Neutrino Masses and Modular Invariance

Prospects of Neutrino Physics

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## Precision Era for Neutrino Physics

| $r \equiv |\Delta m^2_{\text{sol}}/\Delta m^2_{\text{atm}}|$ | IO       | NO       | Stimulating time for models of neutrino masses and mixing angles. |
|----------------|---------|----------|---------------------------------------------------------------|
| 0.0301(8)      | 0.0299(8) | 2.7%     |                                                               |
| $\sin^2 \theta_{12}$ | 0.303(13)  | 0.304(13) | 4.3%                                                          |
| $\sin^2 \theta_{13}$ | 0.0218(8)  | 0.0214(8) | 3.7%                                                          |
| $\sin^2 \theta_{23}$ | 0.56(3)    | 0.55(3)   | 5.4%                                                          |
| $\delta/\pi$   | 1.52(14)  | 1.32(19)  | $\approx 10\%$                                               |

[F. Capozzi, E. Lisi, A. Marrone and A. Palazzo 1804.09678]

| $y_e(m_Z)$ | $2.794745(16) \times 10^{-6}$ |
| $y_\mu(m_Z)$ | $5.899863(19) \times 10^{-4}$ |
| $y_\tau(m_Z)$ | $1.002950(91) \times 10^{-2}$ |

[Antusch and Maurer 1306.6879]
Flavour Symmetry approach [→ Lisa Everett’s talk]

One of the few tools we have, but with several obstacles

high number of free parameters

\[ m_{ij}(\tau) = m_{ij}^0 + m_{ij}^{1\alpha} \tau^\alpha + m_{ij}^{2\alpha\beta} \tau^\alpha \tau^\beta + \ldots \]

lowest order Lagrangian parameters

higher dimensional operators

vacuum alignment in SB sector

SUSY breaking effects RGE corrections \((\Lambda_{UV}, m_{SUSY}, \tan\beta)\)
This proposal [F.F. 1706.08749]

a) neutrino masses and mixings depend on a small number of fields
    [ideally a single complex field τ]

b) dependence of $m_{ij}$ on τ is holomorphic

c) flavour symmetry acts non-linearly
    [to determine all higher dimensional operators]

a) + b) + c) the functional form of $m_{ij}(τ)$ is completely determined

d) the VEV $τ$ is selected by some unknown mechanism
    [anarchy in vacuum selection]

Here: a) + b) + c) from modular invariance as flavour symmetry
Modular Invariance as Flavour Symmetry

String theory in $d=10$ need 6 compact dimensions

Simplest compactification: 3 copies of a torus $T^2$

Lattice left invariant by modular transformations:

$$\tau \rightarrow \gamma \tau \equiv \frac{a\tau + b}{c\tau + d}$$

$a,b,c,d$ integers $ad-bc=1$

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(g(\tau), \varphi)$$

Completely characterized by:

$$\tau = \frac{\omega_2}{\omega_1}, \quad \text{Im}(\tau) > 0$$

[figures from Tatsuishi's talk at Flasy 2018]
they form the (discrete, infinite) modular group $\Gamma$ generated by

\[ S : \tau \rightarrow -\frac{1}{\tau}, \quad T : \tau \rightarrow \tau + 1 \]

- duality
- discrete shift symmetry

\[ S^2 = 1, \quad (ST)^3 = 1 \]

- can be thought of as a gauge symmetry
- with a "gauge choice" $\tau$ can be restricted to a fundamental region

most general transformation on a set of $\mathcal{N}=1$ SUSY chiral multiplets $\varphi^{(I)}$

\[
\begin{align*}
\gamma & \rightarrow \gamma' = a\gamma + b \\
\varphi^{(I)} & \rightarrow (c\gamma + d)kI\rho^{(I)}(\gamma)\varphi^{(I)}
\end{align*}
\]

\[
\Gamma_N \equiv \Gamma_{\mathcal{N}} \equiv \Gamma / \Gamma(\mathcal{N})
\]

$N = 1, 2, 3, \ldots$

\[ \varphi^{(I)} = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} \]

[Ferrara, Lust, Shapere and Theisen, 1989]
N=1 SUSY modular invariant theories

Yukawa interactions in N=1 global SUSY

\[ S = \int d^4 x d^2 \theta \, w(\tau, \varphi) + h.c + \text{kinetic terms} \]

\[ w(\tau, \varphi) = \sum_{n} Y_{I_1 \ldots I_n}(\tau) \varphi^{(I_1)} \ldots \varphi^{(I_n)} \]

invariance of \( w(\Phi) \) guaranteed by an holomorphic \( Y_{I_1 \ldots I_n}(\tau) \) such that

\[ Y_{I_1 \ldots I_n}(\gamma \tau) = (c \tau + d)^{k_Y(n)} \rho(\gamma) Y_{I_1 \ldots I_n}(\tau) \]

1. \( k_Y(n) + k_{I_1} + \ldots + k_{I_n} = 0 \)
2. \( \rho \times \rho^{I_1} \times \ldots \times \rho^{I_n} \supset 1 \)

modular forms of level N and weight \( k_Y \)

form a linear space \( \mathcal{M}_k(\Gamma_N) \)

of finite dimension
$$\Gamma_3 \approx A_4$$

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \rightarrow (c\tau + d)^{-1} \rho(y) \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

$$k_\nu = -1$$

$$w(\tau, \nu) = m_0 \nu Y(\tau) \nu + h. c.$$  

modular form of level 3 
$$k = +2$$ and $$\rho \subset 3 + 1 + 1' + 1''$$

$$d(M_2(\Gamma_3)) = 3$$

$$\rho = 3$$

$$\begin{bmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{bmatrix}$$

mass matrix completely determined in terms of $$\tau$$ up to an overall constant

no corrections from higher order operators in the exact SUSY limit

[F.F. 1706.08749]
**Modular Forms**

$k > 0$ even integer

<table>
<thead>
<tr>
<th>$\Gamma_2 \approx S_3$</th>
<th>$\Gamma_3 \approx A_4$</th>
<th>$\Gamma_4 \approx S_4$</th>
<th>$\Gamma_5 \approx A_5$</th>
<th>$\Gamma_8 \supset \Delta(96)$</th>
<th>$\Gamma_{16} \supset \Delta(384)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k/2 + 1$</td>
<td>$k + 1$</td>
<td>$2k + 1$</td>
<td>$5k + 1$</td>
<td>$k = 2$</td>
<td>$k = 2$</td>
</tr>
<tr>
<td>$2$</td>
<td>$3$</td>
<td>$2 + 3'$</td>
<td>$3 + 3'$</td>
<td>$\rho = 3$</td>
<td>$\rho = 3$</td>
</tr>
<tr>
<td>$1 + 2$</td>
<td>$1 + 1' + 3$</td>
<td>$1 + 2 + 3 + 3'$</td>
<td>$1 + 3 + 3' + 4 + 5 + 5$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Built in terms of
- Dedekind eta function
- Klein forms
- Jacobi theta functions

**References**

[TTT = T. Kobayashi, K. Tanaka and T. H. Tatsuishi, 1803.10391]
[F = F. Feruglio 1706.08749]
[PP = J. T. Penedo and S. T. Petcov 1806.11040]
[NPPT = P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov 1812.02158]
[DKL = G. J. Ding, S. F. King and X. G. Liu 1903.12588]
[KT = T. Kobayashi and S. Tamba, 1811.11384]
Selection of results

- freedom in a bottom-up approach: \( \Gamma_N, \rho^{(I)}, k_I \)
- Majorana neutrinos
- weight \( k = 2 \) in \( Y(\tau) \) of neutrino sector

number of free parameters \( p \):

\[ p = \text{Total} - 3 \text{ (charged fermion masses)} \]

number of observables in neutrino sector = 9
(3 angles + 3 masses + 3 phases)

\( (9-p) \) predictions

\[ p \geq 3 \text{ (always includes } Re(\tau), Im(\tau) \text{ and one overall scale)} \]
Here \( p > 5 \) not considered

- charged lepton sector either diagonal or \( y_e(\tau) \)

<table>
<thead>
<tr>
<th>$\Gamma_3$</th>
<th>$\gamma_e(\tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Weinberg</strong></td>
<td>$p = 3$ not viable</td>
</tr>
<tr>
<td><strong>Seesaw</strong></td>
<td>$p = 5$ NO</td>
</tr>
<tr>
<td></td>
<td>$m_1 = 40$ meV</td>
</tr>
<tr>
<td></td>
<td>$m_2 = 40$ meV</td>
</tr>
<tr>
<td></td>
<td>$m_3 = 60$ meV</td>
</tr>
<tr>
<td></td>
<td>$</td>
</tr>
<tr>
<td></td>
<td>$\delta_{CP} = \pm (0.3 \div 1)\pi$</td>
</tr>
</tbody>
</table>

$\gamma_e$ diagonal not viable

2 extra parameters from Dirac neutrino mass

\[ \begin{array}{|c|c|c|}
\hline
\Gamma_4 & \mathcal{Y}_e(\tau) \\
\hline
\text{Weinberg} & p = 3 & \text{not viable} \\
\hline
\text{Seesaw} & p = 5 & \star \text{ NO} \\
& m_1 & 17 \text{ meV} & 21 \text{ meV} & 18 \text{ meV} \\
& m_2 & 19 \text{ meV} & 22 \text{ meV} & 20 \text{ meV} \\
& m_3 & 53 \text{ meV} & 54 \text{ meV} & 53 \text{ meV} \\
& |m_{ee}| & 17 \text{ meV} & 20 \text{ meV} & 7 \text{ meV} \\
& \delta_{CP} & \pm 1.3 \pi & \pm 1.9 \pi & \pm 1.6 \pi \\
\hline
\end{array} \]

\[ \text{\textit{\small{\begin{center}[P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, 1811.04933]}}}} \]
<table>
<thead>
<tr>
<th>$\Gamma_5$</th>
<th>$\nu_e$ diagonal</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Weinberg</strong></td>
<td>$p = 3$ not viable</td>
</tr>
<tr>
<td><strong>Seesaw</strong></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$p = 5$</th>
<th>NO</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>83 meV</td>
</tr>
<tr>
<td>$m_2$</td>
<td>83 meV</td>
</tr>
<tr>
<td>$m_3$</td>
<td>97 meV</td>
</tr>
<tr>
<td>$</td>
<td>m_{ee}</td>
</tr>
<tr>
<td>$\delta_{CP}$</td>
<td>$\pm 1.7 \pi$</td>
</tr>
</tbody>
</table>

no known solutions with $p \leq 5$ and $\nu_e(\tau)$

2 extra parameters from Dirac neutrino mass

more solutions with IO

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[P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov 1812.02158
G. J. Ding, S. F. King and X. G. Liu 1903.12588]
A different approach [Juan Carlos Criado, F.F., 1807.01125]

charged lepton sector depending on additional flavon \( \varphi = (1,0,\varphi_3) \quad \varphi_3 \ll 1 \) real

all dimensionless neutrino data are determined in terms of 3 vacuum parameters \( \text{Re}(\tau), \text{Im}(\tau), \varphi_3 \)

\[
p = 4
\]

| \( \tau \) | \(-0.2005 + i 1.0578\) |
| \( \varphi_3 \) | \(0.117\) |

Seesaw \(-\text{NO}\)

\[
\Delta \chi^2
\]

\[
\text{1st octant}
\]

\[
m_1 = 1.09(3) \times 10^{-2} \text{eV} \quad , \quad m_2 = 1.39(2) \times 10^{-2} \text{eV} \quad , \quad m_3 = 5.11(4) \times 10^{-2} \text{eV}
\]

\[
|m_{ee}| = 1.04(2) \times 10^{-2} \text{eV}
\]

[see caveats in Lisi’s talk]
most of the solutions with NO prefer a nearly degenerate spectrum $m_1 > 10 \text{ meV}$ and $|m_{ee}|$ on the high side of allowed range

Some trend

[0νββ region from: S. Dell’Oro, S. Marcocci and F. Vissani, 1404.2616]
most of the solutions with NO prefer a nearly degenerate spectrum $m_1 > 10 \text{ meV}$ and $|m_{ee}|$ on the high side of allowed range

higher N $\rightarrow$ more solutions

any preferred value of $\tau$? 

extrema of $V(\tau)$ at the border of the fundamental region and along the $Im(\tau)$ axis?

$\tau$ at points with residual symmetries?

$\tau = i \ Z_2(S)$

$\tau = -\frac{1}{2} + \frac{i\sqrt{3}}{2} \ Z_3(ST)$

Some trend

$\Gamma_4$ solutions


[P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, 1811.04933 and 1812.02158]
Corrections from SUSY breaking

unknown breaking mechanism. Here:

F-component of a chiral supermultiplet, gauge and modular invariant

\[ X = \vartheta^2 F \]

messenger scale \( M \)

SUSY-breaking scale

\[ m_{\text{SUSY}} = \frac{F}{M} \]

most general correction term to lepton masses and mixing angles

\[ \delta S = \frac{1}{M^2} \int d^4 x d^2 \theta d^2 \bar{\theta} \ X^\dagger \ f(\Phi, \bar{\Phi}) + h.c. \]

\( f(\Phi, \bar{\Phi}) \) has dimension 3, determined by gauge invariance and lepton number conservation (treating \( \Lambda \) as spurion with \( L=+2 \))

\[ \frac{\delta W}{W} \approx \frac{\delta y}{y} \approx \frac{m_{\text{SUSY}}}{M} \]

tiny, if sufficient gap between \( m_{\text{SUSY}} \) and \( M \)

\[ 10^{-10} \text{ for } \frac{m_{\text{SUSY}}}{M} = \frac{10^8 \text{ GeV}}{10^{18} \text{ GeV}} \]
Conclusions

neutrino data set a new standard in model building
- accurate predictions
- falsifiable models

modular invariance as flavour symmetry can determine the functional dependence of Yukawa couplings on a modulus field

\[ Y(\tau) \]
- no/less flavons,
- less parameters
- no corrections from higher dimensional operators
- stability against SUSY breaking corrections

can be implemented in a bottom-up approach:
absolute masses, \( m_{ee} \) and phases are predicted

Open questions:
- role of modular symmetry in quark Yukawa couplings
- in GUTs
- vacuum selection
- corrections from Kahler potential

[Kobayashi, Shimizu, Takagi, Tanimoto, Tatsuishi and Uchida, 1812.11072
Okada and Tanimoto, 1812.09677]
[de Anda, King and Perdomo, 1812.05620]
Backup Slides
Corrections from RGE

Model 1 (IO)
- $r$ and $\sin^2 \theta_{12}$ mostly affected, at large $\tan \beta$

$\Lambda = 10^{15}$ GeV

<table>
<thead>
<tr>
<th>$m_{\text{SUSY}}$</th>
<th>Quantity</th>
<th>$\tan \beta = 2.5$</th>
<th>$\tan \beta = 10$</th>
<th>$\tan \beta = 15$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^4$ GeV</td>
<td>$r$</td>
<td>0.0302</td>
<td>0.0292</td>
<td>0.0288</td>
</tr>
<tr>
<td></td>
<td>$\sin^2 \theta_{12}$</td>
<td>0.304</td>
<td>0.345</td>
<td>0.418</td>
</tr>
<tr>
<td></td>
<td>$\chi^2_{\text{min}}$</td>
<td>0.4</td>
<td>12.2</td>
<td>82.0</td>
</tr>
<tr>
<td>$10^8$ GeV</td>
<td>$r$</td>
<td>0.0302</td>
<td>0.0294</td>
<td>0.0286</td>
</tr>
<tr>
<td></td>
<td>$\sin^2 \theta_{12}$</td>
<td>0.303</td>
<td>0.335</td>
<td>0.389</td>
</tr>
<tr>
<td></td>
<td>$\chi^2_{\text{min}}$</td>
<td>0.4</td>
<td>7.0</td>
<td>47.7</td>
</tr>
</tbody>
</table>

Model 2 (NO)
- negligible corrections for $\tan \beta$ up to 25 and $m_{\text{SUSY}}$ as low as $10^4$ GeV
\( \mathcal{N}=1 \) SUSY modular invariant theories

Known since late 1980s


Focus on Yukawa interactions and \( \mathcal{N}=1 \) global SUSY

\[
S = \int d^4x d^2\theta d^2\bar{\theta} \ K(\Phi, \bar{\Phi}) + \int d^4x d^2\theta \ w(\Phi) + h.c.
\]

\( \Phi = (\tau, \varphi) \)

Kahler potential, kinetic terms

Superpotential, holomorphic function of \( \Phi \)

Yukawa interactions

\[
\begin{align*}
    w(\Phi) & \rightarrow w(\Phi) \\
    K(\Phi, \bar{\Phi}) & \rightarrow K(\Phi, \bar{\Phi}) + f(\Phi) + f(\bar{\Phi})
\end{align*}
\]

S invariant if invariance of the Kahler potential easy to achieve. For example:

\[
K(\Phi, \bar{\Phi}) = -h \log(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\varphi^{(I)}|^2
\]

Minimal K

Extension to \( \mathcal{N}=1 \) SUGRA straightforward: ask invariance of \( G = K + \log|w|^2 \)
Action of modular invariance on flavor space

most general transformation on a set of $\mathcal{N}=1$ SUSY chiral multiplets $\varphi^{(I)}$

\[ \begin{align*}
\tau & \rightarrow \frac{a\tau + b}{c\tau + d} \\
\varphi^{(I)} & \rightarrow (c\tau + d)^{-k_I} \rho^{(I)}(\gamma)\varphi^{(I)}
\end{align*} \]

the weight, a real number

unitary representation of the finite modular group

\[ \Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N) \]

$\Gamma_N$ are finite groups

if all $k_I=0$, the construction collapses to the well-known models based on linear, unitary flavor symmetries.
invariance of the superpotential much less trivial. Expand \( w(\Phi) \) in powers of the matter supermultiplets

\[
w(\Phi) = \sum_n Y_{I_1...I_n}(\tau) \phi^{(I_1)}...\phi^{(I_n)}
\]

invariance of \( w(\Phi) \) guaranteed by an holomorphic \( Y \) such that

\[
Y_{I_1...I_n}(\gamma \tau) = (c \tau + d)^{k_Y(n)} \rho(\gamma) Y_{I_1...I_n}(\tau)
\]

\[
k_Y(n) = k_{I_1} + ... + k_{I_n}
\]

The product \( \rho \times \rho^{I_1} \times ... \times \rho^{I_n} \) contains an invariant singlet
**Few facts about (level-N) Modular Forms**

- **Transformation property under the modular group**
  \[ f_i(\gamma \tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau) \]

- **Unitary representation of the finite modular group**
  \[ \Gamma_N \equiv \overline{\Gamma}/\Gamma(N) \]

- **q-expansion**
  \[ f(\tau + N) = f(\tau) \]
  \[ f(\tau) = \sum_{n=0}^{\infty} a_n q_n \]
  \[ q_N = e^{\frac{i2\pi\tau}{N}} \]

- **k < 0**
  \[ f(\tau) = 0 \]

- **k = 0**
  \[ f(\tau) = \text{constant} \]

- **k > 0 (even integer)**
  \[ f(\tau) \in \mathcal{M}_k(\Gamma(N)) \]

- **Ring of modular forms generated by few elements**
  \[ \mathcal{M}(\Gamma(N)) = \bigoplus_{k=0}^{\infty} \mathcal{M}_{2k}(\Gamma(N)) \]

- **Finite-dimensional linear space**

- **An explicit example in a moment**
Table 1: Some properties of modular forms: $g$ is the genus of the space $\mathcal{H}/\Gamma(N)$ after compactification, $d_{2k}(\Gamma(N))$ the dimension of the linear space $\mathcal{M}_{2k}(\Gamma(N))$, $\mu_N$ is the dimension of the quotient group $\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$, which, for $N \leq 5$, is isomorphic to a permutation group.
Modular forms of level 3 \([1706.08749]\)

3 linearly independent modular forms of level 3 and minimal weight \(k_1 = 2\) can be expressed in terms of the Dedekind eta function

\[
\eta(\tau) = \exp\left(\frac{\pi i}{12}\right) q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q \equiv e^{2\pi i \tau}
\]

\[
Y_1(\tau) = \frac{i}{2\pi} \left[ \frac{\eta'(\frac{\tau}{3})}{\eta(\frac{\tau}{3})} + \frac{\eta'(\frac{\tau+1}{3})}{\eta(\frac{\tau+1}{3})} + \frac{\eta'(\frac{\tau+2}{3})}{\eta(\frac{\tau+2}{3})} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right]
\]

\[
Y_2(\tau) = \frac{-i}{\pi} \left[ \frac{\eta'(\frac{\tau}{3})}{\eta(\frac{\tau}{3})} + \omega^2 \frac{\eta'(\frac{\tau+1}{3})}{\eta(\frac{\tau+1}{3})} + \omega \frac{\eta'(\frac{\tau+2}{3})}{\eta(\frac{\tau+2}{3})} \right]
\]

\[
Y_3(\tau) = \frac{-i}{\pi} \left[ \frac{\eta'(\frac{\tau}{3})}{\eta(\frac{\tau}{3})} + \omega \frac{\eta'(\frac{\tau+1}{3})}{\eta(\frac{\tau+1}{3})} + \omega^2 \frac{\eta'(\frac{\tau+2}{3})}{\eta(\frac{\tau+2}{3})} \right]
\]

they transform in a triplet 3 of \(\Gamma_3\)

\[
Y(-1/\tau) = \tau^2 \rho(S)Y(\tau)
\]

\[
Y(\tau + 1) = \rho(T)Y(\tau)
\]

\[
\rho(S) = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}
\]

\[
\rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}
\]

\[
\omega = \frac{-1}{2} + \frac{\sqrt{3}}{2} i
\]

any modular form of level 3 and weight \(2k\) can be written as an homogeneous polynomial in \(Y_i\) of degree \(k\)

dimension of linear space \(\mathcal{M}_k(\Gamma(3))\) is \((k+1)\), \(k > 0\) even integer
Figure 1: Fundamental domain for $\Gamma(3)$. 
Q-expansion

\[ Y_1(\tau) = 1 + 12q + 36q^2 + 12q^3 + \ldots \]
\[ Y_2(\tau) = -6q^{1/3}(1 + 7q + 8q^2 + \ldots) \]
\[ Y_3(\tau) = -18q^{2/3}(1 + 2q + 5q^2 + \ldots) \]

\[ Y_2^2 + 2Y_1Y_3 = 0 \]

some VEVs

\[(Y_1, Y_2, Y_3)\big|_{\tau = i\infty} = (1, 0, 0)\]
\[(Y_1, Y_2, Y_3)\big|_{\tau = i} = Y_1(i)(1, 1 - \sqrt{3}, -2 + \sqrt{3})\]
\[Y(-1/\tau)\big|_{\tau = i} = -\rho(S)Y(\tau)\big|_{\tau = i}\]
As discussed explicitly in Appendix D, the constraint \( (30) \) is essential to recover the correct dimension of the linear space \( \mathcal{M}_{2k}(\Gamma(3)) \). On the one side from table 1 we see that this space has dimension \( 2k + 1 \). On the other hand the number of independent homogeneous polynomial \( Y_1 Y_2 \cdots Y_k \) of degree \( k \) that we can form with \( Y_i \) is \( (k + 1)(k + 2)/2 \). These polynomials are modular forms of weight \( 2k \) and, to match the correct dimension, \( k(k-1)/2 \) among them should vanish. Indeed this happens as a consequence of eq. \( (30) \). Therefore the ring \( \mathcal{M}(\Gamma(3)) \) is generated by the modular forms \( Y_i(\tau) \) \((i = 1, 2, 3)\).
Models 1 and 2 are based on $\Gamma_3$

Why $\Gamma_3$? $\Gamma_3$ is isomorphic to $A_4$, smallest group of the $\Gamma_N$ series possessing a 3-dimensional irreducible representation.

[recent extensions to $\Gamma_2$ and $\Gamma_4$ in Kobayashi, Tanaka, Tatsuishi, 1803.10391; Penedo, Petcov 1806.11040]

### Table 1: Chiral supermultiplets, transformation properties and weights. Model 1 has no gauge singlets $N^c$.

<table>
<thead>
<tr>
<th></th>
<th>$(E^c_1, E^c_2, E^c_3)$</th>
<th>$N^c$</th>
<th>$L$</th>
<th>$H_d$</th>
<th>$H_u$</th>
<th>$\varphi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(2)_L \times U(1)_Y$</td>
<td>(1, +1)</td>
<td>(1, 0)</td>
<td>(2, $-1/2$)</td>
<td>(2, $-1/2$)</td>
<td>(2, $+1/2$)</td>
<td>(1, 0)</td>
</tr>
<tr>
<td>$\Gamma_3 \equiv A_4$</td>
<td>(1, $1''$, 1')</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$k_I$</td>
<td>$(k_{E_1}, k_{E_2}, k_{E_3})$</td>
<td>$k_N$</td>
<td>$k_L$</td>
<td>$k_d$</td>
<td>$k_u$</td>
<td>$k_{\varphi}$</td>
</tr>
</tbody>
</table>

### Table 2: Weights of chiral multiplets. Model 1 has no gauge singlets $N^c$.

<table>
<thead>
<tr>
<th>$k_{E_i}$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Model 1</td>
<td>−2</td>
<td>−1</td>
<td>0</td>
<td>0</td>
<td>+3</td>
</tr>
<tr>
<td>Model 2</td>
<td>−4</td>
<td>−1</td>
<td>+1</td>
<td>0</td>
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</tr>
</tbody>
</table>

Modular invariance broken by

$\tau$

$\varphi = (1, 0, \varphi_3)$
if we go minimal

we get

by scanning $\tau$ VEVs the best agreement is obtained for

<table>
<thead>
<tr>
<th>$\Delta m^2_{sol}$</th>
<th>$\Delta m^2_{atm}$</th>
<th>$\sin^2 \theta_{12}$</th>
<th>$\sin^2 \theta_{13}$</th>
<th>$\sin^2 \theta_{23}$</th>
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<th>$\frac{\alpha_{21}}{\pi}$</th>
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<tbody>
<tr>
<td>Exp</td>
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<td>0.297</td>
<td>0.0215</td>
<td>0.5</td>
<td>1.4</td>
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<td>0.2</td>
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</tr>
<tr>
<td>prediction</td>
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<td>0.0447</td>
<td>0.651</td>
<td>1.55</td>
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</tr>
</tbody>
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2-parameter fit to 5 physical quantities

many $\sigma$ away

8 dimensionless physical quantities independent on any coupling constant!

if we go minimal

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2-parameter fit to 5 physical quantities

many $\sigma$ away

8 dimensionless physical quantities independent on any coupling constant!
neutrino masses from see-saw mechanism

\[ w_\nu = g \left( N_c H_u L \right)_1 + \Lambda \left( N_c N_c Y \right)_1 \]

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & L & N^c & H_u & Y \\
\hline
SU(2) \times U(1) & (2,-1/2) & (1,0) & (2,+1/2) & (1,0) \\
\hline
\Gamma_3 \equiv A_4 & 3 & 3 & 1 & 3 \\
\hline
k_l & k_L & +1 & k_u & +2 \\
\hline
\end{array}
\]

we get the best agreement at \[ \tau = -0.195 + 1.0636i \]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
 & \Delta m_{sol}^2 & \sin^2 \vartheta_{12} & \sin^2 \vartheta_{13} & \sin^2 \vartheta_{23} & \frac{\delta_{CP}}{\pi} & \frac{\alpha_{21}}{\pi} & \frac{\alpha_{31}}{\pi} \\
\hline
Exp & 0.0292 & 0.297 & 0.0215 & 0.5 & 1.4 & - & - \\
\hline
1\sigma & 0.0008 & 0.017 & 0.0007 & 0.1 & 0.2 & - & - \\
\hline
prediction & 0.0280 & 0.291 & 0.0486 & 0.331 & 1.47 & 1.83 & 1.26 \\
\hline
\end{array}
\]

Normal mass ordering is predicted

\[ m_1 = 1.096 \times 10^{-2} \text{ eV} \quad m_2 = 1.387 \times 10^{-2} \text{ eV} \quad m_3 = 5.231 \times 10^{-2} \text{ eV} \]
Status of neutrino oscillations 2018: 3

3σ hint for normal mass ordering and improved CP sensitivity

Charged Lepton Sector

\[ \mathcal{V}_e = \begin{pmatrix}
  a \varphi_1 & a \varphi_3 & a \varphi_2 \\
  b \varphi_2 & b \varphi_1 & b \varphi_3 \\
  c \varphi_3 & c \varphi_2 & c \varphi_1 \\
\end{pmatrix} \]

\[ U_e = \begin{pmatrix}
  1 & \varphi_3 & 0 \\
  0 & -\varphi_3 & 1 \\
 -\varphi_3 & 1 & \varphi_3 \\
\end{pmatrix} + \ldots \]

where dots stand for terms of order \( \varphi_3^2, (m_e^2/m_\mu^2)\varphi_3 \) and \( (m_\mu^2/m_\tau^2)\varphi_3 \).
[from pdg 2017]
Fit to Model 1

| $r \equiv |\Delta m_{sol}^2 / \Delta m_{atm}^2|$ | best value | pull |
|---|---|---|
| 0.0302(11) | +0.13 |

| $m_3/m_2$ | 0.0150(5) | $-$ |
| $\sin^2 \theta_{12}$ | 0.304(17) | +0.08 |
| $\sin^2 \theta_{13}$ | 0.0217(8) | $-$0.13 |
| $\sin^2 \theta_{23}$ | 0.577(4) | +0.67 |
| $\delta/\pi$ | 1.529(3) | +0.07 |
| $\alpha_{21}/\pi$ | 0.135(6) | $-$ |
| $\alpha_{31}/\pi$ | 1.728(18) | $-$ |

Inverted mass Ordering

- no SUSY breaking effects
- no RGE corrections

best fit parameters

| $\tau$ | $0.0117 + i 0.9948$ |
| $\varphi_3$ | $-0.086$ |

$\chi^2_{min} = 0.4$

8 dimensionless physical quantities independent on any coupling constant!

$m_1 = 4.90(3) \times 10^{-2}\text{eV}$, $m_2 = 4.98(2) \times 10^{-2}\text{eV}$, $m_3 = 7.5(3) \times 10^{-4}\text{eV}$

$|m_{ee}| = 4.73(4) \times 10^{-2}\text{eV}$

by reproducing individually $\Delta m_{sol}^2$ and $\Delta m_{atm}^2$
Fit to Yukawa couplings

### Model 1

<table>
<thead>
<tr>
<th>Term</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \cos \beta$</td>
<td>$2.806923 \times 10^{-6}$</td>
</tr>
<tr>
<td>$b \cos \beta$</td>
<td>$9.992488 \times 10^{-3}$</td>
</tr>
<tr>
<td>$c \cos \beta$</td>
<td>$5.899778 \times 10^{-4}$</td>
</tr>
</tbody>
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<tr>
<td>$y_e(m_Z)$</td>
<td>$2.794745 \times 10^{-6}$</td>
</tr>
<tr>
<td>$y_{\mu}(m_Z)$</td>
<td>$5.899864 \times 10^{-4}$</td>
</tr>
<tr>
<td>$y_\tau(m_Z)$</td>
<td>$1.002950 \times 10^{-2}$</td>
</tr>
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### Model 2

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1σ parameter space

Intervals where $\chi^2 \leq \chi^2_{\text{min}} + 1$:

<table>
<thead>
<tr>
<th></th>
<th>IO</th>
<th>NO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re(τ)</td>
<td>[0.0113, 0.0120]</td>
<td>[−0.2023, −0.1987]</td>
</tr>
<tr>
<td>Im(τ)</td>
<td>[0.9944, 0.9951]</td>
<td>[1.0522, 1.0633]</td>
</tr>
<tr>
<td>Re(φ₃)</td>
<td>[−0.090, −0.082]</td>
<td>[0.113, 0.121]</td>
</tr>
</tbody>
</table>
relevant parameters

\[ m_1 < m_2 \quad [\Delta m_{ij}^2 \equiv m_i^2 - m_j^2] \]

\[ \Delta m_{21}^2 < |\Delta m_{32}^2|, |\Delta m_{31}^2| \]

i.e. 1 and 2 are, by definition, the closest levels

two possibilities: NO and IO

Mixing matrix \( U_{PMNS} \) (Pontecorvo, Maki, Nakagawa, Sakata)

\[ L_{CC} = -\frac{g}{\sqrt{2}} W_\mu \bar{e}_L \gamma^\mu U_{PMNS} \nu_L \]

standard parametrization

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau \\
\end{pmatrix}
\begin{pmatrix}
c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\
-s_{12} c_{23} - c_{12} s_{13} s_{23} e^{i\delta} & c_{12} c_{23} - s_{12} s_{13} s_{23} e^{i\delta} & c_{13} s_{23} \\
-c_{12} s_{13} c_{23} e^{i\delta} + s_{12} s_{23} & -s_{12} s_{13} c_{23} e^{i\delta} - c_{12} s_{23} & c_{13} c_{23}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\alpha_{21}} & 0 \\
0 & 0 & e^{i\alpha_{31}}
\end{pmatrix}
\]
Yukawa interactions in $\mathcal{N}=1$ global SUSY

\[ S = \int d^4 xd^2 \theta d^2 \bar{\theta} K(\Phi, \bar{\Phi}) + \int d^4 xd^2 \theta \ w(\Phi) + h.c. \]

$S$ invariant

\[ \begin{align*}
K(\Phi, \bar{\Phi}) &\to K(\Phi, \bar{\Phi}) + f(\Phi) + \bar{f}(\bar{\Phi}) \\
I(\Phi) &\to I(\Phi)
\end{align*} \]

$w(\Phi) = \sum_n Y_{I_1 \ldots I_n}(\tau) \phi^{(I_1)} \cdots \phi^{(I_n)}$

\[ Y_{I_1 \ldots I_n}(\tau) \]

invariance of $w(\Phi)$ guaranteed by an holomorphic $Y_{I_1 \ldots I_n}(\tau)$ such that

\[ Y_{I_1 \ldots I_n}(\gamma \tau) = (c\tau + d)^{k_Y(n)} \rho(\gamma) Y_{I_1 \ldots I_n}(\tau) \]

1. $k_Y(n) + k_{I_1} + \ldots + k_{I_n} = 0$

2. $\rho \times \rho^{I_1} \times \ldots \times \rho^{I_n} \supset 1$

modular forms of level $N$ and weight $k_Y$

form a linear space $\mathcal{M}_k(\Gamma_N)$ of finite dimension

$\Phi = (\tau, \varphi)$

[extension to $\mathcal{N}=1$ SUGRA straightforward]
Flavour Symmetry approach

One of the few tools we have, but with several obstacles

1. predictability
   - high number of free parameters
     - lowest order Lagrangian parameters
     - complicated SB sector
     - EFT: higher dimensional operators
     - SUSY breaking effects
     - RGE corrections ($\Lambda_{\text{UV}}, m_{\text{SUSY}}, \tan \beta$)

2. choice of the vacuum
   - choice of direction in flavour space of
     - dynamically selected? (minimum of energy density)
     - by hand?
     - anthropic selection?