Vertex algebras associated with hypertoric varieties

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Basic Strategy

Hypertoric varieties

$= $ Hamiltonian reduction of $T^*\mathbb{C}^N \times \mathbb{C}^M$

by $(\mathbb{C}^\times)^M$-action

$\rightsquigarrow$ chiralization

Hypertoric VOAs

$= (\frac{\infty}{2}$-$)$ BRST reduction of

$\beta\gamma$-systems $\otimes$ a Heis. VOA

by the action of a Heisenberg VA $V_0(\mathbb{C}^M)$.

Hypertoric varieties

Hypertoric varieties

$= $ Hamiltonian reduction of $T^*\mathbb{C}^N$ by torus $(\mathbb{C}^*)^M$.

$G = (\mathbb{C}^*)^M \circlearrowright V = \mathbb{C}^N$

$\rightsquigarrow G \circlearrowright T^*V$, $\mathbb{C}[T^*V]$, Hamiltonian action

Induced action of $\mathfrak{g} = \mathbb{C}^M$ is given by comoment map

$$\mu^*(A_i) = \sum_{k=1}^{N} \Delta_{ik} x_k y_k \qquad (i = 1, \ldots, M)$$

Assumption: Matrix $(\Delta_{ik})_{i,k}$ is unimodular.

Hypertoric varieties

$$X_0 = \mu^{-1}(0) /\!\!/ G$$
$$= \mathrm{Spec}(\mathbb{C}[T^*V]/\langle \mu^*(A_i) | i = 1, \ldots, M \rangle)^G$$

Hypertoric varieties (cont.)

Hypertoric varieties

$$\begin{aligned} X_0 &= \mu^{-1}(0)/\!\!/ G \\ &= \mathrm{Spec}(\mathbb{C}[T^*V]/\langle \mu^*(A_i) | i = 1, \dots, M\rangle)^G \\ X &= X_\delta = \mu^{-1}(0)/\!\!/_\delta G, \end{aligned}$$

GIT quotient wrt a stability param. δ

We have resolution of singularity $X \longrightarrow X_0$, and X is symplectic manifold. (symplectic resolution)
X_0 is known as conical symplectic singularity.
X is its symplectic resolution.

Hypertoric varieties (cont.)

Fact (Nagaoka, Losev)

(1) $^{\exists}\widetilde{X} \longrightarrow \mathfrak{g}^*$, *univ. family of symplectic (Poisson) deform. of* X.
(2) $^{\exists}\mathbb{W}$, *Namikawa-Weyl group (finite group)*
$c,\ c' \in \mathfrak{g}^*$, *if* $c' \in \mathbb{W}c$, $^{\exists}$ *isom.* $\mathbb{C}[(\widetilde{X})_c] \simeq \mathbb{C}[(\widetilde{X})_{c'}]$ *as Poisson alg, and*

$$\mathbb{C}[\mathfrak{g}^*]^{\mathbb{W}} \xrightarrow{\mu^*} \mathbb{C}[\widetilde{X}]^{\mathbb{W}} = (\mathbb{C}[T^*V]^G)^{\mathbb{W}}$$

is univ. family of Poisson deform. of $\mathbb{C}[X_0] = \mathbb{C}[X]$.

$\mathbb{C}[\mathfrak{g}^*]^{\mathbb{W}}$ is Poisson center of $\mathbb{C}[\widetilde{X}]^{\mathbb{W}}$.

Hypertoric VAs

We consider VA analog (chiralization) of the above construction.

$\mathbb{C}[T^*V] \rightsquigarrow \mathcal{D}^{ch}(V)$, $\beta\gamma$-system (CDO)

$$y_i(z)x_j(w) \sim \frac{\delta_{ij}}{z-w} \qquad (i,j=1,\ldots,N)$$

$\mathbb{C}[\mathfrak{g}^*] \rightsquigarrow V_{\langle\,,\,\rangle}(\mathfrak{g})$, Heisenberg VA

$$c_i(z)c_j(w) \sim \frac{\langle c_i, c_j\rangle}{(z-w)^2} \qquad (i,j=1,\ldots,M)$$

where $\langle c_i, c_j\rangle = \sum_{k=1}^{N} \Delta_{ik}\Delta_{jk}$, symm. bilinear form.

Hypertoric VAs (cont.)

Chiral comoment map

$\mu^* : \mathbb{C}[\mathfrak{g}^*] \longrightarrow \mathbb{C}[T^*V]$, comoment map

$$\mu^*(A_i) = \sum_{k=1}^{N} \Delta_{ik} x_k y_k \qquad (i = 1, \ldots, M)$$

$\rightsquigarrow \mu_{ch} : V_0(\mathfrak{g}) \longrightarrow \mathcal{D}^{ch}(V) \otimes V_{\langle\,,\,\rangle}(\mathfrak{g})$, VA hom.

$$\mu_{ch}(A_i(z)) = \sum_{k=1}^{N} \Delta_{ik} {}^\circ_\circ x_k(z) y_k(z) {}^\circ_\circ - c_i(z)$$

Since $V_{\langle\,,\,\rangle}(\mathfrak{g})$ has nontrivial OPEs, μ_{ch} is a hom. of (commutative) VAs.

$\frac{\infty}{2} -$ *reduction*

$C^{\frac{\infty}{2}+\bullet} = \mathcal{D}^{ch}(V) \otimes V_{\langle\, , \rangle}(\mathfrak{g}) \otimes Cl^{\bullet}(\bigoplus_{i=1}^{d} \mathbb{C}\psi_i^* \oplus \mathbb{C}\psi_i)$

where

$Cl^{\bullet}(\bigoplus_{i=1}^{d} \mathbb{C}\psi_i^* \oplus \mathbb{C}\psi_i)$: Clifford VA gen. by ψ_i^*, ψ_i

$\psi_i(z)\psi_j^*(w) \sim \delta_{ij}\frac{1}{z-w}$, $\psi_i^*(z)\psi_j^*(w) \sim \psi_i(z)\psi^j(w) \sim 0$

$Cl^{\bullet}$ and thus $C^{\frac{\infty}{2}+\bullet}$ is $\mathbb{Z}$-graded VA by the grading

$$\deg \psi_i^* = +1, \deg \psi_i = -1.$$

The odd element of degree 1

$Q(z) = \sum_{i=1}^{d} \mu_{ch}(A_i(z)) \otimes \psi_i^*(z) \in C^{\frac{\infty}{2}+1}$ satisfies

$(Q_{(0)})^2 = 0$, where $Q_{(0)} = \oint_{z=0} Q(z)dz$.

$\Longrightarrow$

$(C^{\frac{\infty}{2}+\bullet}, Q_{(0)})$ is a cochain complex.

$\mathrm{D}^{ch}(\widetilde{X}) = H^0(C^{\frac{\infty}{2}+\bullet}, Q_{(0)})$ and $\mathrm{D}^{ch}(\widetilde{X})^{\mathbb{W}}$, hypertoric VA

Fundamental Properties

Proposition

The cohomology $H^n(C^{\frac{\infty}{2}+\bullet}, Q_{(0)})$ vanishes unless $n \geq 0$.

Proposition

$\mathrm{D}^{ch}(\widetilde{X})$ (and $\mathrm{D}^{ch}(\widetilde{X})^{\mathbb{W}}$) is a VOA
(of central charge $-\frac{M+N}{2}$).

$\mathrm{D}^{ch}(\widetilde{X})$ is $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded wrt the conformal weight.

Remark

$^{\exists}$ other choice of conformal vector.

Fundamental Properties (cont.)

Theorem

$\mathrm{D}^{ch}(\widetilde{X})$ *is localized as a sheaf of* $\hbar$*-adic VAs over* $\widetilde{X}$*. Namely, we can construct a sheaf of* $\hbar$*-adic VAs over* $\widetilde{X}$ *by the same* $\frac{\infty}{2}$*-reduction, and* $\mathrm{D}^{ch}(\widetilde{X})$ *is the VA of its global sections "specialized at* $\hbar = 1$*."*

Corollary

The sheaf is locally isomorphic to $\mathcal{D}^{ch}(\mathbb{C}^{N-M}) \otimes V_{\langle\, , \rangle}(\mathfrak{g})$.
$\Longrightarrow$ free field (Wakimoto) realization (embedding).

Remark: The sheaf is microlocal analog of CDO [Gorbounov-Malikov-Schechtman], and was first introduced in [Arakawa-K-Malikov]

Zhu algebra

For a VOA V, we have an associative algebra called Zhu algebra,

$$A(V) = V/V \circ V, \qquad A \circ B = \sum_{j \geq 0} \binom{\deg A}{j} A_{(j-2)} B$$

with product $A * B = \sum_{j \geq 0} \binom{\deg A}{j} A_{(j-1)} B$.

Proposition

Zhu algebra of the hypertoric VOA $\mathrm{D}^{ch}(\widetilde{X})$ is a subalgebra of the universal family of (filtered) quantizations of the Poisson algebra $\mathbb{C}[X_0]$.

Analog to $\mathcal{W}$-algebras

The hypertoric VAs are analog of $\mathcal{W}$-algebras.

$\mathfrak{m}_f \subset \mathfrak{g}$, nilp. Lie subalgebra of simple Lie algebra $\mathfrak{g}$.

$V(\mathfrak{m}_f) \circlearrowright V^k(\mathfrak{g})$

$\rightsquigarrow$ By $\infty/2$-reduction ([Feigin-Frenkel], [Kac-Roan-Wakimoto])

$$\mathcal{W}^k(\mathfrak{g}, f), \quad \text{affine } \mathcal{W}\text{-algebra}$$

Zhu algebra = finite $\mathcal{W}$-algebra ([De Sole-Kac])

$\mathfrak{m}_f \circlearrowright U(\mathfrak{g})$

$\rightsquigarrow$ By quantum Hamiltonian reduction ([Premet])

$$U(\mathfrak{g}, f), \text{finite } \mathcal{W}\text{-algebra (algebra over } Z(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{h}]^{\mathbb{W}})$$

$U(\mathfrak{g}, f) \otimes_{Z(\mathfrak{g})} \mathbb{C}_\chi$ is a quantization of Slodowy variety $\mathbb{S} \cap \mathcal{N}$.

$U(\mathfrak{g}, f)$ is the univ. family of quantizations of $\mathbb{C}[\mathbb{S} \cap \mathcal{N}]$.

Example 1: $X = T^*\mathbb{P}^{N-1}$ $(X_0 = \overline{\mathbb{O}^{min}})$

$$G = \mathbb{C}^* \circlearrowright \mathbb{C}^1 \longrightarrow \mathbb{C}^N$$

$$T^*V = \mathrm{Hom}(\mathbb{C}^1, \mathbb{C}^N) \oplus \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^1) \simeq \mathbb{C}^{2N}$$

$$G = \mathbb{C}^*$$

G acts by the comoment map $\mu^*(A) = \sum_{i=1}^N x_i y_i$.

$\rightsquigarrow$ Hamilton red.

$$\begin{array}{ccc}
X = T^*\mathbb{P}^{N-1} & \longrightarrow & X_0 = \overline{\mathbb{O}^{min}} \subset \mathfrak{sl}(N) \\
\downarrow & & \downarrow \\
\widetilde{X} = (T^*\mathbb{P}^{N-1})^{tw} & \longrightarrow & \widetilde{X}_0 = \text{family of } \overline{\mathbb{O}} \subset \mathfrak{gl}(N)
\end{array}$$

$$\begin{array}{ccc}
\widetilde{X} & & \widetilde{X}_0 \\
& \searrow \quad \swarrow & \\
& \mathbb{C}^1 &
\end{array}$$

$$\mathbb{W} = \{1\}$$

Example 1: $X = T^*\mathbb{P}^{N-1}$ $(X_0 = \overline{\mathbb{O}^{min}})$

Chiralization

$$\mu_{ch} : V_0(\mathbb{C}^1) \longrightarrow \mathcal{D}^{ch}(\mathbb{C}^N) \otimes V_N(\mathbb{C}^1)$$

$$\mu_{ch}(A(z)) = \sum_{i=1}^{N} {}^{\circ}_{\circ} x_i(z) y_i(z) {}^{\circ}_{\circ} - c(z)$$

with $c(z)c(w) \sim N/(z-w)^2$.

$\rightsquigarrow$ By $\infty/2$-reduction, hypertoric VOA $\mathrm{D}^{ch}(\widetilde{X})$.

Remark

$\mathrm{D}^{ch}(\widetilde{X})$ is localized as a sheaf on $\widetilde{X} = (T^*\mathbb{P}^{N-1})^{tw}$.

Example 1: $X = T^*\mathbb{P}^{N-1}$ $(X_0 = \overline{\mathbb{O}^{min}})$

Generators of $\mathrm{D}^{ch}(\widetilde{X})$

$$E_{ij}(z) = x_i(z)y_j(z) \qquad (i \neq j)$$
$$H_i(z) = {}^\circ_\circ x_i(z)y_i(z)^\circ_\circ - {}^\circ_\circ x_{i+1}(z)y_{i+1}(z)^\circ_\circ \; (i = 1, \ldots, N-1)$$

Note: These elements commute with $\mu_{ch}(A(z))$.

$\Longrightarrow$

$$\begin{array}{ccc} V^{-1}(\mathfrak{sl}_N) & \twoheadrightarrow & \mathrm{D}^{ch}(\widetilde{X}),\ \text{hypertoric VOA} \\ \downarrow \text{assoc. variety} & & \downarrow \\ \mathfrak{sl}_N^* & \underset{\supset}{\longleftarrow} & \text{a Dixmier sheet, smaller} \end{array}$$

Example 1: $X = T^*\mathbb{P}^{N-1}$ ($X_0 = \overline{\mathbb{O}^{min}}$)

Proposition

(1) For $N \geq 4$, $\mathrm{D}^{ch}(\widetilde{X}) = \mathrm{D}^{ch}(\widetilde{X})^{\mathbb{W}}$ is a simple affine VOA $L_{-1}(\mathfrak{sl}_N)$ at level -1. Thus, $L_{-1}(\mathfrak{sl}_N)$ is localized as a sheaf on $(T^*\mathbb{P}^{N-1})^{tw}$.
(2) Zhu algebra $\simeq \mathcal{D}^{tw}(\mathbb{P}^{N-1})\backslash\mathbb{C}^* c^1$

(Proof.) The simplicity is most non-trivial.
When $N \geq 4$, by [Arakawa-Moreau, 2018].
$(E_{1,N,(-1)}E_{2,N-1} - E_{2,N,(-1)}E_{1,N-1} = 0$ in $\mathrm{D}^{ch}(\widetilde{X})$.)
(When $N = 3$, the same trick of [AM] with using geometry of $(T^*\mathbb{P}^2)^{tw}$.)

Example 1: $X = T^*\mathbb{P}^{N-1}$ ($X_0 = \overline{\mathbb{O}^{min}}$)

local coord. = free field (Wakimoto) realization:

$$E_{i,i+1}(z) = a^*_{i-1}(z)a_i(z) \qquad (i \geq 2)$$

$$E_{i+1,i}(z) = a^*_i(z)a_{i-1}(z)$$

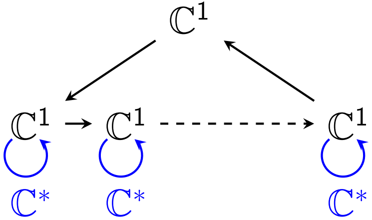
$$H_i(z) = {}^\circ_\circ a^*_{i-1}(z)a_{i-1}(z){}^\circ_\circ - {}^\circ_\circ a^*_i(z)a_i(z){}^\circ_\circ$$

$$E_{12}(z) = a_1(z)$$

$$H_1(z) = -2{}^\circ_\circ a^*_1(z)a_1(z){}^\circ_\circ - \sum_{i=2}^{N-1} {}^\circ_\circ a^*_i(z)a_i(z){}^\circ_\circ + b(z)$$

$$E_{21}(z) = -{}^\circ_\circ a^*_1(z)^2 a_1(z){}^\circ_\circ - \sum_{i=2}^{N-1} {}^\circ_\circ a^*_1(z)a^*_i(z)a_i(z){}^\circ_\circ + {}^\circ_\circ a^*_1(z)b(z){}^\circ_\circ - \partial a^*_1(z)$$

Example 2: $X_0 =$ Klein sing. of type A



affine Dynkin quiver
of type $A_{N-1}^{(1)}$

$$T^*V = \bigoplus_{i \to i+1} \mathrm{Hom}(\mathbb{C}^1, \mathbb{C}^1) \oplus \bigoplus_{i \leftarrow i+1} \mathrm{Hom}(\mathbb{C}^1, \mathbb{C}^1) \simeq \mathbb{C}^{2N}$$

$$G = \prod_{i=1}^{N-1} \mathbb{C}^* \circlearrowleft T^*V$$

Example 2: $X_0 =$ Klein sing. of type A

comoment map

$$\mu^*(A_i) = x_i y_i - x_{i+1} y_{i+1} \qquad (i = 1, \dots, N-1)$$

$\rightsquigarrow$ Hamilton red.

$$\begin{array}{ccc}
X = \text{minimal resolution} & \longrightarrow & X_0 = \mathbb{C}^2/(\mathbb{Z}/N\mathbb{Z}) \\
\big\downarrow & & \big\downarrow \\
\widetilde{X} & \longrightarrow & \widetilde{X_0} \\
& \searrow \quad \swarrow & \\
& \mathbb{C}^{N-1} &
\end{array}$$

$\mathbb{W} = \mathfrak{S}_{N-1}$

Example 2: $X_0 =$ Klein sing. of type A

Chiralization

$$\mu_{ch} : V_0(\mathbb{C}^{N-1}) \longrightarrow \mathcal{D}^{ch}(\mathbb{C}^N) \otimes V_{\langle\,,\,\rangle}(\mathbb{C}^{N-1})$$

$$\mu_{ch}(A_i(z)) = {:}x_i(z)y_i(z){:} - {:}x_{i+1}(z)y_{i+1}(z){:} - c_i(z)$$

$$c_i(z)c_j(w) \sim \frac{\langle c_i, c_j\rangle}{(z-w)^2}, \quad (\langle c_i, c_j\rangle)_{i,j} : \text{Cartan of type } A_{N-1}$$

$c_i(z)c_j(w) \sim N/(z-w)^2$.
$\rightsquigarrow$ By $\infty/2$-reduction, we obtain hypertoric VOA $\mathrm{D}^{ch}(\widetilde{X})$, and $\mathrm{D}^{ch}(\widetilde{X})^{\mathbb{W}}$.

Remark

$\mathrm{D}^{ch}(\widetilde{X})$ is localized as a sheaf on $\widetilde{X}$.

Example 2: $X_0 =$ Klein sing. of type A

Generators of $\mathrm{D}^{ch}(\widetilde{X})^{\mathbb{W}}$

$$G^+(z) = x_1(z)x_2(z)\dots x_N(z),$$
$$G^-(z) = y_1(z)y_2(z)\dots y_N(z),$$
$$J(z) = \frac{-1}{N}\sum_{i=1}^{N} {}^{\circ}_{\circ} x_i(z)y_i(z) {}^{\circ}_{\circ}$$
$$= -{}^{\circ}_{\circ} x_1(z)y_1(z) {}^{\circ}_{\circ} + \frac{N-1}{N}c_1(z) + \cdots + \frac{1}{N}c_{N-1}(z)$$

These elements commute with all $\mu_{ch}(A_i(z))$'s, and it satisfies the same OPEs as ones of $\mathcal{W}^{-N+1}(\mathfrak{sl}_N, f_{subreg})$.

Example 2: X_0 = Klein sing. of type A

Proposition

(1) $\mathcal{W}^{-N+1}(\mathfrak{sl}_N, f_{subreg}) \xrightarrow{\simeq} \mathrm{D}^{ch}(\widetilde{X})^{\mathbb{W}}$, $^{\exists}$ an isom of VOAs.
(2) Zhu algebra of $\mathrm{D}^{ch}(\widetilde{X})^{\mathbb{W}}$ is
$\simeq$ the finite $\mathcal{W}$-algebra $U(\mathfrak{sl}_N, f_{subreg})$.

Remark

Again, the description by local coord. gives a free field realization of $\mathrm{D}^{ch}(\widetilde{X})$ in $\mathcal{D}^{ch}(\mathbb{C}) \otimes V_{\langle\,,\,\rangle}(\mathbb{C}^{N-1})$.
(Essentially the same as one in [Feigin-Semikhatov])

Example 2: X_0 = Klein sing. of type A

The isomorphism $\mathcal{W}^{-N+1}(\mathfrak{sl}_N, f_{subreg}) \xrightarrow{\simeq} \mathrm{D}^{ch}(\widetilde{X})^{\mathbb{W}}$ is compatible with Miura transform:

$$\begin{array}{ccc}
\mathcal{W}^{-N+1}(\mathfrak{sl}_N, f_{subreg}) & \xrightarrow{\simeq} & \mathrm{D}^{ch}(\widetilde{X}),\ \text{hypertoric VOA} \\
\Big\downarrow \text{Miura} & & \Big\downarrow \mathrm{Res}^{\widetilde{X}}_{(T^*\mathbb{P}^1)^{tw}\times\mathbb{C}^{N-2}} \\
V^{-1}(\mathfrak{sl}_2)\otimes V_{\langle\,,\rangle}(\mathbb{C}^{N-2}) & \longrightarrow & \Gamma((T^*\mathbb{P}^1)^{tw}\times\mathbb{C}^{N-2}, \widetilde{D}^{ch}_{X,\hbar})
\end{array}$$

Proposition

(3) The above diagram commutes.

Example 3: $X_0 = \overline{\mathbb{O}^{min}}(\{\ell_1, 1^{N-1}\})$

$\overline{\mathbb{O}^{min}}(\{\ell_1, 1^{N-1}\})$: "generalized minimal nilpotent orbit closure" [Nagaoka]

Remark

It generalizes Example 1 and 2.
$\ell_1 = 1$: Example 1, $N = 2$: Example 2

Chiral comoment map

$$\mu_{ch}(A_i(z)) = {}^{\circ}_{\circ} x_{1i}(z) y_{1i}(z) {}^{\circ}_{\circ} - {}^{\circ}_{\circ} x_{1i+1}(z) y_{1i+1}(z) {}^{\circ}_{\circ} - c_i(z)$$

$$\mu_{ch}(A_0(z)) = \sum_{i=2}^{N} {}^{\circ}_{\circ} x_i(z) y_i(z) {}^{\circ}_{\circ} + {}^{\circ}_{\circ} x_{11}(z) y_{11}(z) {}^{\circ}_{\circ} - c_0(z)$$

Example 3: $X_0 = \overline{\mathbb{O}^{min}}(\{\ell_1, 1^{N-1}\})$

Generators of $\mathrm{D}^{ch}(\widetilde{X})^{\mathbb{W}}$

$$E_{1i}(z) = x_{11}(z)\dots x_{1\ell_1}(z)y_i(z)$$
$$E_{i1}(z) = x_i(z)y_{11}(z)\dots y_{1\ell_1}(z)$$
$$E_{ij}(z) = x_i(z)y_j(z)$$
$$H_1(z) = \sum_{k=1}^{\ell_1} {}^\circ_\circ x_{1k}(z)y_{1k}(z){}^\circ_\circ - {}^\circ_\circ x_2(z)y_2(z){}^\circ_\circ$$
$$H_i(z) = {}^\circ_\circ x_i(z)y_i(z){}^\circ_\circ - {}^\circ_\circ x_{i+1}(z)y_{i+1}(z){}^\circ_\circ$$

for $i \neq j \geq 2$. ↰ Combination of $\mathcal{W}^{-\ell_1}(\mathfrak{sl}_{\ell_1+1}, f_{subreg})$ and $L_{-1}(\mathfrak{sl}_N)$!?

Problems

1. Representation Theory of $D^{ch}(\widetilde{X})$
   construction of simple module etc.
2. Symplectic duality vs. "Koszul duality"
   $T^*\mathbb{P}^{N-1} \leftrightarrow \mathbb{C}^2/(\mathbb{Z}/N\mathbb{Z})$, symplectic dual
   $\rightsquigarrow L_{-1}(\mathfrak{sl}_N) \leftrightarrow \mathcal{W}^{-N+1}(\mathfrak{sl}_N, f_{subreg})$, duality?
3. "Level deformation" by the same trick as Chebotarov's transitive vertex algebroid with twisting by Courant algebroid.