Inflation and pre-inflation: recent developments

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International conference "Accelerating Universe in the Dark"

YITP, Kyoto, Japan, 05.03.2019

Inflation and two new fundamental parameters

The simplest one-parametric inflationary models

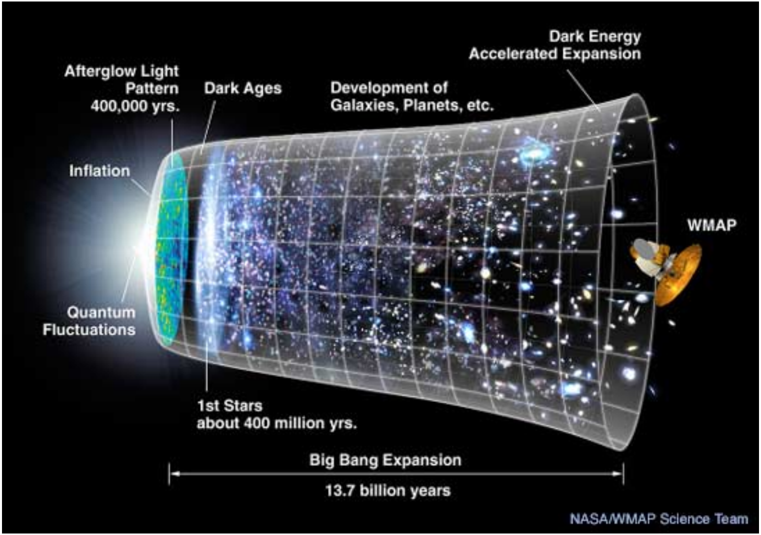
R^2 inflation as a dynamical attractor for scalar-tensor models

Quantum corrections to the simplest model

Generality of inflation

Formation of inflation from generic curvature singularity

Conclusions


Dark Energy
Accelerated Expansion
Afterglow Light
Pattern
400,000 yrs.
Dark Ages
Development of
Galaxies, Planets, etc.
Inflation
WMAP
Quantum
Fluctuations
1st Stars
about 400 million yrs.
Big Bang Expansion
13.7 billion years
NASA/WMAP Science Team

Inflation

The (minimal variant of the) inflationary scenario is based on the two cornerstone independent ideas (hypothesis):

1. Existence of inflation (or, quasi-de Sitter stage) – a stage of accelerated, close to exponential expansion of our Universe in the past preceding the hot Big Bang with decelerated, power-law expansion.

2. The origin of all inhomogeneities in the present Universe is the effect of gravitational creation of pairs of particles - antiparticles and field fluctuations during inflation from the adiabatic vacuum (no-particle) state for Fourier modes covering all observable range of scales (and possibly somewhat beyond).

NB. This effect is the same as particle creation by black holes, but no problems with the loss of information, 'firewalls', trans-Planckian energy etc. in cosmology, as far as observational predictions are calculated.

Outcome of inflation

In the super-Hubble regime ($k \ll aH$) in the coordinate representation:

$$ds^2 = dt^2 - a^2(t)(\delta_{lm} + h_{lm})dx^l dx^m, \quad l, m = 1, 2, 3$$

$$h_{lm} = 2\mathcal{R}(\mathbf{r})\delta_{lm} + \sum_{a=1}^{2} g^{(a)}(\mathbf{r})\, e_{lm}^{(a)}$$

$$e_l^{l(a)} = 0, \;\; g^{(a)}_{,l}\, e_m^{l(a)} = 0, \;\; e_{lm}^{(a)}\, e^{lm(a)} = 1$$

$\mathcal{R}$ describes primordial scalar perturbations, g – primordial tensor perturbations (primordial gravitational waves (GW)).

The most important quantities:

$$n_s(k) - 1 \equiv \frac{d \ln P_{\mathcal{R}}(k)}{d \ln k}, \quad r(k) \equiv \frac{P_g}{P_{\mathcal{R}}}$$

In fact, metric perturbations h_{lm} are quantum (operators in the Heisenberg representation) and remain quantum up to the present time. But, after omitting of a very small part, decaying with time, they become commuting and, thus, equivalent to classical (c-number) stochastic quantities with the Gaussian statistics (up to small terms quadratic in $\mathcal{R}, \ g$).

In particular:

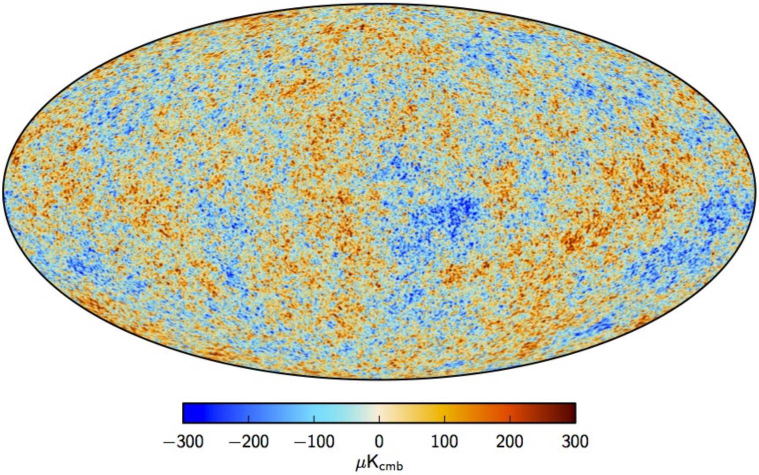
$$\hat{\mathcal{R}}_k = \mathcal{R}_k \, i(\hat{a}_{\mathbf{k}} - \hat{a}_{\mathbf{k}}^\dagger) + \mathcal{O}\left((\hat{a}_{\mathbf{k}} - \hat{a}_{\mathbf{k}}^\dagger)^2\right) + ... + \mathcal{O}(10^{-100})(\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}^\dagger) + , , ,$$

The last term is time dependent, it is affected by physical decoherence and may become larger, but not as large as the second term.

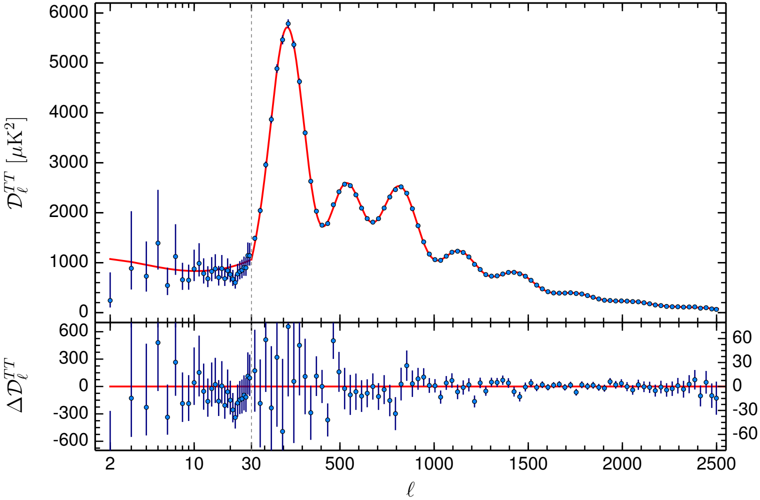
Remaining quantum coherence: deterministic correlation between $\mathbf{k}$ and $-\mathbf{k}$ modes - shows itself in the appearance of acoustic oscillations (primordial oscillations in case of GW).

CMB temperature anisotropy

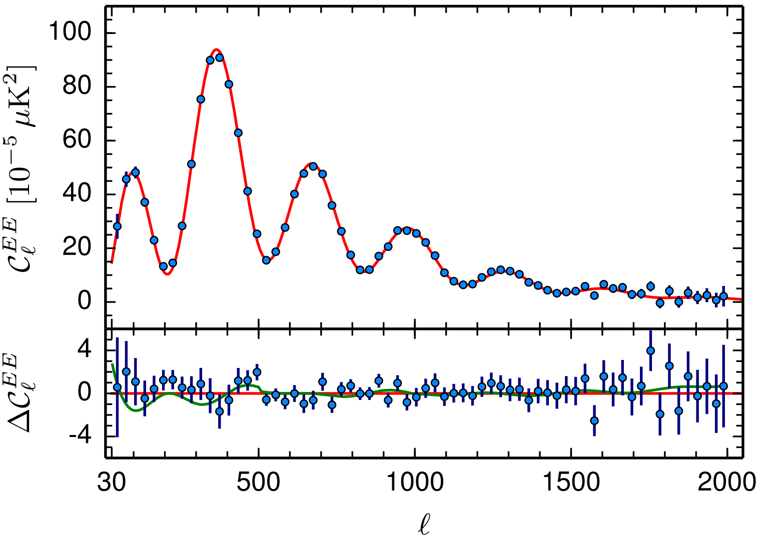
Planck-2015: P. A. R. Ade et al., arXiv:1502.01589



CMB temperature anisotropy multipoles



CMB E-mode polarization multipoles



New cosmological parameters relevant to inflation

Now we have numbers: N. Agranim et al., arXiv:1807.06209

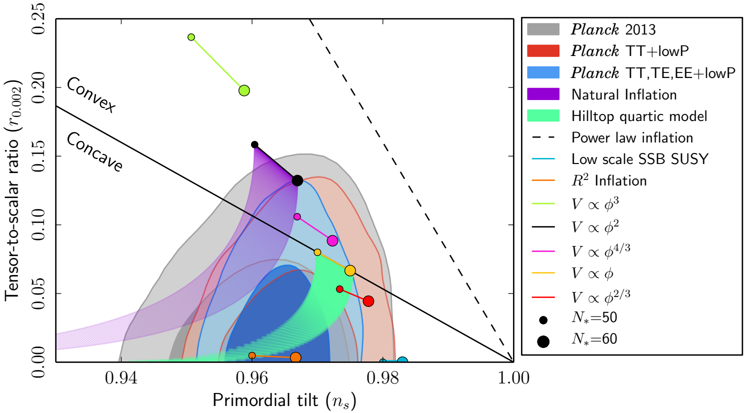
The primordial spectrum of scalar perturbations has been measured and its deviation from the flat spectrum $n_s = 1$ in the first order in $|n_s - 1| \sim N_H^{-1}$ has been discovered (using the multipole range $\ell > 40$):

$$< \mathcal{R}^2(\mathbf{r}) >= \int \frac{P_{\mathcal{R}}(k)}{k}\, dk, \;\; P_{\mathcal{R}}(k) = (2.10 \pm 0.03)\cdot 10^{-9} \left(\frac{k}{k_0}\right)^{n_s - 1}$$

$$k_0 = 0.05 \text{ Mpc}^{-1}, \quad n_s - 1 = -0.035 \pm 0.004$$

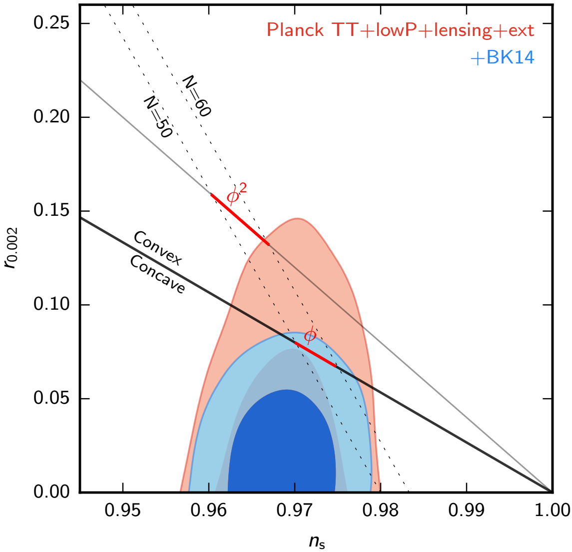
Two fundamental observational constants of cosmology in addition to the three known ones (baryon-to-photon ratio, baryon-to-matter density and the cosmological constant). Existing inflationary models can predict (and predicted, in fact) one of them, namely $n_s - 1$, relating it finally to $N_H = \ln \frac{k_B T_\gamma}{\hbar H_0} \approx 67.2$. (note that $(1 - n_s)N_H \sim 2$).

Direct approach: comparison with simple smooth models



Combined results from Planck/BISEP2/Keck Array

P. A. R. Ade et al., Phys. Rev. Lett. 116, 031302 (2016)



The simplest models producing the observed scalar slope

1. The $R + R^2$ model (Starobinsky, 1980):

$$\mathcal{L} = \frac{f(R)}{16\pi G}, \quad f(R) = R + \frac{R^2}{6M^2}$$

$$M = 2.6 \times 10^{-6} \left(\frac{55}{N}\right) M_{Pl} \approx 3.1 \times 10^{13}\,\mathrm{GeV}$$

$$n_s - 1 = -\frac{2}{N} \approx -0.036, \quad r = \frac{12}{N^2} \approx 0.004$$

$$N = \ln\frac{k_f}{k} = \ln\frac{T_\gamma}{k} - \mathcal{O}(10), \quad H_{dS}(N = 55) = 1.4 \times 10^{14}\,\mathrm{GeV}$$

2. The same prediction from a scalar field model with $V(\phi) = \frac{\lambda\phi^4}{4}$ at large ϕ and strong non-minimal coupling to gravity $\xi R\phi^2$ with $\xi < 0, \ |\xi| \gg 1$, including the Higgs inflationary model (Bezrukov and Shaposhnikov, 2008).

The simplest purely geometrical inflationary model

$$\mathcal{L} = \frac{R}{16\pi G} + \frac{N^2}{288\pi^2 P_{\mathcal{R}}(k)} R^2 + (\text{small rad. corr.})$$

$$= \frac{R}{16\pi G} + 5.1 \times 10^8 \, R^2 + (\text{small rad. corr.})$$

The quantum effect of creation of particles and field fluctuations works twice in this model:

a) at super-Hubble scales during inflation, to generate space-time metric fluctuations;

b) at small scales after inflation, to provide scalaron decay into pairs of matter particles and antiparticles (AS, 1980, 1981).

Weak dependence of the time t_r when the radiation dominated stage begins:

$$N(k) \approx N_H + \ln \frac{a_0 H_0}{k} - \frac{1}{3} \ln \frac{M_{\rm Pl}}{M} - \frac{1}{6} \ln(M_{\rm Pl} t_r)$$

Evolution of the $R + R^2$ model

1. During inflation ($H \gg M$):

$$H = \frac{M^2}{6}(t_f - t), \quad |\dot{H}| \ll H^2$$

.

2. After inflation ($H \ll M$):

$$a(t) \propto t^{2/3}\left(1 + \frac{2}{3Mt}\sin M(t - t_1)\right)$$

The most effective decay channel: into minimally coupled scalars with $m \ll M$. Then the formula

$$\frac{1}{\sqrt{-g}}\frac{d}{dt}(\sqrt{-g}\,n_s) = \frac{R^2}{576\pi}$$

(Ya. B. Zeldovich and A. A. Starobinsky, JETP Lett. 26, 252 (1977)) can be used for simplicity, but the full integral-differential system of equations for the Bogoliubov α_k, β_k coefficients and the average EMT was in fact solved in AS (1981). Scalaron decay into graviton pairs is suppressed (A. A. Starobinsky, JETP Lett. 34, 438 (1981)).

For this channel of the scalaron decay:

$$N(k) \approx N_H + \ln\frac{a_0 H_0}{k} - \frac{5}{6}\ln\frac{M_{\rm Pl}}{M}$$

Possible microscopic origins of this phenomenological model.

1. Follow the purely geometrical approach and consider it as the specific case of the fourth order gravity in 4D

$$\mathcal{L} = \frac{R}{16\pi G} + AR^2 + BC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + (\text{small rad. corr.})$$

for which $A \gg 1,\ A \gg |B|$. Approximate scale (dilaton) invariance and absence of ghosts in the curvature regime $A^{-2} \ll (RR)/M_P^4 \ll B^{-2}$.

One-loop quantum-gravitational corrections are small (their imaginary parts are just the predicted spectra of scalar and tensor perturbations), non-local and qualitatively have the same structure modulo logarithmic dependence on curvature.

2. Another, completely different way:

consider the $R + R^2$ model as an approximate description of GR + a non-minimally coupled scalar field with a large negative coupling ξ ($\xi_{conf} = \frac{1}{6}$) in the gravity sector::

$$\mathcal{L} = \frac{R}{16\pi G} - \frac{\xi R \phi^2}{2} + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi), \quad \xi < 0, \quad |\xi| \gg 1 \ .$$

Geometrization of the scalar:

for a generic family of solutions during inflation and even for some period of non-linear scalar field oscillations after it, the scalar kinetic term can be neglected, so

$$\xi R \phi = -V'(\phi) + \mathcal{O}(|\xi|^{-1}) \ .$$

No conformal transformation, we remain in the the physical (Jordan) frame!

These solutions are the same as for $f(R)$ gravity with

$$\mathcal{L} = \frac{f(R)}{16\pi G}, \ f(R) = R - \frac{\xi R \phi^2(R)}{2} - V(\phi(R)).$$

For $V(\phi) = \frac{\lambda(\phi^2 - \phi_0^2)^2}{4}$, this just produces $f(R) = \frac{1}{16\pi G}\left(R + \frac{R^2}{6M^2}\right)$ with $M^2 = \lambda/24\pi\xi^2 G$ and $\phi^2 = |\xi| R/\lambda$.

The same theorem is valid for a multi-component scalar field.

More generally, R^2 inflation (with an arbitrary n_s, r) serves as an intermediate dynamical attractor for a large class of scalar-tensor gravity models.

Inflation in the mixed Higgs-R^2 Model

M. He, A. A. Starobinsky and J. Yokoyama, JCAP **1805**, 064 (2018).

$$\mathcal{L} = \frac{1}{16\pi G}\left(R + \frac{R^2}{6M^2}\right) - \frac{\xi R\phi^2}{2} + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - \frac{\lambda\phi^4}{4}, \ \ \xi < 0, \ \ |\xi| \gg 1$$

In the attractor regime during inflation (and even for some period after it), we return to the $f(R) = R + \frac{R^2}{6M^2}$ model with the renormalized scalaron mass $M \to \tilde{M}$:

$$\frac{1}{\tilde{M}^2} = \frac{1}{M^2} + \frac{24\pi\xi^2 G}{\lambda}$$

Background FRW equations in $f(R)$ gravity

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$$H \equiv \frac{\dot{a}}{a}\,, \quad R = 6(\dot{H} + 2H^2)$$

The trace equation (4th order)

$$\frac{3}{a^3}\frac{d}{dt}\left(a^3\frac{df'(R)}{dt}\right) - Rf'(R) + 2f(R) = 8\pi G(\rho_m - 3p_m)$$

The 0-0 equation (3d order)

$$3H\frac{df'(R)}{dt} - 3(\dot{H} + H^2)f'(R) + \frac{f(R)}{2} = 8\pi G\rho_m$$

Reduction to the first order equation

In the absence of spatial curvature and $\rho_m = 0$, it is always possible to reduce these equations to a first order one using either the transformation to the Einstein frame and the Hamilton-Jacobi-like equation for a minimally coupled scalar field in a spatially flat FLRW metric, or by directly transforming the 0-0 equation to the equation for $R(H)$:

$$\frac{dR}{dH} = \frac{(R - 6H^2)f'(R) - f(R)}{H(R - 12H^2)f''(R)}$$

See, e.g. H. Motohashi amd A. A. Starobinsky, Eur. Phys. J. C **77**, 538 (2017), but in the special case of the $R + R^2$ gravity this was found and used already in the original AS (1980) paper.

Analogues of large-field (chaotic) inflation: $F(R) \approx R^2 A(R)$ for $R \to \infty$ with $A(R)$ being a slowly varying function of R, namely

$$|A'(R)| \ll \frac{A(R)}{R} \ , \ |A''(R)| \ll \frac{A(R)}{R^2} \ .$$

Analogues of small-field (new) inflation, $R \approx R_1$:

$$F'(R_1) = \frac{2F(R_1)}{R_1} \ , \ F''(R_1) \approx \frac{2F(R_1)}{R_1^2} \ .$$

Thus, all inflationary models in $f(R)$ gravity are close to the simplest one over some range of R.

Perturbation spectra in slow-roll $f(R)$ inflationary models

Let $f(R) = R^2 A(R)$. In the slow-roll approximation $|\ddot{R}| \ll H|\dot{R}|$:

$$P_{\mathcal{R}}(k) = \frac{\kappa^2 A_k}{64\pi^2 A_k'^2 R_k^2}, \quad P_g(k) = \frac{\kappa^2}{12 A_k \pi^2}, \quad \kappa^2 = 8\pi G$$

$$N(k) = -\frac{3}{2}\int_{R_f}^{R_k} dR\, \frac{A}{A' R^2}$$

where the index k means that the quantity is taken at the moment $t = t_k$ of the Hubble radius crossing during inflation for each spatial Fourier mode $k = a(t_k)H(t_k)$.

Quantum corrections to the simplest model

Due to the scale-invariance of the $R + R^2$ model for $R \gg M^2$, one may expect logarithmic running of the dimensionless coefficient in front of the R^2 term for large energies and curvatures. The concrete 'asymptotically safe' model with

$$f(R) = R + \frac{R^2}{6M^2 \left[1 + b \ln\left(\frac{R}{\mu^2}\right)\right]}$$

was recently considered in L.-H. Liu, T. Prokopec, A. A. Starobinsky, Phys. Rev. D **98**, 043505 (2018); arXiv:1806.05407.

However, comparison with CMB observational data shows that b is small by modulus: $|b| \lesssim 10^{-2}$. Thus, from the observational point of view this model can be simplified to

$$f(R) = R + \frac{R^2}{6M^2}\left[1 - b\ln\left(\frac{R}{\mu^2}\right)\right],$$

for which the analytic solution exists:

$$n_s - 1 = -\frac{4b}{3}\left(e^{\frac{2bN}{3}} - 1\right)^{-1}$$

$$r = \frac{16b^2}{3}\frac{e^{\frac{4bN}{3}}}{\left(e^{\frac{2bN}{3}} - 1\right)^2}$$

For $|b|N \ll 1$, these expressions reduce to those for the $R + R^2$ model.

Second type: terms with higher derivatives of R

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R + \alpha R^2 + \gamma R\Box R\right], \quad \alpha = \frac{1}{6M^2}$$

An inflationary regime in this model was first considered in S. Gottlöber, H.-J. Schmidt and A. A. Starobinsky, Class. Quant. Grav. **7**, 803 (1990). But this model, if taken in full, has a scalar ghost in addition to a physical massive scalar and the massless graviton.

Its recent re-consideration avoiding ghosts:
A. R. R. Castellanos, F. Sobreira, I. L. Shapiro and A. A. Starobinsky, JCAP **1812**, 007 (2018); arXiv:1810.07787.

The idea is to treat the $\gamma R\Box R$ term perturbatively with respect to the $R + R^2$ gravity, i.e., to consider only those solutions which reduce to the solutions of the $R + R^2$ gravity in the limit $\gamma - 0$. Then the second (ghost) scalar degree of freedom does not appear.

Results:

1. $|k| \lesssim 0.3$ where $k = \frac{\gamma}{6\alpha^2}$.

2. In the limit $kN \ll 1$, leading corrections $\propto kN$ to $n_s - 1$ and r vanish. The first result is in the agreement with that in a more general non-local gravity model without ghosts constructed in A. S. Koshelev, L. Modesto, L. Rachwal and A. A. Starobinsky, JHEP **1611**, 067 (2016); arXiv:1604.03127 which contains an infinite number of R derivatives.

Third type: terms arising from the conformal (trace) anomaly

The tensor producing the $\propto \left(R_{\mu\nu}R^{\mu\nu} - \frac{R^2}{3}\right)$ term in the trace anomaly:

$$T_\mu^\nu = \frac{k_2}{2880\pi^2}\left(R_\mu^\alpha R_\alpha^\nu - \frac{2}{3}RR_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{4}\delta_\mu^\nu R^2\right)$$

It is covariantly conserved in the isotropic case only! Can be generalized to the weakly anisotropic case by adding a term proportional to the first power of the Weyl tensor.

$$T_0^0 = \frac{3H^4}{\kappa^2 H_1^2}, \quad T = -\frac{1}{\kappa^2 H_1^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{R^2}{3}\right), \quad H_1^2 = \frac{2880\pi^2}{\kappa^2 k_2}$$

The spectrum of scalar and tensor perturbations in this case was calculated already in A. A. Starobinsky, Sov. Astron. Lett. **9**, 302 (1983).

$$n_s - 1 = -2\beta \frac{e^{\beta N}}{e^{\beta N} - 1}, \quad \beta = \frac{M^2}{3H_1^2}$$

If $n_s > 0.957$ and $N = 55$, then $H_1 > 7.2M$.

Generality of inflation

Some myths (or critics) regarding inflation and its onset:

1. In the Einstein frame, inflation begins with $V(\phi) \sim \dot{\phi}^2 \sim M_{Pl}^2$.
2. As a consequence, its formation is strongly suppressed in models with a plateau-type potentials in the Einstein frame (including $R + R^2$ inflation) favored by observations.
3. Beginning of inflation in some patch requires causal connection throughout the patch.
4. "De Sitter (both the exact and inflationary ones) has no hair".
5. One of weaknesses of inflation is that it does not solve the singularity problem, i.e. that its models admit generic anisotropic and inhomogeneous solutions with much higher curvature preceding inflation.

Theorem. In inflationary models in GR and $f(R)$ gravity, there exists an open set of classical solutions with a non-zero measure in the space of initial conditions at curvatures much exceeding those during inflation which have a metastable inflationary stage with a given number of e-folds.

For the GR inflationary model this follows from the generic late-time asymptotic solution for GR with a cosmological constant found in A. A. Starobinsky, JETP Lett. 37, 55 (1983). For the $R + R^2$ model, this was proved in A. A. Starobinsky and H.-J. Schmidt, Class. Quant. Grav. 4, 695 (1987). For the power-law and $f(R) = R^p, p < 2$, $2 - p \ll 1$ inflation – in V. Müller, H.-J. Schmidt and A. A. Starobinsky, Class. Quant. Grav. 7, 1163 (1990).

Generic late-time asymptote of classical solutions of GR with a cosmological constant Λ both without and with hydrodynamic matter (also called the Fefferman-Graham expansion):

$$ds^2 = dt^2 - \gamma_{ik} dx^i dx^k$$

$$\gamma_{ik} = e^{2H_0 t} a_{ik} + b_{ik} + e^{-H_0 t} c_{ik} + \dots$$

where $H_0^2 = \Lambda/3$ and the matrices a_{ik}, b_{ik}, c_{ik} are functions of spatial coordinates. a_{ik} contains two independent physical functions (after 3 spatial rotations and 1 shift in time + spatial dilatation) and can be made unimodular, in particular. b_{ik} is unambiguously defined through the 3-D Ricci tensor constructed from a_{ik}. c_{ik} contains a number of arbitrary physical functions (two - in the vacuum case, or with radiation) – tensor hair.

A similar but more complicated construction with an additional dependence of H_0 on spatial coordinates in the case of $f(R) = R^p$ inflation – scalar hair.

Consequences:

1. (Quasi-) de Sitter hair exist globally and are partially observable after the end of inflation.

2. The appearance of an inflating patch does not require that all parts of this patch should be causally connected at the beginning of inflation.

Similar property in the case of a generic curvature singularity formed at a spacelike hypersurface in GR and modified gravity. However, 'generic' does not mean 'omnipresent'.

What was before inflation?

Duration of inflation was finite inside our past light cone. In terms of e-folds, difference in its total duration in different points of space can be seen by the naked eye from a smoothed CMB temperature anisotropy map.

ΔN formalism: $\Delta\mathcal{R}(\mathbf{r}) = \Delta N_{tot}(\mathbf{r})$ where
$N_{tot} = \ln\left(\frac{a(t_{fin})}{a(t_{in})}\right) = N_{tot}(\mathbf{r})$ (AS, 1982,1985).

For $\ell \lesssim 50$, neglecting the Silk and Doppler effects, as well as the ISW effect due the presence of dark energy,

$$\frac{\Delta T(\theta,\phi)}{T_\gamma} = -\frac{1}{5}\Delta\mathcal{R}(r_{LSS},\theta,\phi) = -\frac{1}{5}\Delta N_{tot}(r_{LSS},\theta,\phi)$$

For $\frac{\Delta T}{T} \sim 10^{-5}$, $\Delta N \sim 5\times 10^{-5}$, and for $H \sim 10^{14}\,\mathrm{GeV}$, $\Delta t \sim 5 t_{Pl}$!

Different possibilities were considered historically:
1. Creation of inflation "from nothing" (Grishchuk and Zeldovich, 1981).
One possibility among infinite number of others.
2. De Sitter "Genesis": beginning from the exact contracting full de Sitter space-time at $t \to -\infty$ (AS, 1980).
Requires adding an additional term

$$R_i^l R_l^k - \frac{2}{3} R R_i^k - \frac{1}{2}\delta_i^k R_{lm} R^{lm} + \frac{1}{4}\delta_i^k R^2$$

to the rhs of the gravitational field equations. Not generic.
May not be the "ultimate" solution: a quantum system may not spend an infinite time in an unstable state.
3. Bounce due to a positive spatial curvature (AS, 1978).
Generic, but probability of a bounce is small for a large initial size of a universe $W \sim 1/Ma_0$.

Formation of inflation from generic curvature singularity

In classical gravity (GR or modified $f(R)$): space-like curvature singularity is generic. Generic initial conditions near a curvature singularity in modified gravity models (the $R+R^2$ and Higgs ones): anisotropic and inhomogeneous (though quasi-homogeneous locally).

Recent analytical and numerical investigation for $f(R)$ gravity in the Bianchi I type model in D. Muller, A. Ricciardone, A. A. Starobinsky and A. V. Toporensky, Eur. Phys. J. C **78**, 311 (2018). Two types of singularities in $f(R)$ gravity with the same structure at $t \to 0$:

$$ds^2 = dt^2 - \sum_{i=1}^{3} |t|^{2p_i}\, a_l^{(i)} a_m^{(i)}\, dx^l dx^m,\ 0 < s \le 3/2,\ u = s(2-s)$$

where $p_i < 1,\ s = \sum_i p_i,\ u = \sum_i p_i^2$ and $a_l^{(i)},\ p_i$ are functions of $\mathbf{r}$. Here $R^2 \ll R_{\alpha\beta}R^{\alpha\beta}$.

Bianchi I type models with inflation in R^2 gravity

Type A. $1 \le s \le 3/2,\ R \propto |t|^{1-s} \to +\infty$

Type B. $0 < s < 1,\ R \to R_0 < 0,\ f'(R_0) = 0$

For $f(R) = R^2$ even an exact solution can be found.

$$ds^2 = \tanh^{2\alpha}\left(\frac{3H_0 t}{2}\right)\left(dt^2 - \sum_{i=1}^{3} a_i^2(t)dx_i^2\right)$$

$$a_i(t) = \sinh^{1/3}(3H_0 t)\tanh^{\beta_i}\left(\frac{3H_0 t}{2}\right),\ \sum_i \beta_i = 0,\ \sum_i \beta_i^2 < \frac{2}{3}$$

$$\alpha^2 = \frac{\frac{2}{3} - \sum_i \beta_i^2}{6},\quad \alpha > 0$$

Next step: relate arbitrary functions of spatial coordinates in the generic solution near a curvature singularity to those in the quasi-de Sitter solution. Spatial gradients may become important for some period before the beginning of inflation.

The same structure of generic singulariry for a non-minimally coupled scalar field (scalar-tensor gravity)

$$S = \int \left(f(\phi)R + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi) \right) \sqrt{-g}\, d^4x + S_m$$

$$f(\phi) = \frac{1}{2\kappa^2} - \xi\phi^2$$

Type A. $\xi < 0, |\phi| \to \infty$
Type B. $\xi > 0, |\phi| \to 1/\sqrt{2\xi}\kappa$

The asymptotic regimes and a number of exact solutions in the Bianchi type I model are presented in A. Yu. Kamenshchik, E. O. Pozdeeva, A. A. Starobinsky, A. Tronconi, G. Venturi and S. Yu. Vernov, Phys. Rev. D **97**, 023536 (2018) with some of them borrowed from A. A. Starobinsky, MS Degree thesis, Moscow State University, 1971, unpublished.

What is sufficient for beginning of inflation in classical (modified) gravity, is:

1) the existence of a sufficiently large compact expanding region of space with the Riemann curvature much exceeding that during the end of inflation ($\sim M^2$) – realized near a curvature singularity;

2) the average value $< R >$ over this region positive and much exceeding $\sim M^2$, too, – type A singularity;

3) the average spatial curvature over the region is either negative, or not too positive.

On the other hand, causal connection is certainly needed to have a "graceful exit" from inflation, i.e. to have practically the same amount of the total number of e-folds during inflation N_{tot} in some sub-domain of this inflating patch.

Conclusions

- The typical inflationary predictions that $|n_s - 1|$ is small and of the order of N_H^{-1}, and that r does not exceed $\sim 8(1 - n_s)$ are confirmed. Typical consequences following without assuming additional small parameters: $H_{55} \sim 10^{14}\,\mathrm{GeV}, \; m_{infl} \sim 10^{13}\,\mathrm{GeV}$.
- In $f(R)$ gravity, the simplest $R + R^2$ model is one-parametric and has the preferred values $n_s - 1 = -\frac{2}{N}$ and $r = \frac{12}{N^2} = 3(n_s - 1)^2$. The first value produces the best fit to present observational CMB data.
- Inflation in $f(R)$ gravity represents a dynamical attractor for slow-rolling scalar fields strongly coupled to gravity.

- Comparison with observational data shows that logarithmic high-curvature quantum corrections to the $R + R^2$ model in the observable part of inflation are small, no more than a few percents. The same refers to higher-derivative and conformal anomaly corrections.
- Inflation is generic in the $R + R^2$ inflationary model and close ones. Thus, its beginning does not require causal connection of all parts of an inflating patch of space-time (similar to spacelike singularities). However, graceful exit from inflation requires approximately the same number of e-folds during it for a sufficiently large compact set of geodesics. To achieve this, causal connection inside this set is necessary (though still may appear insufficient).
- Inflation can form generically and with not a small probability from generic space-like curvature singularity.