Goldstone counting and Inhomogeneous Ground States at Large Global Charge

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Bootstrap Approach to Conformal Field Theories and Applications

Based on
[1505.01537] with
Simeon Hellerman, Domenico Orlando, Susanne Reffert
[1705.05825] and work *almost done* with
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Theory of Big-J

- In order to solve quantum gravity it is inevitable that we understand strongly-coupled QFT systematically.
- Semiclassical analysis can sometimes help us do so and if you are lucky you even have a weakly-coupled Lagrangian description.
- Giving system large charges, J, we can sometimes analyse strongly-coupled theory in the semi-classical regime, where the full Lagrangian is then weakly coupled in units of $1/J$.
- We consider strongly-coupled QFT on the spatial slice S^2 with radius R (or T^2 with periods $R_{1,2}$) in this talk. We give charge density ρ to the state and mostly set $R = 1$ by rescaling.

Theory of Big-J

- Let me tell you how Big-J works. You give large dimensionful VEV to the fields associated with the symmetry. Then there is a large hierarchy between UV and IR energies.
- In this case, we can expect $\Lambda_{\mathrm{UV}} = \sqrt{\rho}$ and $\Lambda_{\mathrm{IR}} = 1/R$ by dimensional analysis. This happens when the ground state configuration for the effective Lagrangian is homogeneous.
- Incidentally, this is only an assumption right at this moment, but it can be shown that this homogeneity assumption is all consistent in some cases.
- Now, when we take the limit of $J \sim \rho/R^{-2} \to \infty$, small ratio of $\Lambda_{\mathrm{IR}}/\Lambda_{\mathrm{UV}}$ should render the theory weakly-coupled!

RG flow of the $O(2)$ model at large charge

- Let me start with the simplest model of all to be analysed at large charge, J.
- We consider a theory constructed from a complex field $\phi \equiv a \times e^{i\chi}$, which (by fine-tuning or whatever) flows to the conformal $O(2)$ Wilson-Fischer fixed point in the IR.
- We can break the $O(2)$ symmetry spontaneously by giving $a = |\phi|$ a large dimensionful VEV that goes as $\sqrt{\rho} \gg 1$ at $\Lambda = \Lambda_{\mathrm{UV}} = \sqrt{\rho}$.
- So by scaling everything far below Λ_{UV}, all the quantum fluctuations are suppressed, and we are left with a classically scale invariant Lagrangian at leading order in $1/J$.

RG flow of the $O(2)$ model at large charge

- I will now give you the leading order $O(2)$ invariant IR Lagrangian, which is also classically conformally invariant,

$$\mathcal{L}_{\mathrm{IR}} = -\frac{1}{2}(\partial a)^2 - \frac{1}{2}\kappa a^2 (\partial\chi)^2 - \frac{h^2}{12} a^6 + \circ\circ\circ$$

- You can also integrate out the a field, whose mass is of order $\sqrt{\rho} \gg 1$, resulting in

$$\mathcal{L}_{\mathrm{IR}} = (\mathrm{const.})|\partial\chi|^3 + \circ\circ\circ$$

- By virtue of Noether theorem, we have $\rho \propto |\partial\chi|^2$ and $J = 4\pi R^2 \rho$, for homogeneous configurations, like the lowest one.

Sorting operators at Big-J

- You can list classically conformal invariant operators in the effective action.
- After integrating out the a field, you are free to put its mass to the denominator of effective operators, which is, in this case, $|\partial\chi|$.
- In order to do this, we have to know the EOM and its classical solution for χ. Because the lowest energy solution of the EOM of χ is homogeneous (either by direct computation or by an argument given later), we can solve it for $\chi = \omega t$.
- Here, we have $\omega \propto \sqrt{J}$.

Operator dimension at large charge

- According to the operator listing,

$$\mathcal{L}_{\rm IR} = c_{3/2}|\partial\chi|^3 + c_{1/2}\texttt{Ric}_3|\partial\chi| + O(J^{-1/4})$$

  This IR Lagrangian is universal when the theory is in the WF fixed point.This universality class even includes $\mathcal{N}=2$ SUSY theory with superpotential $W \propto \Phi^3$.
- Let us calculate the operator dimension at large charge.
- The classical piece is given by just

$$k_{3/2}J^{3/2} + k_{1/2}J^{1/2} + O(J^{-1/4})$$

- Notice this leading J-scaling, $3/2$.This is far above the BPS bound!What does this mean holographically? It might point that extremal RN-AdS black holes can be unstable.

Operator dimension at large charge

- The leading quantum piece of the operator dimension is given by the one-loop vacuum contribution from the $O(J^{3/2})$ piece.
- This gives

$$E_0 = -0.094$$

- There were no counter-terms available at $O(J^0)$, so this is the only universal contribution to the dimension at this order. We therefore get

$$k_{3/2}J^{3/2} + k_{1/2}J^{1/2} - 0.094 + O(J^{-1/4})$$

Comments

- By separating $\chi = \chi_0 + |\partial\chi|^{-1/2}\hat{\chi}$ into VEV and (normalised) fluctuations, we have

$$\mathcal{L}_{\text{leading}}/b_\chi = |\partial\chi_0^3| + \frac{3}{2}\hat{\chi}\left(\partial_t^2 + \frac{1}{2}\triangle_{S^2}\right)\hat{\chi} + \circ\circ\circ$$

- So you can see the speed of the GB is $1/\sqrt{2}$ times the speed of light. This follows directly from conformal symmetry.
- All the states whose dimensions are $O(1)$ above Big-J ground state can also be written down – at spin ℓ, the energy of the excited state increases by $\Delta E(\ell) = \sqrt{\ell(\ell+1)/2}$
- Therefore the spin $\ell = 1$ state is just the descendent of the ground state. Others are just new primaries.

Monte Carlo numerics

- Now you get a nontrivial universal number, you should check this numerically too.
- You can use Monte Carlo simulations to verify the sum rule for the operator dimension.
- The result, done by Banerjee, Chandrasekharan, and Orlando (one of the authors of the original $O(2)$ paper!), suggests the remarkable fit even up to $J \sim 1$!

Monte Carlo numerics

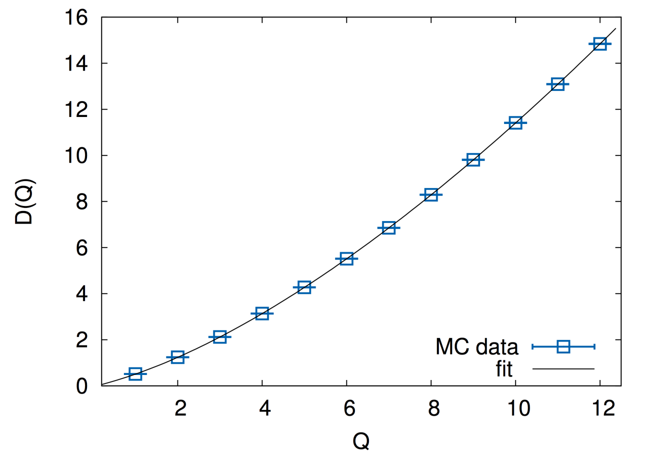

Figure 1: Domenico said on the unit sphere the fitted number is $k_{3/2} = 1.195/\sqrt{4\pi}$ and $k_{1/2} = 0.075\sqrt{4\pi}$. $E_0 \approx -0.094$ comes out right.

Bootstrap at large charge?

- It is clear from the construction that the straightforward numerical bootstrap program slows down at larger charges.
- Analytic bootstrap should be interesting, but not exactly parallel with analytic bootstrap at large spin. The theory is not going to be (generalised) free at leading order (the dimension scales as $J^{3/2}$ instead of J). But maybe vacuum moduli can help as in this case the theory is liberated at large charge.
- Recently Jafferis, Mukhametzhanov and Zhiboedov have put out a nice paper studying Big-J bootstrap.
- It shows that the EFT we derived is the only possible EFT when there is only one Regge trajectory ($O(1)$ excitations with spin more than 2, c.f., [Caron-Huot]).

Comments on cases with moduli space of vacua

- When there is a moduli space of vacua, the story is totally different.
- The curvature of the moduli space vanishes, so that we just have free Lagrangian as the leading-order effective Lagrangian.
- You can also construct superconformally invariant operators, to see sub-leading effects.
- You will find leading such operators scaling at $O(1/J^3)$,which is very small. But more importantly, the form of this operator is $(\partial\phi)^4$ on flat space,so you can use the Arkani-Hamed's argument to derive negative definiteness of the anomalous dimension on the first non-protected operator(above BPS and semi-short operators).This means the attractiveness of the gravitational potential, holographically.

Comments on cases with moduli space of vacua

- You can also compute $2n$-point functions of BPS operators using this method.
- We computed such $2n$-point functions in $D = 4$, $\mathcal{N} = 2$ theories with a one-dimensional Coulomb branch.
- The symmetry group is so big it constrains the IR Lagrangian heavily – there seems to be no contributions, on conformally flat space, to $2n$-point functions aside from the leading and the Weyl-anomaly pieces.
- This gives us universality to all orders in $1/n$ perturbative expansion. Non-perturbative pieces are non-universal and depends on UV details of the theory.
- Will appear on arXiv tomorrow morning. Stay tuned!

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Inhomogeneity of the $O(4)$ large charge grouns states

- Now finally we have all the tools to study the cases where the symmetry algebra has more than one Cartans.
- Why is this important? This is because we used homogeneity to say that Λ_{IR} is small compared to Λ_{UV}. This was a key fact establishing the classical scale invariant EFT.
- In the first part of the talk I just set this ansatz and later proved homogeneity by explicitly computing the classical ground state configuration.

Inhomogeneity of the $O(4)$ large charge ground states

- In 2017, Alvarez-Gaume, Loukas, Orlando and Reffert proved that if we persist in having the homogeneous ground state configuration at large charge in the $O(N)$ model, you can only consider cases where you excite only one Cartan of the symmetry group.
- You can intuitively observe this fact even when you consider free $SU(2)$ bosonic Lagrangian – In the language of the $O(4)$ model, when you excite two Cartans by the same amount
- You can then see what operator describes the large charge ground state with spin 0 – the option is just $\left|\epsilon_{ab}q^a\partial_\mu q^b\right|^n$, where q transform as a doublet in $SU(2)$. Classical configuration cannot be homogeneous on the spatial slices then.

Inhomogeneity of the $O(4)$ large charge ground states

- So inhomogeneity really is a problem when you want to study the large charge expansion of the $O(4)$ theory with generic ρ_1 and ρ_2, eigenvalues of the charge density matrix.
- Why? If the configuration has the instability towards inhomogeneity in the scale of the charge density itself, $1/\sqrt{\rho} = 1/\Lambda_{\mathrm{UV}}$, the EFT is definitely going to break down.
- On the other hand, if it's leaned towards the IR side, the EFT is still applicable even for generic charge density ratios, because you still have the large hierarchy between UV and IR.
- This time, for simplicity, let us put the system on $T^2 \times \mathbb{R}$, where the volume of this torus is $\mathcal{V}$.

Looking for the $O(4)$ large charge ground states

- Let us now look for the large charge ground state configuration of the theory.
- It just the same as in the $O(2)$ case. You decompose the doublet Q appearing in the UV Lagrangian as $Q = A \times q$ and give VEV to A.
- Here we require $q^\dagger q = 1$, so along with the helical ansatz, let's parametrise the helical solution as

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} e^{i\omega_1 t} \sin(p(x)) \\ e^{i\omega_2 t} \cos(p(x)) \end{pmatrix}$$

- The leading IR Lagrangian is

$$\mathcal{L}_{\mathrm{IR}} = b_q \left(\partial q^\dagger \partial q \right)^{3/2}$$

as you might have already rightly guessed from the $O(2)$ case.

Looking for the $O(4)$ large charge ground states

- The EOM for this Lagrangian just reduces to the following equation,

$$-\frac{\kappa^6}{4} = \left(p'(x)^2 - V(p(x))\right)\left(p'(x)^2 + \frac{V(p(x))}{2}\right)^2$$

where $V(p) = \omega_2^2 + (\omega_1^2 - \omega_2^2)\sin^2(p)$

- When $\omega_1 = \omega_2$, the solution just should come down to a homogeneous one, for any values of ω.
- So let's take $\omega_{1,2} \sim O(\sqrt{J})$ and $p'(x) \sim \omega_1 - \omega_2 \sim O(1)$, using which the EOM above can be simplified dramatically,

$$(p'(x))^2 = 2\omega_2(\omega_1 - \omega_2)\left(\sin^2(p_0) - \sin^2(p(x))\right)$$

where p_0 is the maximal value of $p(x)$.

$O(4)$ large charge ground states

- This differential equation is actually the same as the EOM for a classical pendulum problem in a uniform gravitational field. And, we know the answer to this type of differential equation very well.
- The solution to the EOM writes

$$\frac{\sin(p(x))}{\sin(p_0)} = \mathrm{sn}\left(\frac{x}{\ell}; \sin(p_0)\right)$$

where $\ell$is proportional to the aptial period of the solution.

Ground state has preference for homogeneity

- You can now use this solution to the EOM to compute the energy associated with each solution with different periods, or equivalently, different ℓ.
- The explicit result is complicated, but like this,

$$\mathcal{E} = \frac{3\sqrt{3}}{8\sqrt{2b_q}}(\rho_1 + \rho_2)^{3/2} \times \left(1 + \frac{A}{\ell^2}\right)$$

where

$$A \equiv \frac{2b_q}{3(\rho_1 + \rho_2)}\left(\sin^2(p_0) + \frac{\rho_1}{\rho_1 + \rho_2}\right) > 0$$

- So making the period larger is energetically favourable! The period for the ground state configuration becomes the (longer) period of the torus!

Comments

- You can compute the lowest energy and it scales like $J^{3/2}$ again.
- You can also compute two-point functions to see the direct effect on inhomogeneity on observables.

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Inhomogeneity of the ground states and Goldstone counting

- We can actually prove several cute facts about inhomogeneity using Goldstone counting.
- The symmetry is first explicitly broken by adding chemical potential, which then is spontaneously broken by the solution to the EOM itself.
- For example, we can actually prove that the ground state for the $O(2)$ model at large charge must be homogeneous.

Homogeneity: $O(2)$ at large charge

- We can prove the homogeneity of the ground state at large charge of the $O(2)$ model even without resorting to a complicated argument like explicit breaking or anything.
- Assume otherwise; then in the EFT there are two or more GBs, namely, the axion and the GB(s) from the translational symmetry breaking.
- But you started from a theory of a complex scaler, whose dof is two.
- Then the EFT should contain less than two dof, which would contradict with the presence of the translational GB(s).

Inhomogeneity: $O(4)$ at large charge

- We prove that the ground state configuration is still homogeneous if you only excite only one of the two Cartans, i.e., $\rho_1 = \rho$ and $\rho_2 = 0$.
- One easy way to see the breaking pattern caused by this constraint is to think of what transformations preserve the condition $\rho_2 = 0$.
- The condition is equivalent to $q_1 = q_2$. The symmetry actions that preserve this condition is just either overall phase rotation or the elements of diagonal $\mathrm{SU}(2)$.

Inhomogeneity: $O(4)$ at large charge

- So theexplicit breaking from the chemical potential becomes

$$\mathrm{SU}(2) \times \mathrm{SU}(2) \xrightarrow{\text{explicit breaking}} \mathrm{U}(1) \times \mathrm{SU}(2)$$

- Then we solve the EOM to find $\omega_1 = \omega_2$. Then one of the vacuum configuration becomes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so the spontaneous breaking pattern is

$$\mathrm{U}(1) \times \mathrm{SU}(2) \xrightarrow{\text{spontaneous breaking}} \mathrm{U}(1)$$

assuming the homogeneity of the configuration.

Inhomogeneity: $O(4)$ at large charge

- Because $\dim(\mathrm{U}(1) \times \mathrm{SU}(2)/\mathrm{U}(1)) = 3$, if the translational symmetry is further broken, there are four or more Goldstone modes in the system.
- But there are only three light real fields in the spectrum, so this cannot happen.
- We now have proven that the large charge ground state configuration where only one Cartan is excited is homogeneous!

Inhomogeneity: $O(4)$ at large charge

- We now prove that the ground state configuration becomes inhomogeneous only in one direction even if you generically excite two Cartans, i.e., $\rho_1,\ \rho_2 \neq 0$
- The explicit breaking from the chemical potential is

$$\mathrm{SU}(2) \times \mathrm{SU}(2) \xrightarrow{\text{explicit breaking}} \mathrm{U}(1) \times \mathrm{U}(1)$$

- Then we solve the EOM, but you already know from Gaume et.al., that the configuration cannot be homogeneous.
- Assume the inhomogeneity in only one direction. The spontaneous breaking pattern is then

$$\mathrm{U}(1) \times \mathrm{U}(1) \times \{\text{translation}\} \xrightarrow{\text{spontaneous breaking}} \{\text{trivial}\}$$

Inhomogeneity: $O(4)$ at large charge

- The dimension of the coset is, again, 3. So now if the translational symmetry is further broken, there are four or more Goldstone modes in the system
- But there are only three light real fields in the spectrum, so this cannot happen.
- So we now have established that the large charge ground state configuration where two Cartans are excited is only homogeneous in one direction!

Take-home messages

- Large charge expansion is interesting, and makes it possible to analyse a strongly-coupled theory like a weakly-coupled one.
- This analysis is largely dependent on large separation of UV and IR energies.
- Sometimes the ground state configuration at large charge is inhomogeneous, but in our examples the inhomogeneity is at the scale of the underlying geometry itself, and the EFT is still applicable.
- You can directly extract interesting information from the inhomogeneity, such as two-point functions.
- The inhomogeneity is the spontaneous breaking of the translation symmetry, and it is possible to analyse the breaking pattern by matching the number of Goldstones with the available light modes in EFT.