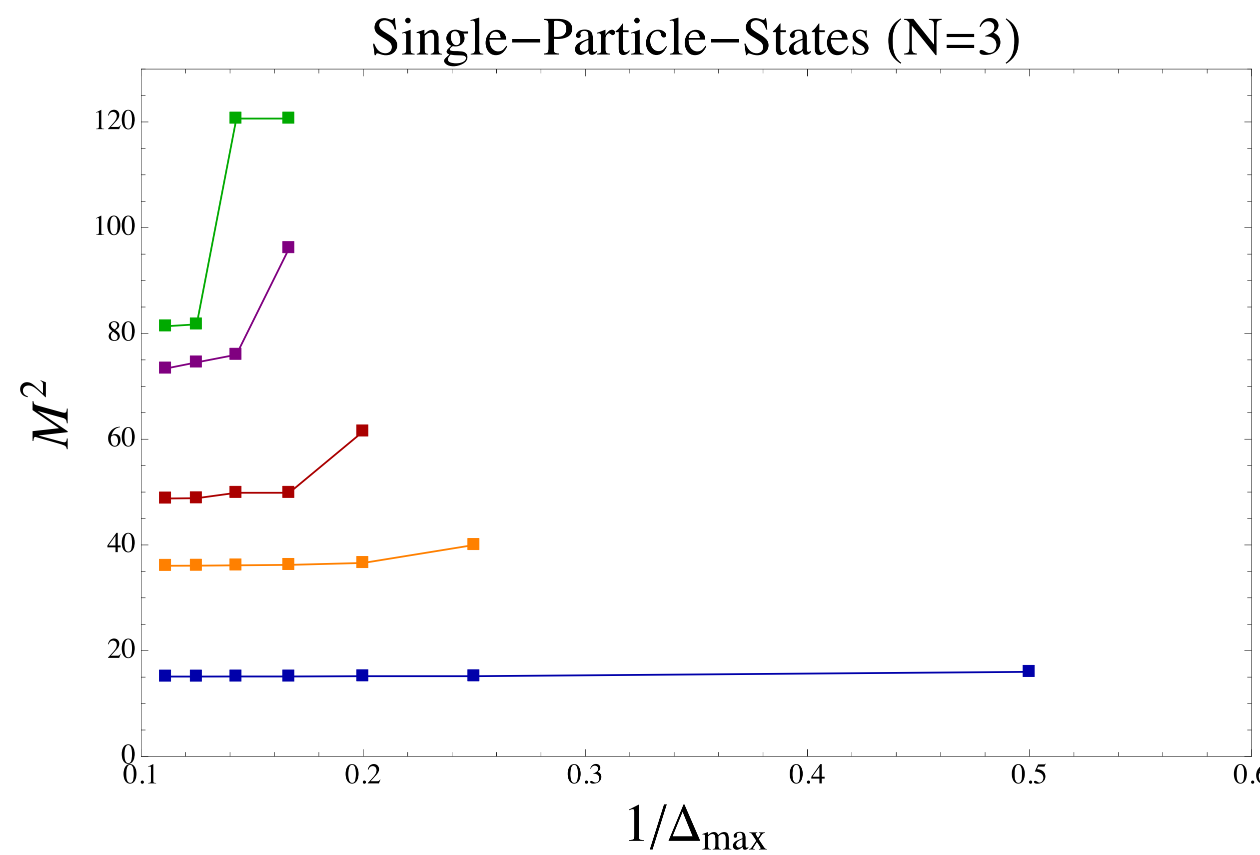
$$P_+ = -g^2 \int dx^- \psi^{\dagger}T^a\psi \frac{1}{\partial_-^2}\psi^{\dagger}T^a\psi$$

$$[g] \sim (Mass)$$

2D QCD

w/ G. Marques-Tavares & Y. Xu





Ex N=3: $|B_1\rangle = 0.81\left(\sqrt{3}(\partial\psi^\dagger\psi - \psi^\dagger\partial\psi)\right)|\Omega\rangle - 0.57\left(\frac{3}{\sqrt{2}}(\psi^\dagger\psi)^2\right)|\Omega\rangle$

(up to ~1% corrections)

2D scalar: $$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

Flows to the 2D Ising for $\lambda \to \lambda_*$:

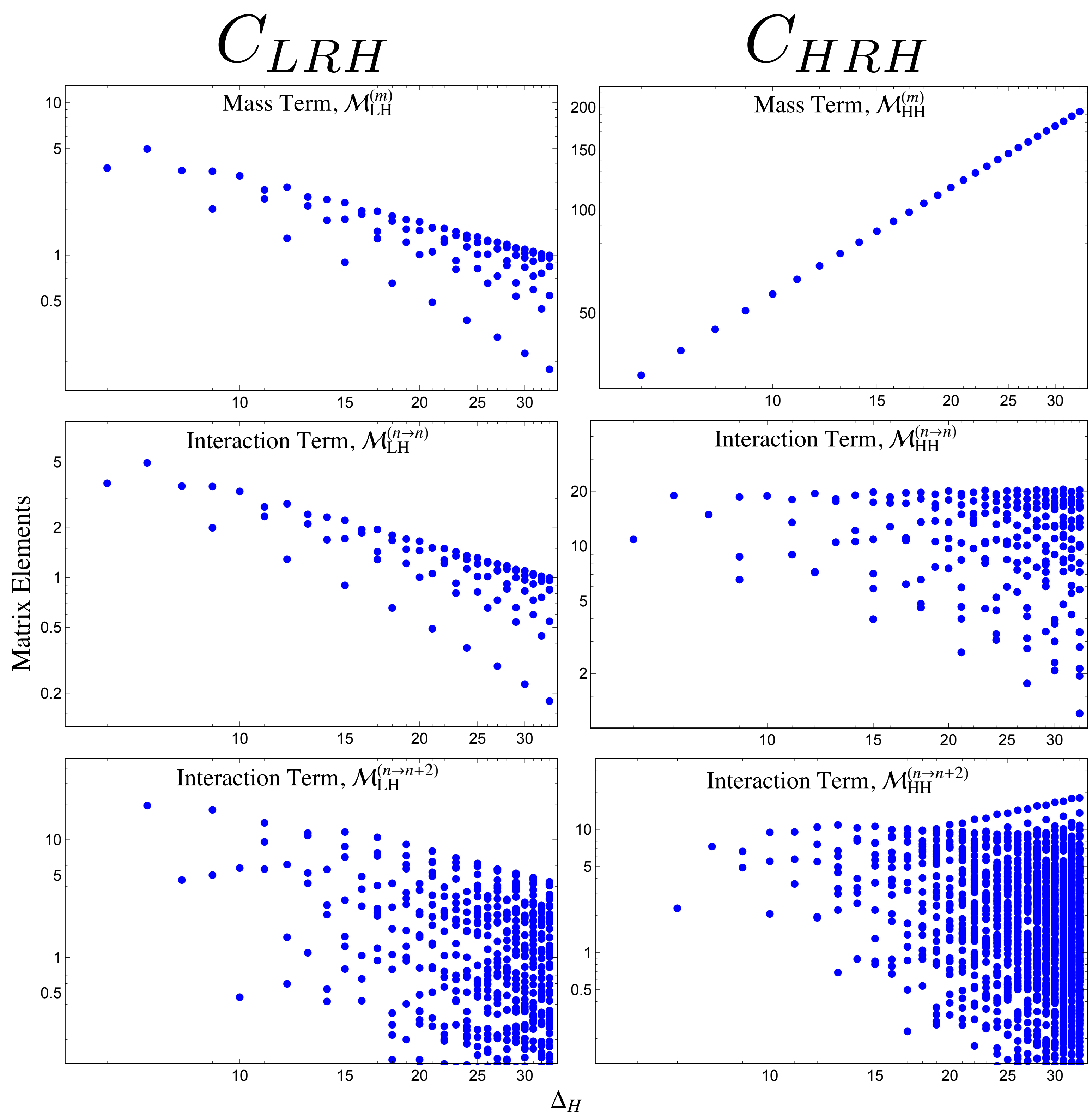
$$\mathcal{L}_{IR} = \frac{1}{2}\psi i\partial_+\psi + \frac{1}{2}\chi i\partial_-\chi - m_f\chi\psi$$

2D CFT basis in this case is chiral:

$$\mathcal{O} = \sum c_{n_1,\cdots,n_m}\partial_-^{n_1}(\partial_-\phi)\cdots\partial_-^{n_m}(\partial_-\phi)$$

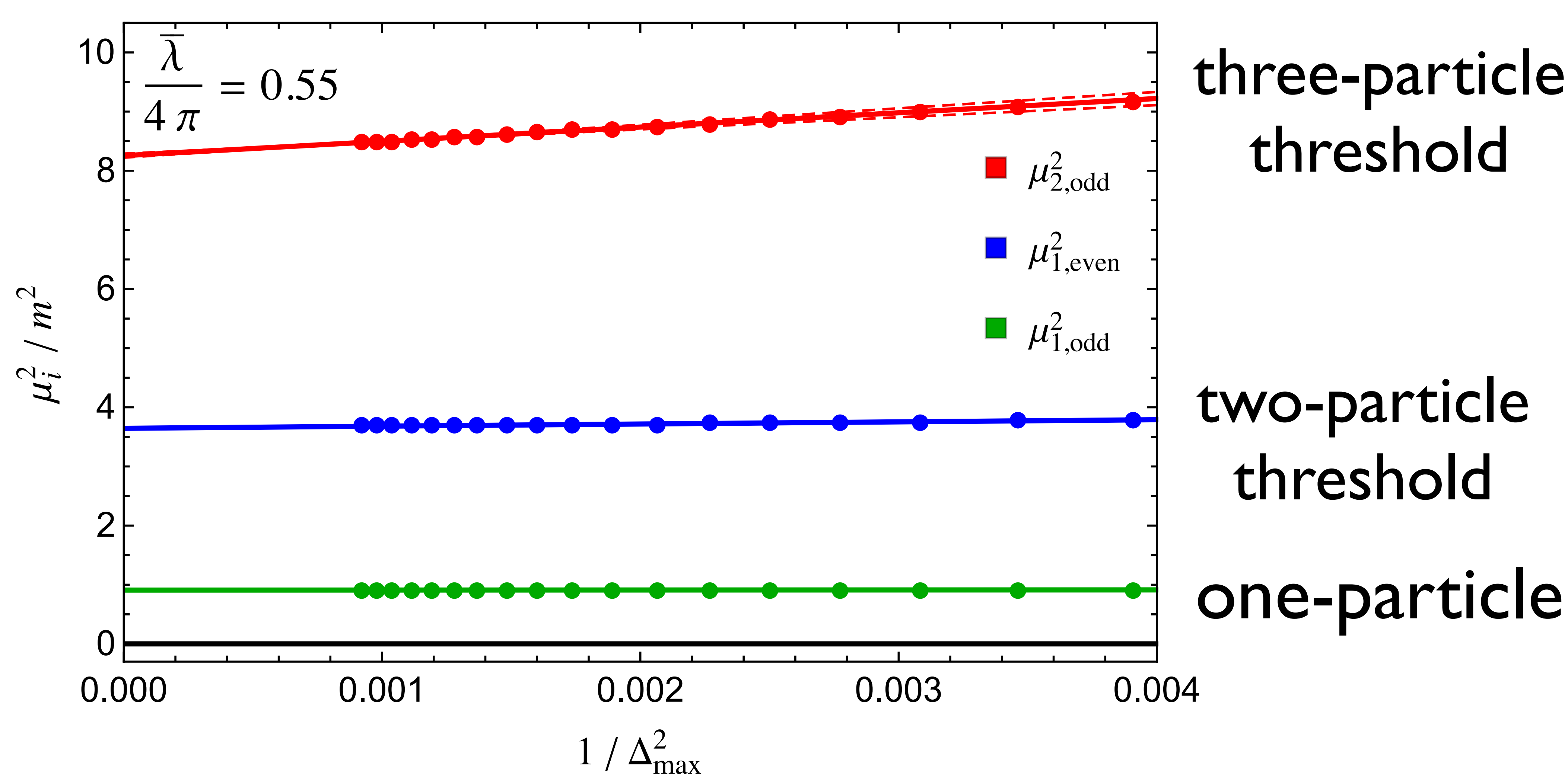
$P^2\mathcal{O} = 0$ → no μ / k_{max}

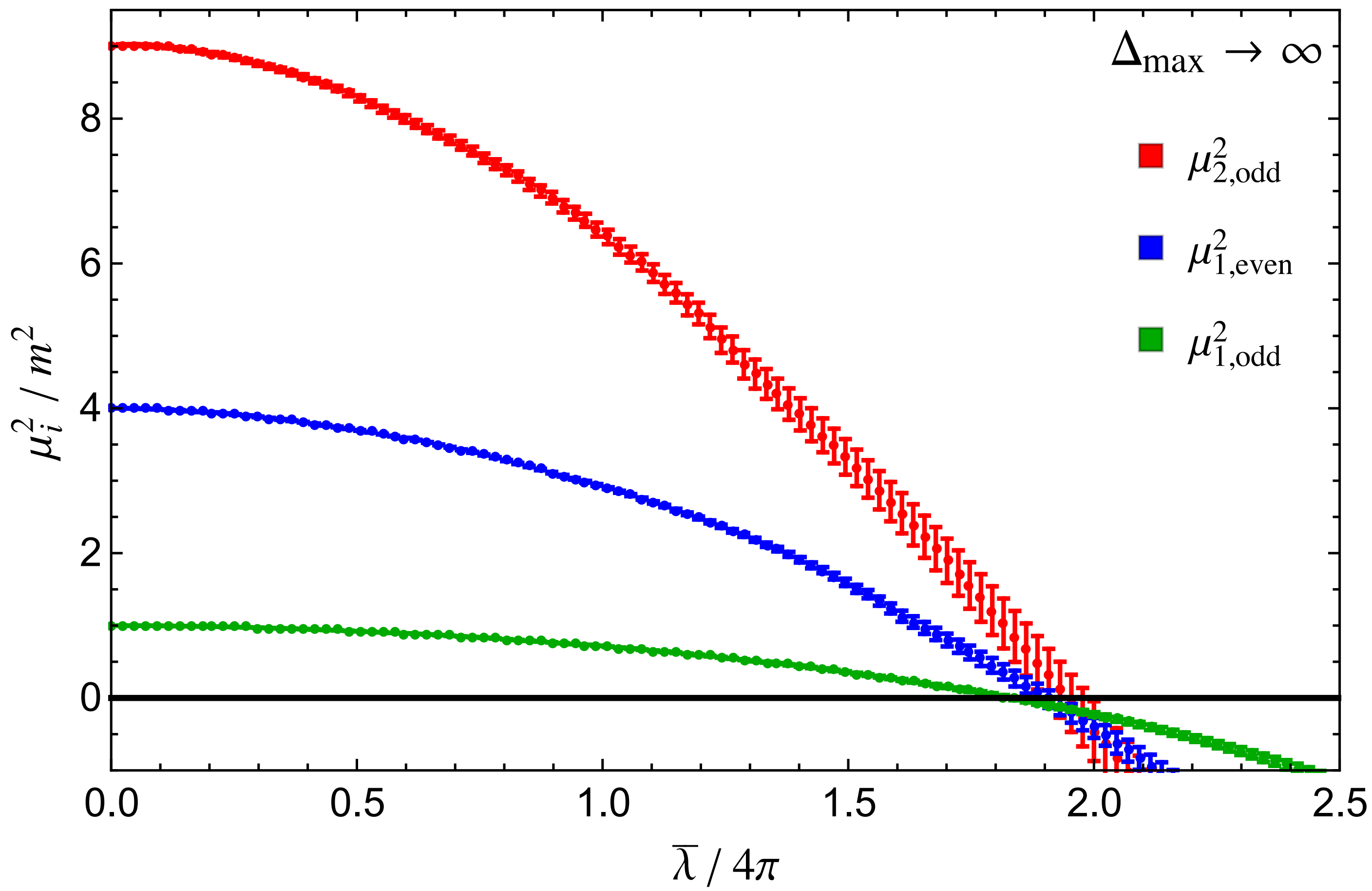
Conjectured behavior (leading to convergence)



$$\sum_{\Delta_i=\Delta_H} \frac{C^2_{\mathcal{O}_L\mathcal{O}_R\mathcal{O}_{H_i}}}{C_{\mathcal{O}_{H_i}\mathcal{O}_R\mathcal{O}_{H_i}}} \sim \frac{1}{(\Delta_H)^q}$$

Spectrum @ random strong coupling





Gap closes!

(critical coupling roughly consistent w/ Chabysheva et al.)

Dynamical observable: 2pt function from the spectral density

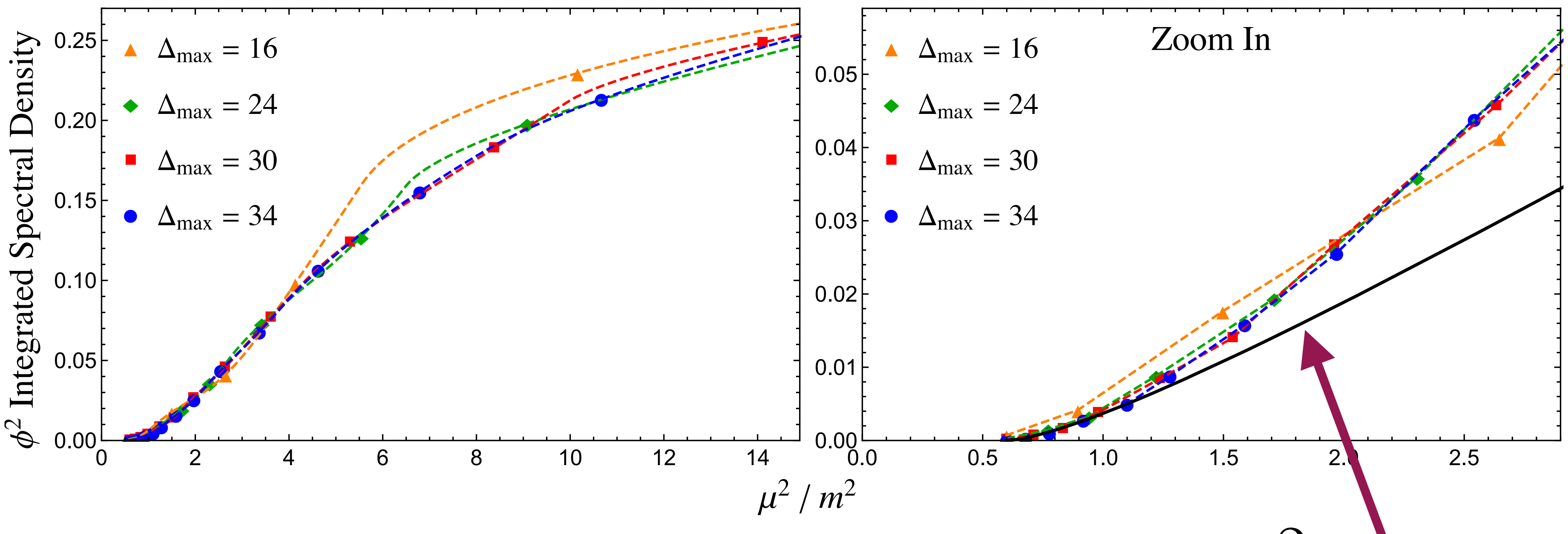
$$\langle \mathcal{O}(r)\mathcal{O}(0)\rangle = \int d\mu^2 \, \boxed{\rho_{\mathcal{O}}(\mu)} \Delta_0(\mu, r)$$

Note: density is difficult for lattice to extract.

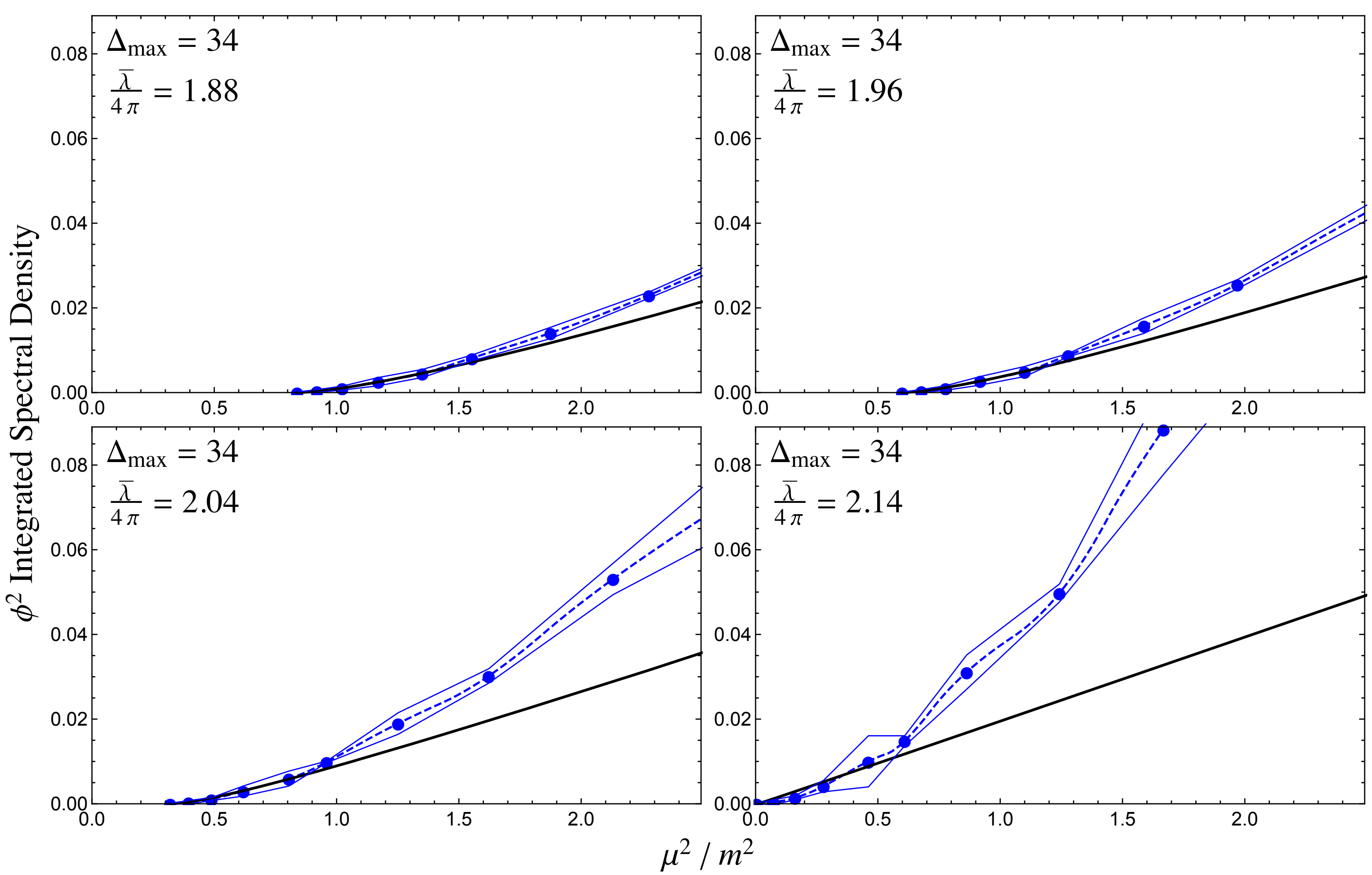
In practice:

$$I_{\mathcal{O}}(\mu^2) \equiv \int_0^{\mu^2} d\mu'^{\,2} \, \rho_{\mathcal{O}}(\mu'^{\,2}) = \sum_{\mu_i \leq \mu} |\langle \mathcal{O}(0)|\mu_i\rangle|^2$$

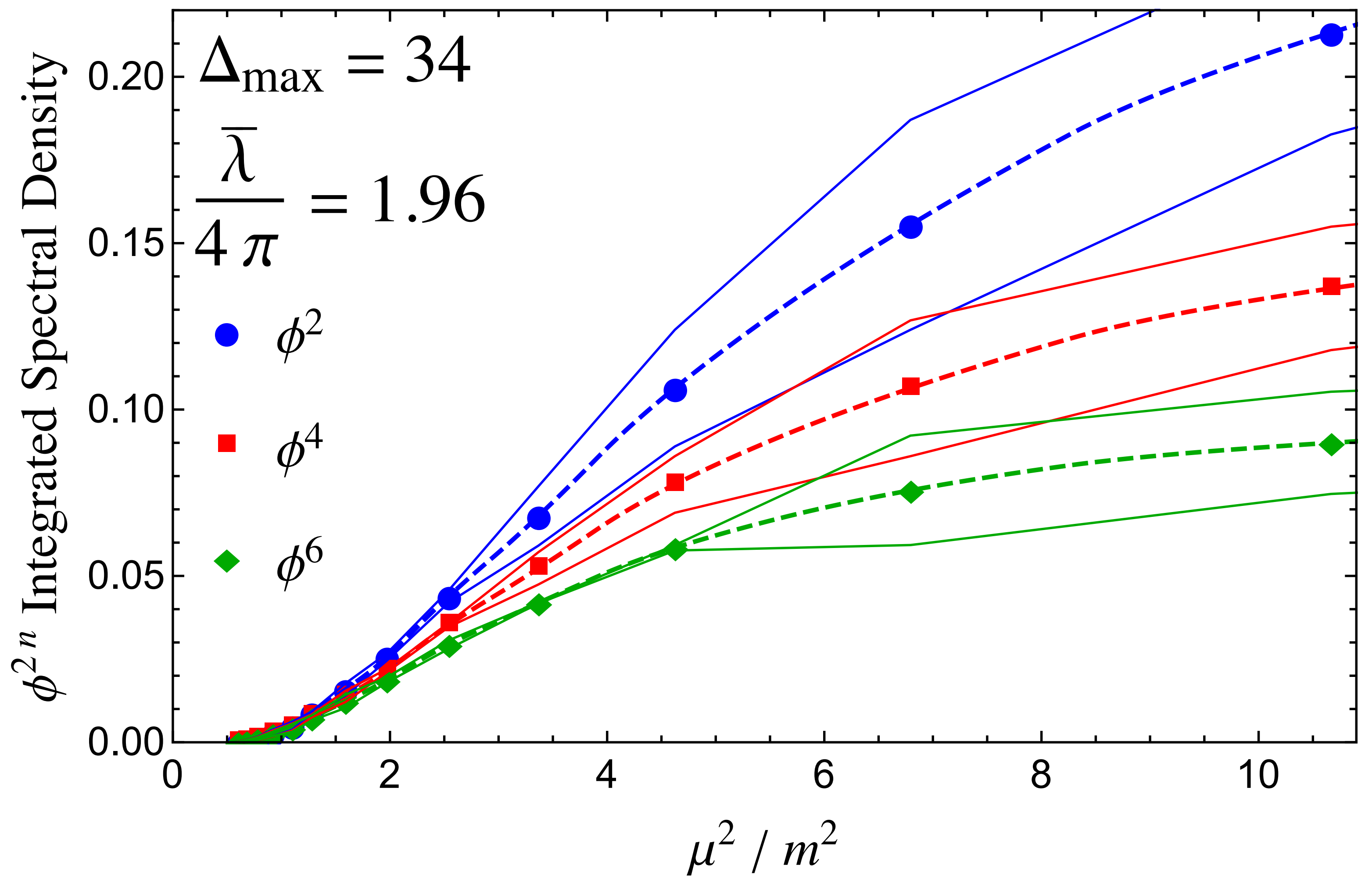
Spectral Density of ϕ^2



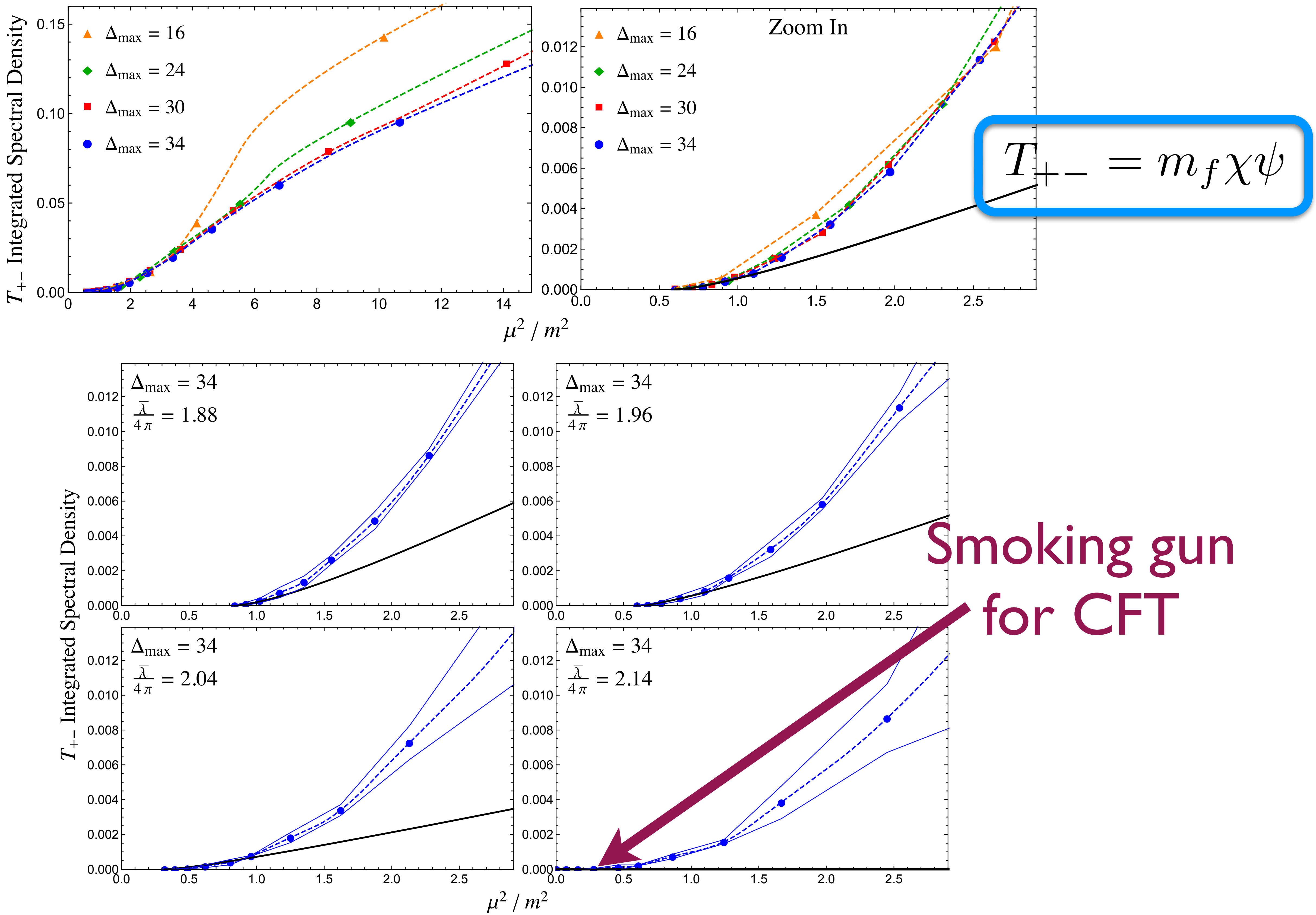
$$\phi^2 \sim \epsilon = \chi\psi$$



Spectral Densities: Universality in even sector

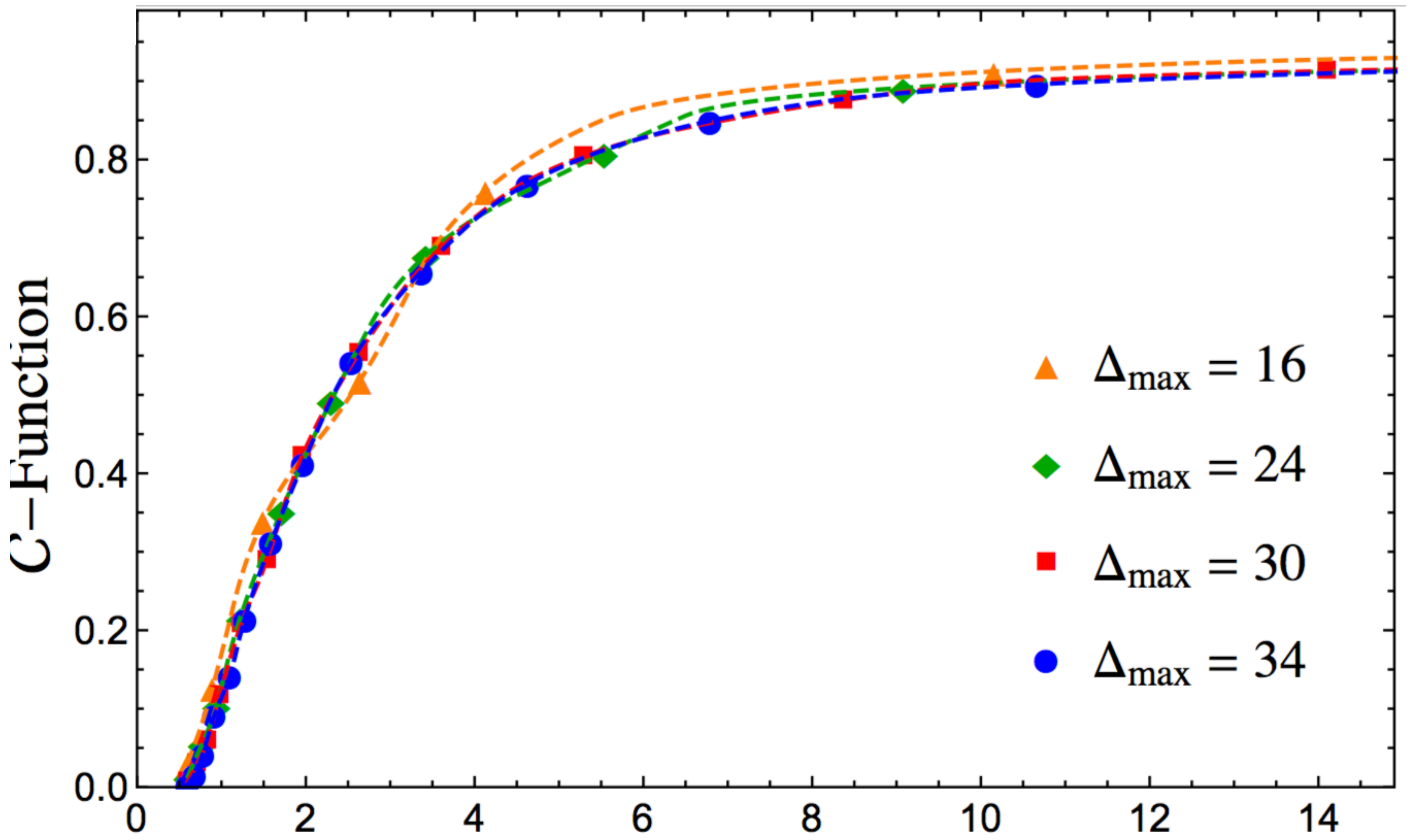


Spectral Density of the trace of stress-tensor



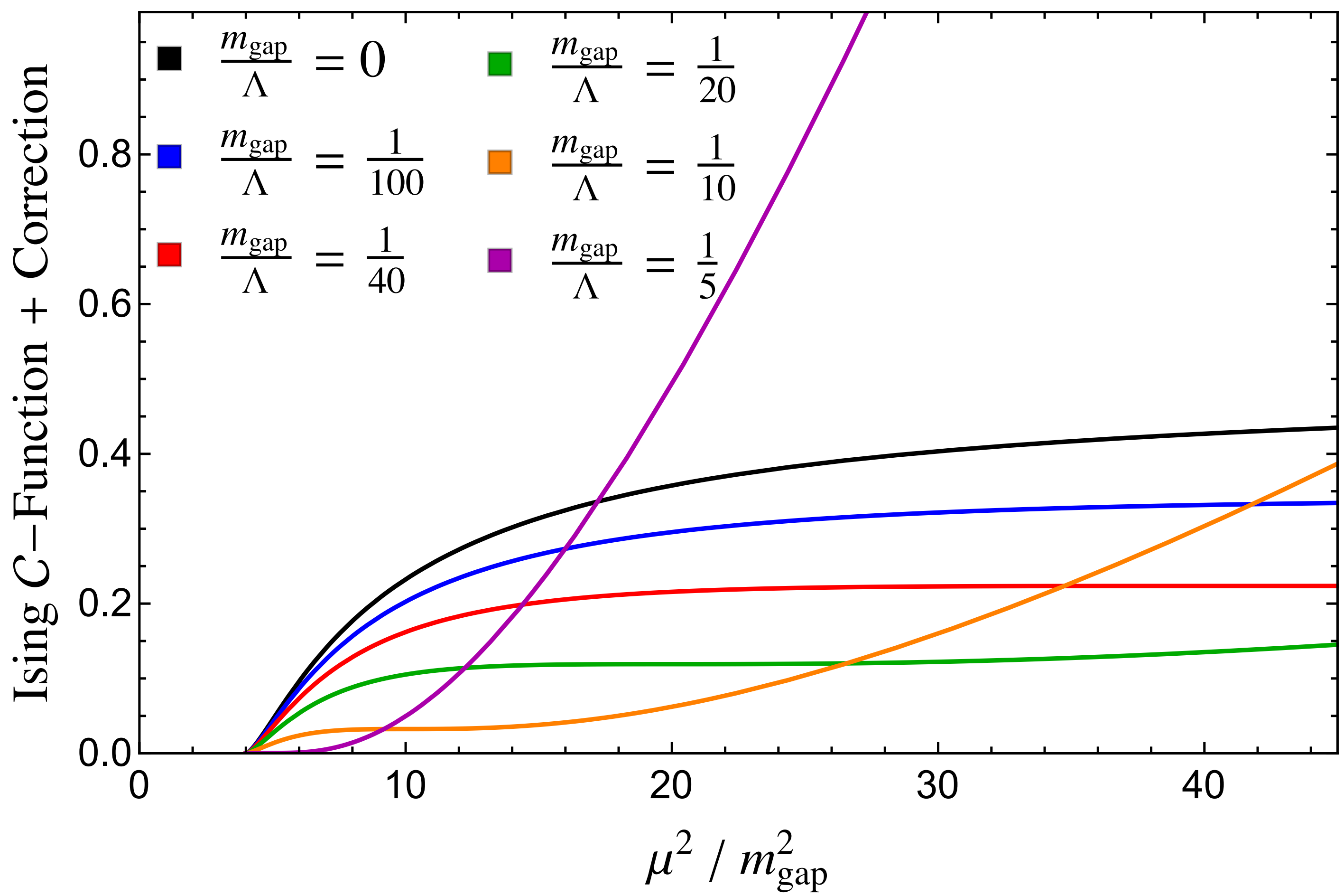
The C-Function

(also spectral integral of T_{--})



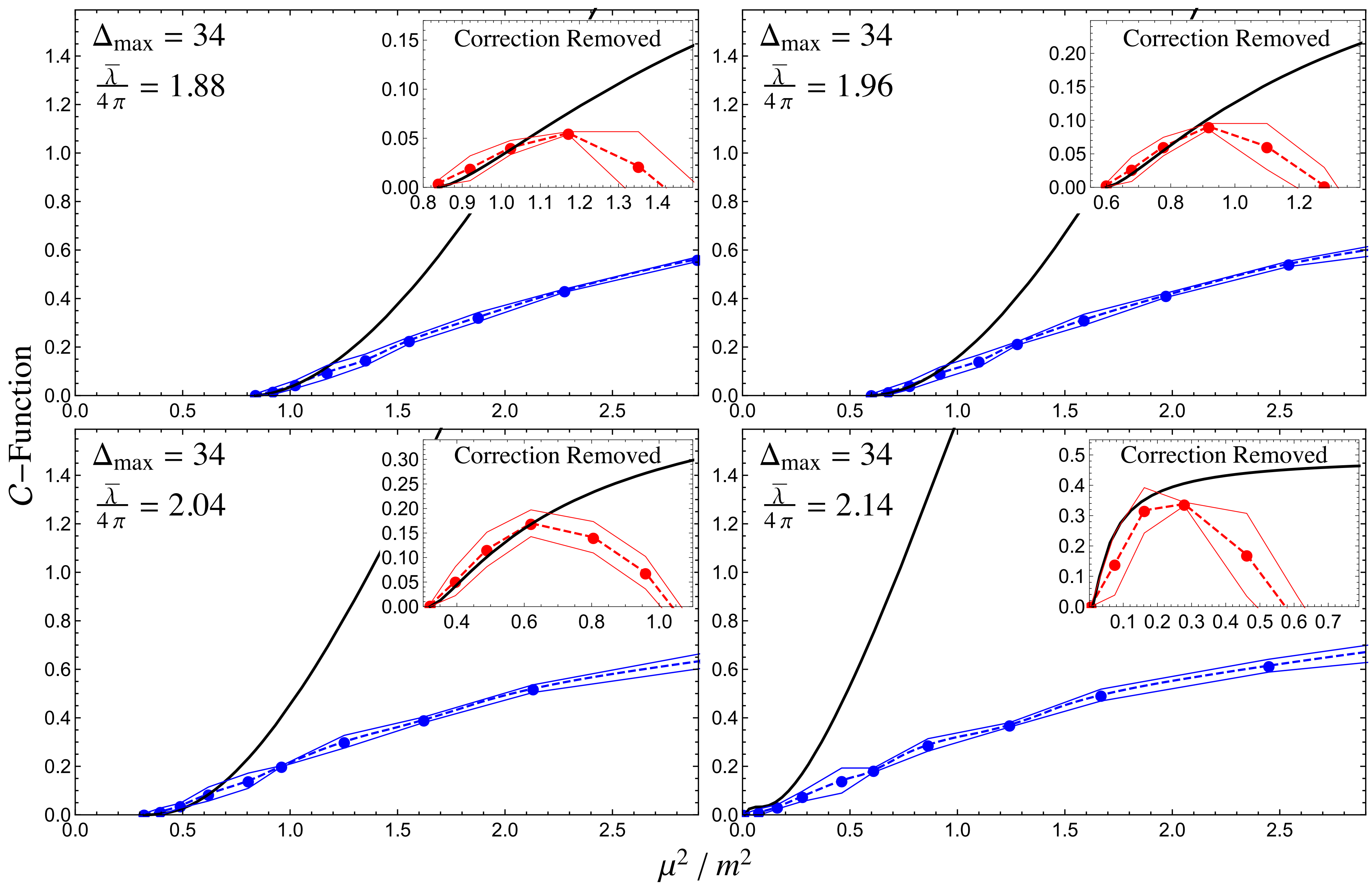
The C-Function and correction

$$T_{+-} \approx m_{gap}\epsilon - \frac{\partial^2 \epsilon}{\Lambda} + \cdots$$



The corrected C-Function

Numerically: $\Lambda \approx 1 \sim \frac{\lambda}{4\pi}$



A sketch of application to Gauge theories

Traditionally: $\mathcal{L} = \mathcal{L}_A + \mathcal{L}_\psi - g A_\mu J^\mu_\psi(x)$

Problem: $g A_\mu J^\mu_\psi(x)$ is not a local gauge-inv. op!

Quantization becomes sensitive choosing an appropriate regulator!

No robust non-perturbative regulator for any known Hamiltonian method respecting space-time symmetries.

Moreover - log running isn't really ever a CFT!

Alternative idea:

1. "Banks-Zaks the theory":

Add vector-like matter to make the theory a weakly coupled CFT.

Ex. QCD w/ some flavors (2-6): Add more flavors.

Nf=16: $\alpha_s * \approx .04$ Nf=15: $\alpha_s * \approx .15$ Nf=14: $\alpha_s * \approx .25$

2. Remove the extra matter:

Ex. QCD: $\delta P_+ = \int d^{d-1}x \; m_q \, \bar{\psi}_i \psi_i(\vec{x})$

Gauge Inv. local Op.

Chiral Ex. SU(5) w/ 10 + $\bar{5}$

Add 4 Adjoint fermions: $\alpha_* \sim .16$

Tension:

Computability of CFT data: small coupling. (conformal bootstrap can help!)

↔

Wanting a smaller basis or shorter RG-flows: larger coupling. (but still allowing for separation of scales)

A possible easier start:

$\mathcal{N} = 4$ SYM @ large-N → Large-N Yang-Mills

LC and Large-N

Single-trace: $\langle \mathcal{O}_i \mathcal{O}_R \mathcal{O}_j \rangle \sim \frac{1}{N}$, $V = N\lambda \int d^{d-1}x\, \mathcal{O}_R(x)$

Generic double-trace: $\langle \mathcal{O}_i \mathcal{O}_R [\mathcal{O}_j \mathcal{O}_k] \rangle \sim \frac{1}{N^2}$

Problematic double-traces: $\langle \mathcal{O}_i \mathcal{O}_R [\mathcal{O}_i \mathcal{O}_R] \rangle \sim 1 + \frac{1}{N^2}$

Naively all planar multi-trace data matters !

On the LC:

$$\langle \mathcal{O}, \vec{P}, \mu | V_{LC} | \mathcal{O}', \vec{P}', \mu' \rangle = 0 \qquad (\Delta' = \Delta + \Delta_R + 2n)$$

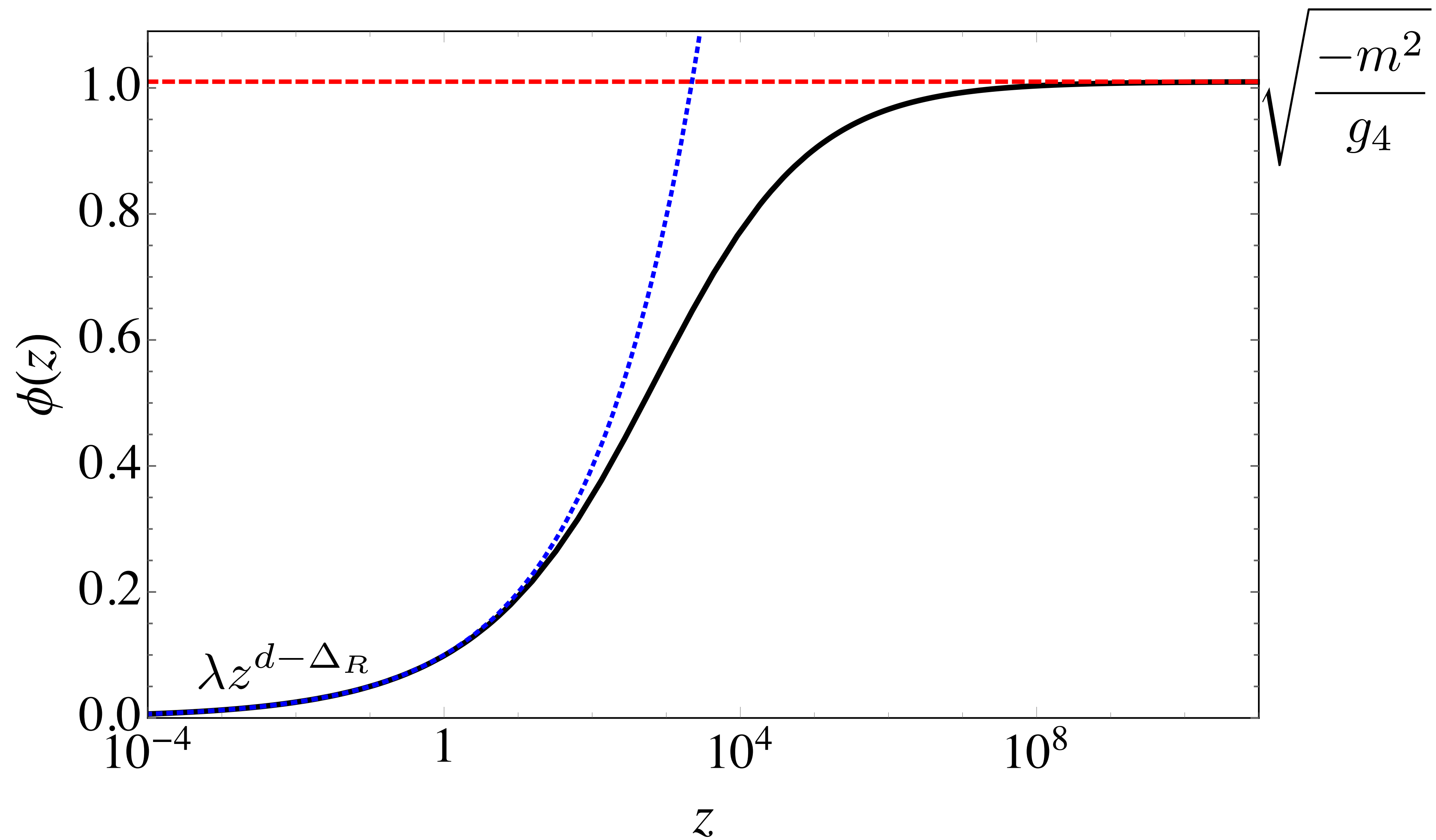
$$\langle \mathcal{O}_i \mathcal{O}_R [\mathcal{O}_i \mathcal{O}_R] \rangle \sim \cancel{1} + \frac{1}{N^2}$$

Only single-trace data matters!

But are we being too quick here?

Cautionary Tale: AdS Bulk Toy Bulk Model (no gravity)

$$\mathcal{L}_{\rm bulk} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\frac{g_4}{N^2}\phi^4$$



LC "Zero-modes"

Field modes with $P_{-} = 0$ thrown out by naive LC

(Ex: bulk profile of our toy made of such modes)

We need a method of integrating such modes out properly!

Prescription:

$$U(x^{+}, 0) \equiv \mathcal{T}\{e^{-i\int_{0}^{x^{+}} dy^{+} V_{LC}(y^{+})}\}$$

$$H_{eff} \equiv \lim_{x^{+}\to 0} i\partial_{+}U(x^{+}, 0)$$

$$\langle \mathcal{O}, P_-, \mu | U(x^+, 0) | \mathcal{O}, P_-, \mu' \rangle = \langle \mathcal{O}, P_-, \mu | \mathcal{O}, P_-, \mu' \rangle - i \int_0^{x^+} dy_1^+ \langle \mathcal{O}, P_-, \mu | V(y_1^+) | \mathcal{O}, P_-, \mu' \rangle$$

$$-\frac{1}{2} \int_0^{x^+} dy_1^+ dy_2^+ \langle \mathcal{O}, P_-, \mu | \mathcal{T}\{V(y_1^+) V(y_2^+)\} | \mathcal{O}, P_-, \mu' \rangle + \cdots$$

Zero-modes: Associated with $\delta(y_{ij}^+)$

$$\int_0^{x^+} dy_1^+ dy_2^+ \langle \mathcal{O}, P_-, \mu | \mathcal{T}\{V(y_1^+) V(y_2^+)\} | \mathcal{O}, P_-, \mu' \rangle \sim \int_0^{x^+} dy_1^+ \langle \mathcal{O}, P_-, \mu | \delta H_{eff}(y_1^+) | \mathcal{O}, P_-, \mu' \rangle$$

Higher-point CFT correlation functions can contribute to H_{eff}

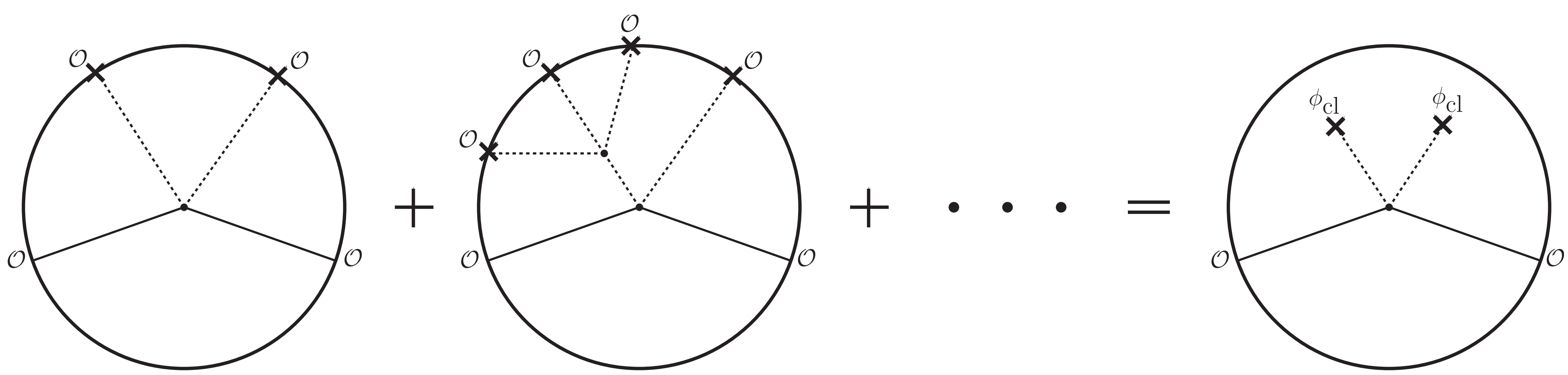
Only contribute if 3pt-functions vanish on the LC:

$$\Delta' = \Delta + \Delta_R + 2n$$

AdS Bulk Toy Model

$$\mathcal{L}_{\rm bulk} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\frac{g_4}{N^2}\phi^4$$

Prescription adds missing contributions to $H_{\rm eff}$



Advantages of the Prescription

1. Demonstrates that correct LC treatment includes a vacuum energy.

2. Shows that in PT the zero modes often only induce a change in bare parameters!

   2D scalar theory: $m_{LC}^2 = m_{ET}^2 + \frac{\lambda}{2}\langle\phi^2\rangle$

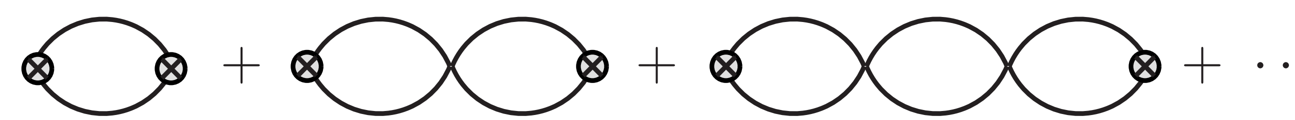
3. Integrates out non-dynamical fields. (Ex: fermions)

4. Allows us to check when RG-flows starting with Large-N CFTs require multi-trace operator data.

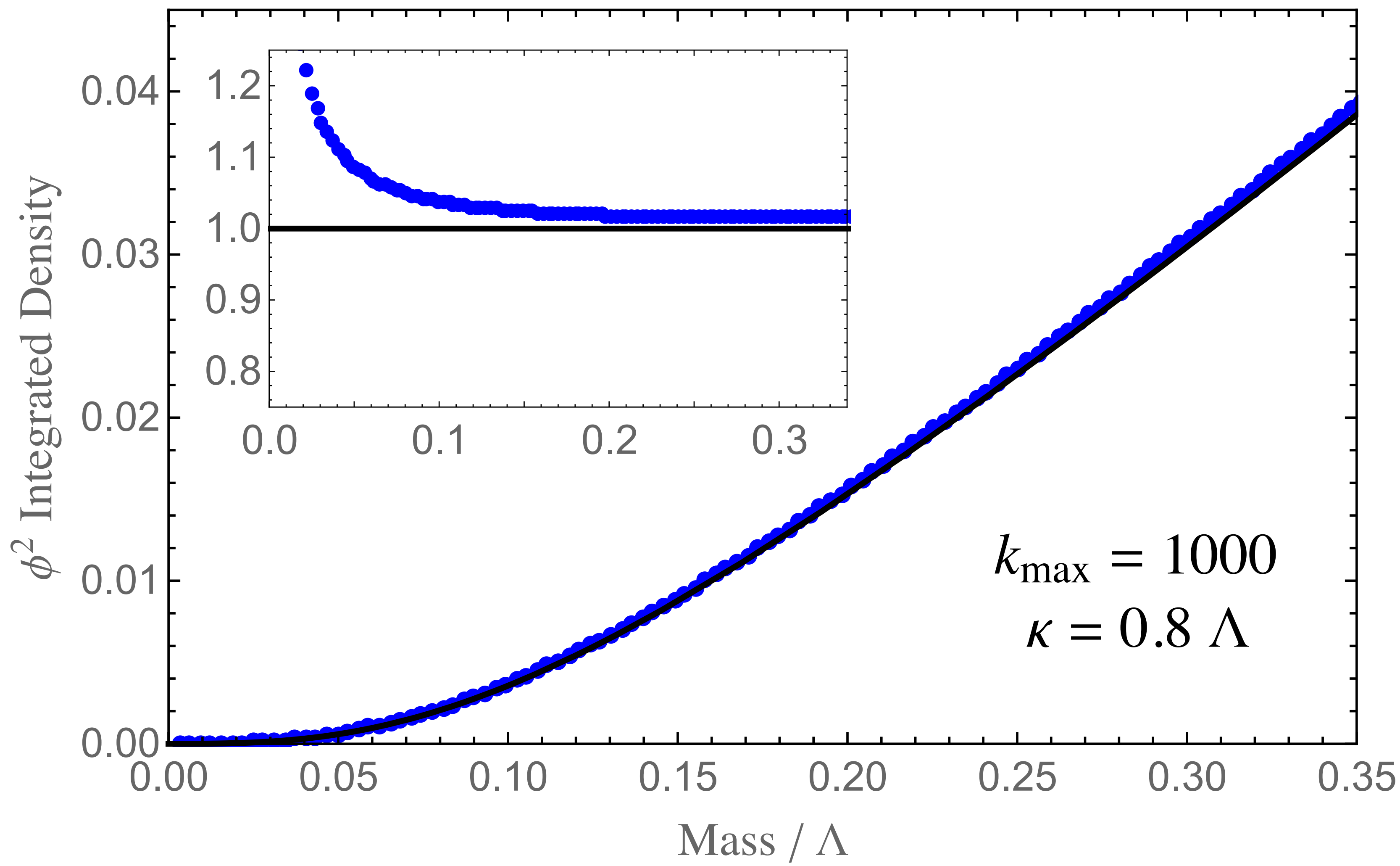
A simple 3D Large-N theory where naive LC works

O(N): $\mathcal{L} = \frac{1}{2}\left(\partial\vec{\phi}\right)^2 - \frac{\lambda}{4!}\left(\vec{\phi}^2\right)^2$ at Large-N

RG-flow takes: $\Delta_{UV} = 1 \to \Delta_{IR} = 2$ (keeping same Casimir)

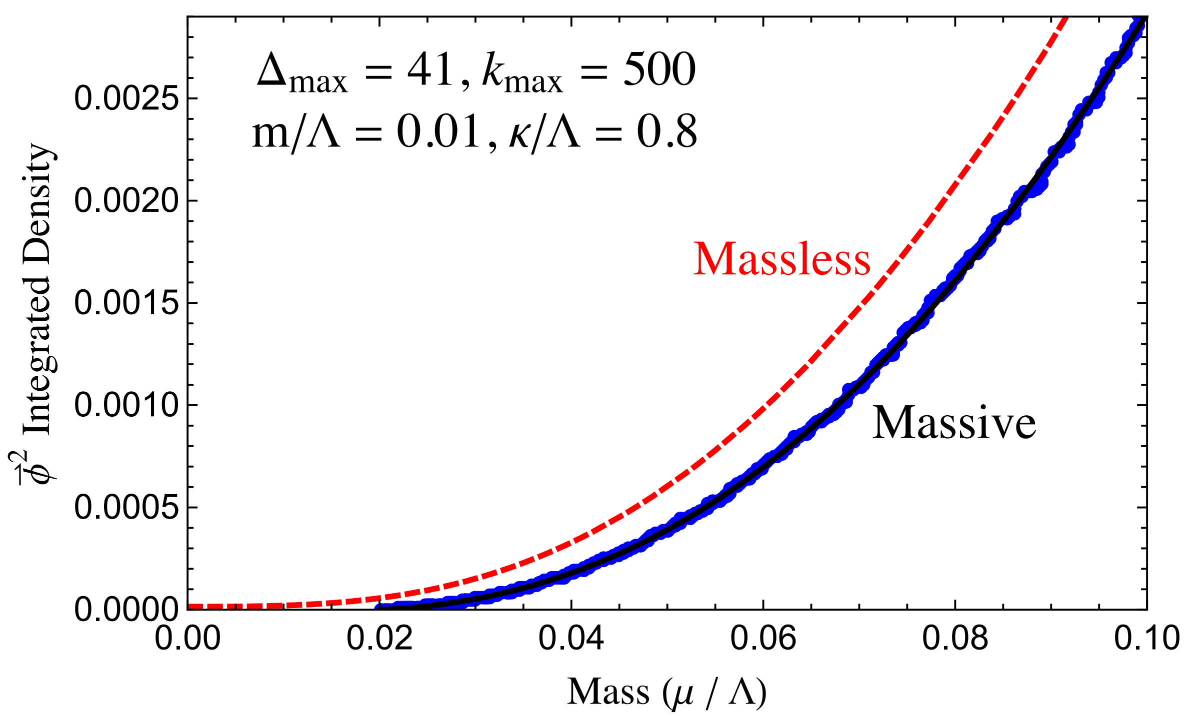
This follows from bubble sum: 

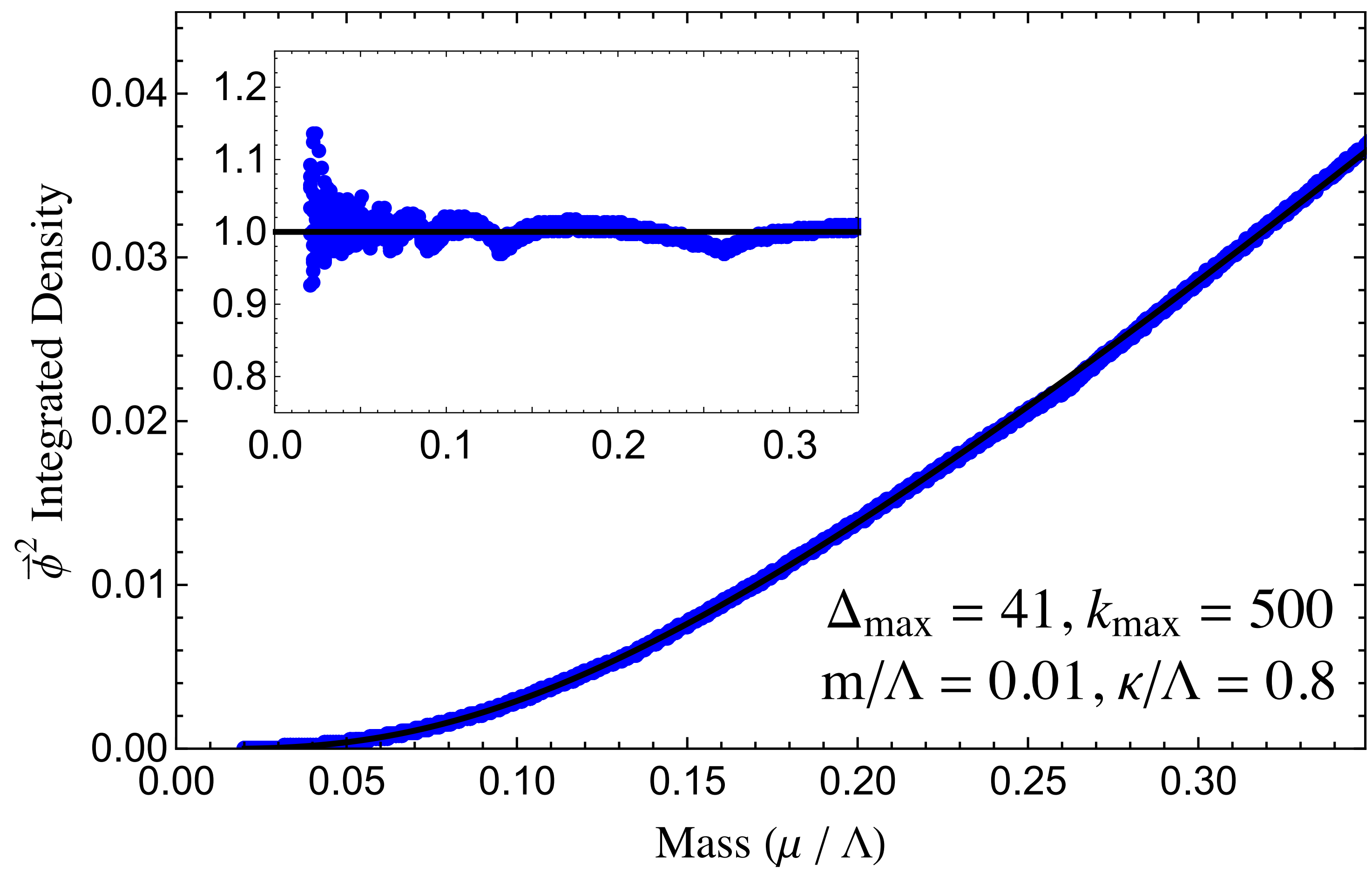
$$\rho_{\vec{\phi}^2}(\mu^2) = \frac{\frac{1}{4\pi\mu}}{1 + \left(\frac{\kappa}{8\mu}\right)^2}$$



Large-N RG-flow in the presence of mass

$$\rho_{\vec{\phi}^2}(\mu^2) = \frac{\frac{1}{4\pi\mu}}{\left(1 + \frac{\kappa}{8\pi\mu}\log\left(\frac{\mu+2m}{\mu-2m}\right)\right)^2 + \left(\frac{\kappa}{8\mu}\right)^2}$$





Conclusions

1. There's new approach to solving/quantizing a QFT using conformal structure on the LC which uses only UV CFT data.

2. It is based on the decoupling of high Casimir states from the low-E spectrum (motivated by AdS/CFT).

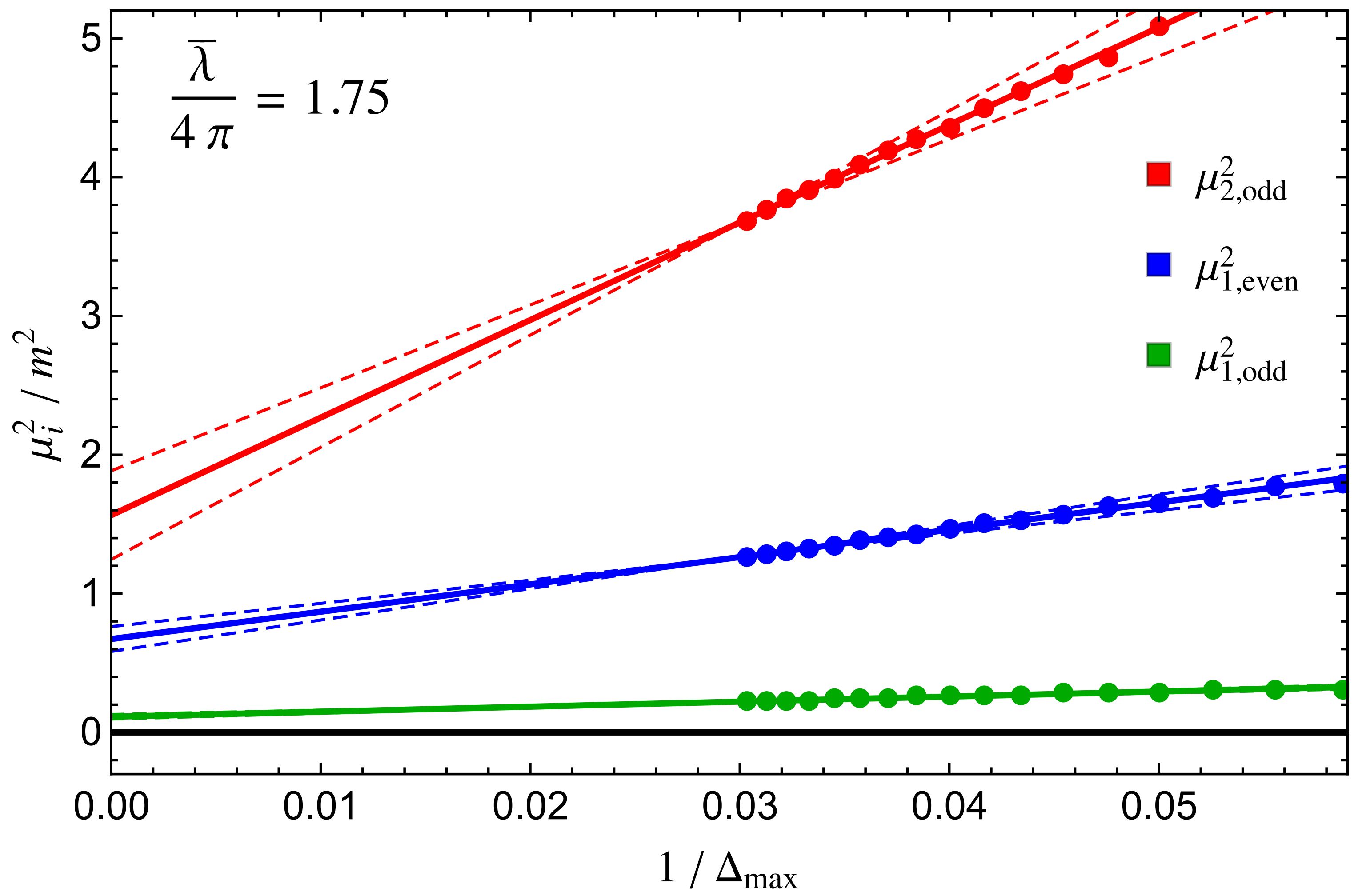
3. Allows calculation of dynamical quantities like the spectral density difficult to obtain with other numerical methods.

4. It has passed certain tests in 2D and has provided new RG-flow results (including the c-function).

5. Should be possible to extend it to theories with gauge-bosons and fermions (including chiral gauge theories in 4D).

6. In some cases LC greatly simplifies large-N RG-flows and we now have a diagnostic as to when this happens. N=4 SYM naively appears to simplify.

Spectrum near the critical coupling

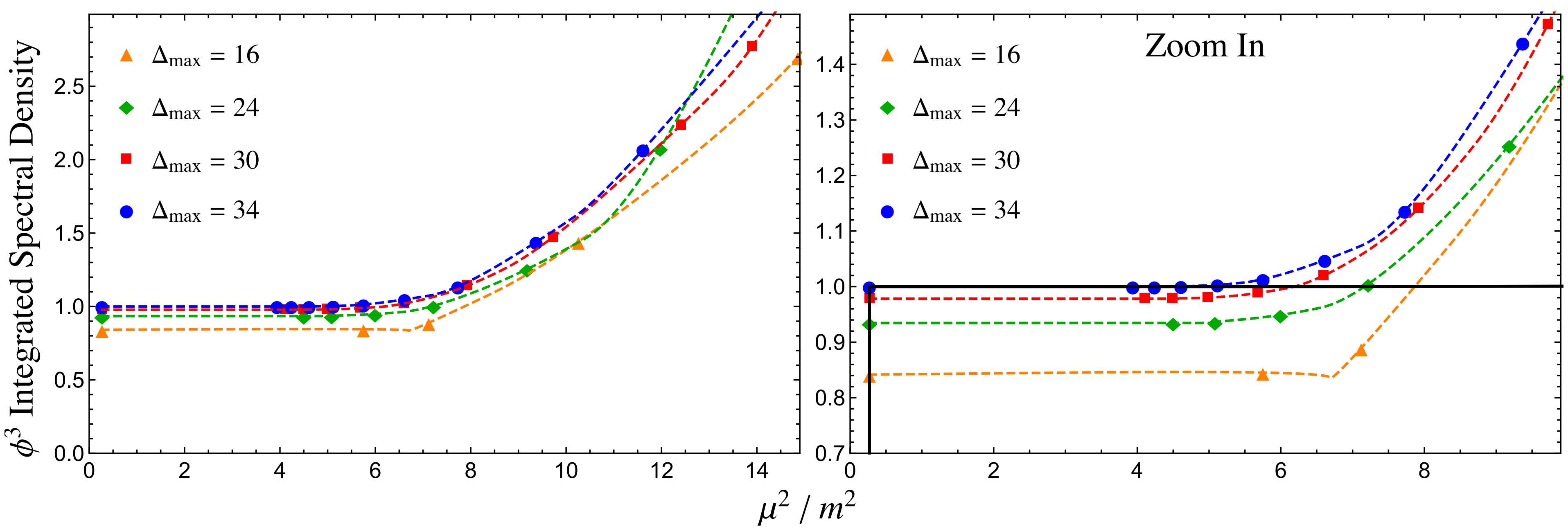


three-fermion threshold

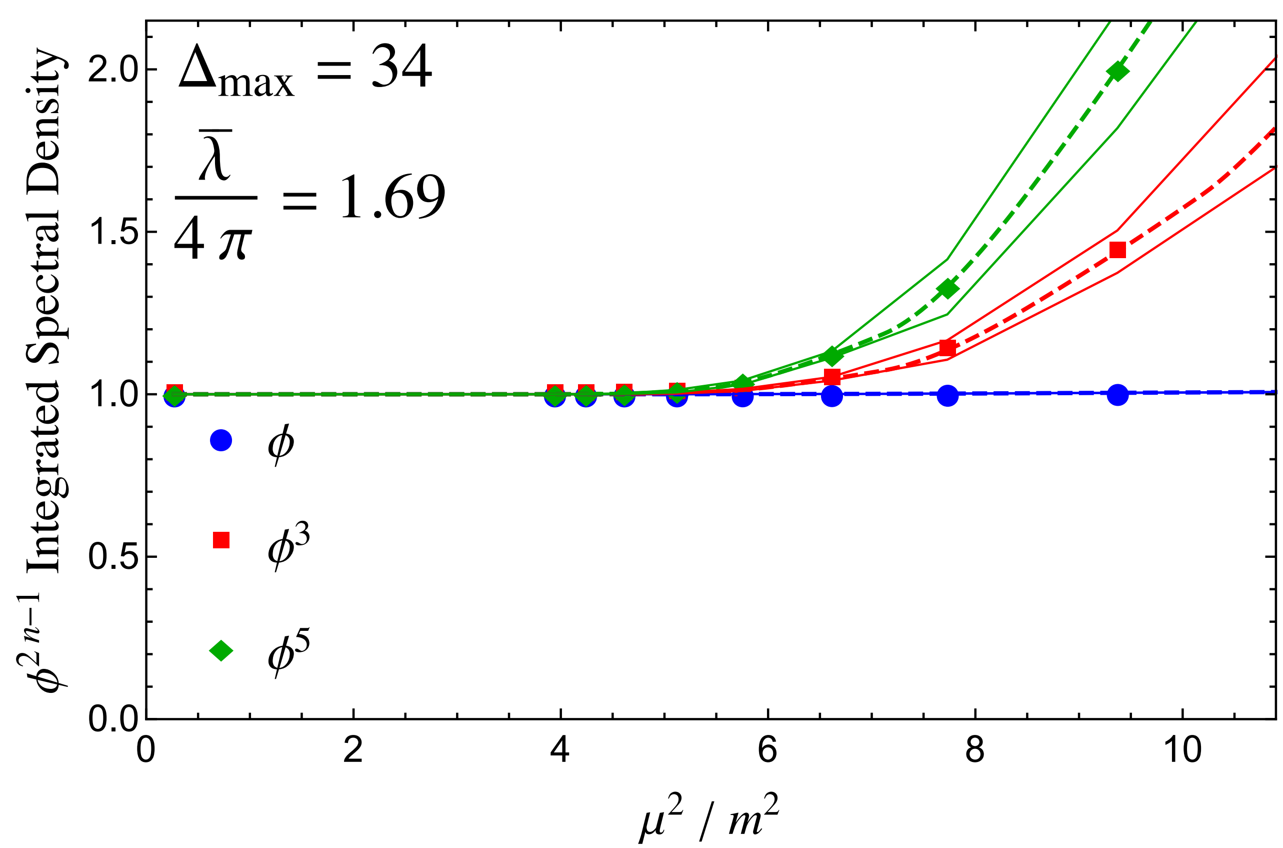
two-fermion threshold

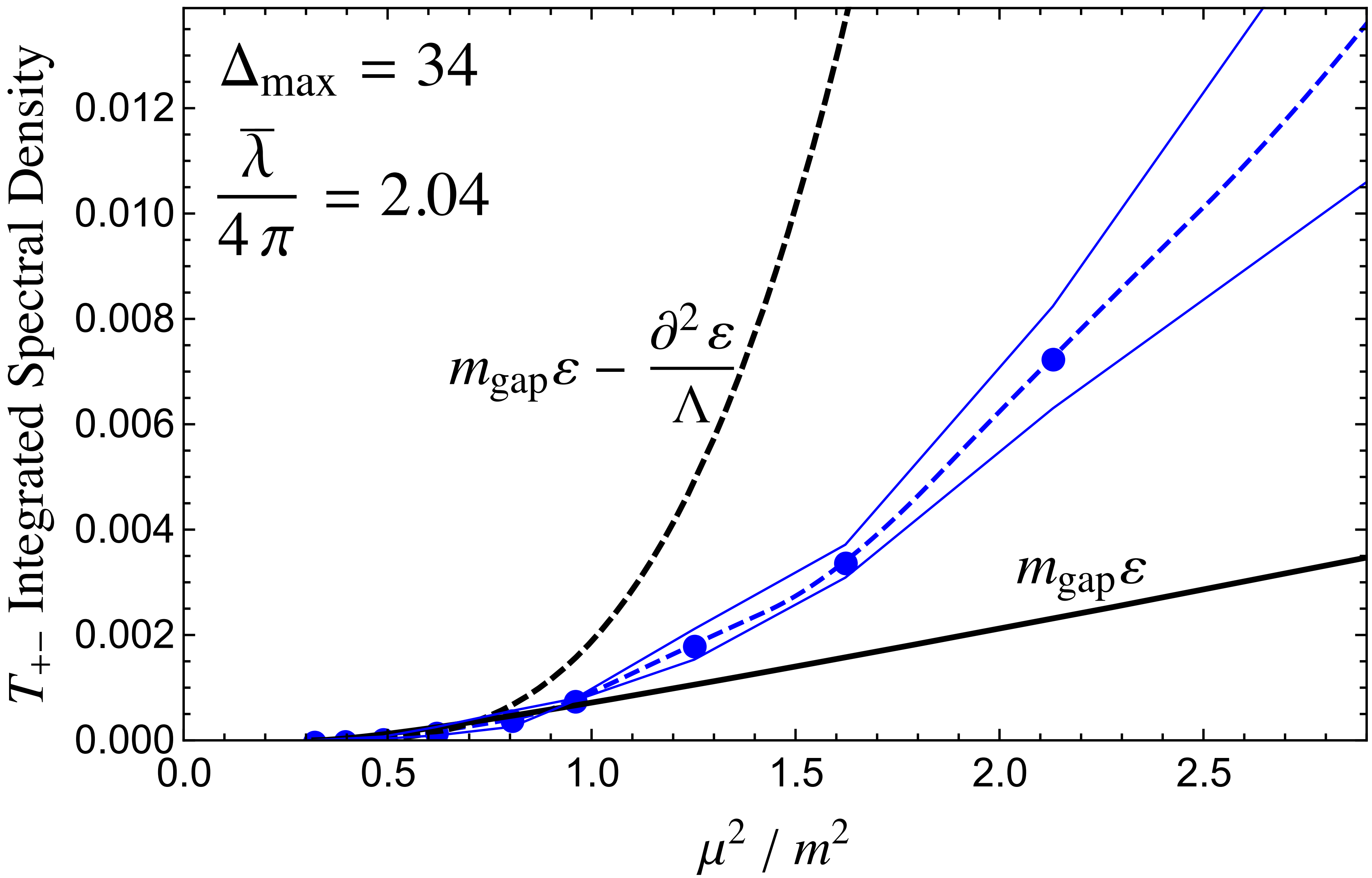
one-fermion

Spectral Densities: Universality in odd sector



$\phi^3 \sim \sigma$





$\Delta_{\max} = 34$
$\frac{\bar{\lambda}}{4\pi} = 2.04$
$m_{\text{gap}}\varepsilon - \frac{\partial^2 \varepsilon}{\Lambda}$
$m_{\text{gap}}\varepsilon$
T_{+-} Integrated Spectral Density
0.012
0.010
0.008
0.006
0.004
0.002
0.000
0.0
0.5
1.0
1.5
2.0
2.5
μ^2 / m^2