Matter Effect in LBL Experiments

Sanjib Kumar Agarwalla

sanjib@iopb.res.in

Institute of Physics, Bhubaneswar, India
Neutrino Oscillations in 3 Flavors

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix} \begin{pmatrix}
1 & 0 & s_{13}e^{-i\delta} \\
0 & 1 & 0 \\
-s_{13}e^{i\delta} & 0 & c_{13}
\end{pmatrix} \begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix}
\]

\(\theta_{23}: P(\nu_\mu \to \nu_\mu)\) by Atoms. \(v\) and \(v\) beam
\(\theta_{13}: P(\nu_e \to \nu_e)\) by Reactor \(v\)
\(\theta_{12}: P(\nu_\mu \to \nu_e)\) by Reactor and solar \(v\)

Three mixing angles: \(\theta_{23}, \theta_{13}, \theta_{12}\) and one CP violating (Dirac) phase \(\delta_{CP}\)

\[
\tan^2 \theta_{12} = \frac{|U_{e2}|^2}{|U_{e1}|^2}; \quad \tan^2 \theta_{23} = \frac{|U_{\mu3}|^2}{|U_{\tau3}|^2}; \quad U_{e3} = \sin \theta_{13}e^{-i\delta}
\]

3 mixing angles simply related to flavor components of 3 mass eigenstates

Over a distance \(L\), changes in the relative phases of the mass states may induce flavor change

\[P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4 \sum_{i>j} \text{Re}[U_{\alpha i}^* U_{\alpha j} U_{\beta i}^* U_{\beta j}^*] \sin^2 \Delta_{ij} - 2 \sum_{i>j} \text{Im}[U_{\alpha i}^* U_{\alpha j} U_{\beta i} U_{\beta j}^*] \sin 2\Delta_{ij}\]

\[\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E_v}\]

2 independent mass splittings \(\Delta m_{21}^2\) and \(\Delta m_{32}^2\), for anti-neutrinos replace \(\delta_{CP}\) by \(-\delta_{CP}\)
Neutrino Oscillations in Matter

Neutrino propagation through matter modify the oscillations significantly

Coherent forward elastic scattering of neutrinos with matter particles

Charged current interaction of $\nu_e$ with electrons creates an extra potential for $\nu_e$

MSW matter term: 

\[
A = \pm 2\sqrt{2}G_F N_e E \quad \text{or} \quad A(eV^2) = 0.76 \times 10^{-4} \rho (g/cc) E(\text{GeV})
\]

$N_e =$ electron number density, $+(-)$ for neutrinos (anti-neutrinos), $\rho =$ matter density in Earth

Matter term changes sign when we switch from neutrino mode to anti-neutrino mode

\[
P(\nu_\alpha \rightarrow \nu_\beta) - P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) \neq 0 \quad \text{even if } \delta_{CP} = 0, \text{ causes fake CP asymmetry}
\]

Matter term modifies oscillation probability differently depending on the sign of $\Delta m^2$

\[
\Delta m^2 \approx A \Leftrightarrow E_{\text{Earth}}^{\text{res}} = 6 - 8 \text{ GeV}
\]

\[
\begin{array}{c|cc}
\Delta m^2 > 0 & \nu & \bar{\nu} \\
\hline
\Delta m^2 < 0 & \text{MSW} & - \\
\end{array}
\]

\[
\text{Resonant conversion – Matter effect}
\]

\[
\text{Resonance occurs for neutrinos (anti-neutrinos) if } \Delta m^2 \text{ is positive (negative)}
\]
Neutrino Mass Ordering: Important Open Question

The sign of $\Delta m_{31}^2 (m_3^2 - m_1^2)$ is not known

Neutrino mass spectrum can be normal or inverted ordered

We only have a lower bound on the mass of the heaviest neutrino

$$\sqrt{2.5 \cdot 10^{-3} \text{eV}^2} \sim 0.05 \text{ eV}$$

We currently do not know which neutrino is the heaviest

Mass Ordering Discrimination: A Binary yes-or-no type question

$|U_{e1}|^2 > |U_{e2}|^2 > |U_{e3}|^2$

$\nu_e$ component of $\nu_1 > \nu_e$ component of $\nu_2 > \nu_e$ component of $\nu_3$
→ In \( \nu_\mu \) survival probability, the dominant term is mainly sensitive to \( \sin^2 2\theta_{23} \)

→ If \( \sin^2 2\theta_{23} \) differs from 1 (recent hints), we get two solutions for \( \theta_{23} \)

→ One in lower octant (LO: \( \theta_{23} < 45 \text{ degree} \))

→ Other in higher octant (HO: \( \theta_{23} > 45 \text{ degree} \))

\( \nu_\mu \) to \( \nu_e \) oscillation channel can break this degeneracy preferred value would depend on the choice of neutrino mass ordering

\[ \theta_{23} = 40^\circ \quad \theta_{23} = 50^\circ \]

Octant ambiguity of \( \theta_{23} \)

Fogli and Lisi, hep-ph/9604415
Is CP violated in the neutrino sector, as in the quark sector?

Mixing can cause CPV in \( \nu \) sector, provided \( \delta_{CP} \neq 0^\circ \) and \( 180^\circ \)

Need to measure the CP-odd asymmetries:

\[
\Delta P_{\alpha\beta} \equiv P(\nu_{\alpha} \rightarrow \nu_{\beta}; L) - P(\bar{\nu}_{\alpha} \rightarrow \bar{\nu}_{\beta}; L) \quad (\alpha \neq \beta)
\]

\[
\Delta P_{e\mu} = \Delta P_{\mu\tau} = \Delta P_{\tau e} = 4J_{CP} \times \left[ \sin \left( \frac{\Delta m_{21}^2}{2E} L \right) + \sin \left( \frac{\Delta m_{32}^2}{2E} L \right) + \sin \left( \frac{\Delta m_{13}^2}{2E} L \right) \right]
\]

Jarlskog CP-odd Invariant \( \rightarrow \) \[ J_{CP} = \frac{1}{8} \cos \theta_{13} \sin 2\theta_{13} \sin 2\theta_{23} \sin 2\theta_{12} \sin \delta_{CP} \]

Three-flavor effects are key for CPV, need to observe interference

Conditions for observing CPV:  
1) Non-degenerate masses \( \checkmark \)  
2) Mixing angles \( \neq 0^\circ \) and \( 90^\circ \) \( \checkmark \)  
3) \( \delta_{CP} \neq 0^\circ \) and \( 180^\circ \) (Hints)
Traditional approach: Neutrino beam from pion decay
Accelerator Long-baseline Neutrino Experiments

Appearance: \( (\nu_\mu \rightarrow \nu_e) \) and \( (\text{anti-}\nu_\mu \rightarrow \text{anti-}\nu_e) \)
(essential for MO, CPV, Octant)

Disappearance: \( (\nu_\mu \rightarrow \nu_\mu) \) and \( (\text{anti-}\nu_\mu \rightarrow \text{anti-}\nu_\mu) \)
(key for precise measurement of \( \Delta m^2_{32} \) and \( \theta_{23} \))

Present: T2K & NOvA

Future: DUNE, T2HK, T2HKK, ESS\nuSB

S. K. Agarwalla, Prospects of Neutrino Physics, Kavli IPMU, Japan, 12th April, 2019
Matter Effect in LBL Experiments

\[ P_{\mu e}(\text{matter}) - P_{\mu e}(\text{vacuum}) \]

\[ \delta_{CP} \text{ (true)} = -90^\circ \]

SKA, Soumya C., Masoom Singh, in preparation

S. K. Agarwalla, Prospects of Neutrino Physics, Kavli IPMU, Japan, 12th April, 2019
Matter Effect in LBL Experiments

$P_{\mu e}(\text{matter}) - P_{\mu e}(\text{vacuum})$

$\delta_{CP} \text{ (true)} = 0^\circ$

SKA, Soumya C., Masoom Singh, in preparation
Three Flavor Effects in $\nu_\mu \rightarrow \nu_e$ oscillation probability

The appearance probability $(\nu_\mu \rightarrow \nu_e)$ in matter, up to second order in the small parameters $\alpha \equiv \Delta m^2_{21}/\Delta m^2_{31}$ and $\sin 2\theta_{13}$,

\[ P_{\mu e} \simeq \sin^2 2\theta_{13} \frac{\sin^2[\Delta]}{(1-\Delta)^2} - \alpha \sin 2\theta_{13} \sin \delta_{CP} \sin(\Delta) \frac{\sin(\hat{A}\Delta)}{\hat{A}} \frac{\sin[(1-\hat{A})\Delta]}{(1-\hat{A})} + \alpha \sin 2\theta_{13} \xi \cos \delta_{CP} \cos(\Delta) \frac{\sin(\hat{A}\Delta)}{\hat{A}} \frac{\sin[(1-\hat{A})\Delta]}{(1-\hat{A})} + \alpha^2 \cos^2 \theta_{23} \sin^2 2\theta_{12} \frac{\sin^2(\hat{A}\Delta)}{\hat{A}^2} ; \]

where $\Delta \equiv \Delta m^2_{31} L/(4E)$, $\xi \equiv \cos \theta_{13} \sin 2\theta_{21} \sin 2\theta_{23}$, and $\hat{A} \equiv \pm (2\sqrt{2}G_F n_e E)/\Delta m^2_{31}$

changes sign with $\text{sgn}(\Delta m^2_{31})$ key to resolve hierarchy!

changes sign with polarity causes fake CP asymmetry!

Cervera et al., hep-ph/0002108
Freund, hep-ph/0103300

This channel suffers from: (Hierarchy – $\delta_{CP}$) & (Octant – $\delta_{CP}$) degeneracy! How can we break them?

S. K. Agarwalla, Prospects of Neutrino Physics, Kavli IPMU, Japan, 12th April, 2019
Analytical approximation of the neutrino oscillation matter effects at large $\theta_{13}$

Sanjib Kumar Agarwalla,$^{a,1}$ Yee Kao$^b$ and Tatsu Takeuchi$^{c,d}$

$^a$Institute of Physics, Sachivalaya Marg, Sainik School Post, Bhubaneswar 751005, Orissa, India
$^b$Department of Chemistry and Physics, Western Carolina University, Cullowhee, NC 28723, U.S.A.
$^c$Center for Neutrino Physics, Physics Department, Virginia Tech, Blacksburg, VA 24061, U.S.A.
$^d$Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, Kashiwa-shi, Chiba-ken 277-8583, Japan

E-mail: sanjib@iopb.res.in, ykao@email.wcu.edu, takeuchi@vt.edu

ABSTRACT: We argue that the neutrino oscillation probabilities in matter are best understood by allowing the mixing angles and mass-squared differences in the standard parametrization to ‘run’ with the matter effect parameter $a = 2\sqrt{2}G_F N_e E$, where $N_e$ is the electron density in matter and $E$ is the neutrino energy. We present simple analytical approximations to these ‘running’ parameters. We show that for the moderately large
Matter Effect Parameter \( a \)

\[ a = 2\sqrt{2}G_F N_e E = 7.63 \times 10^{-5} (\text{eV}^2) \left( \frac{\rho}{\text{g/cm}^3} \right) \left( \frac{E}{\text{GeV}} \right) \]

- Matter effects play an important role
- Mixing angles and mass-squared differences run with the matter effect parameter ‘\( a \)’
- We present simple analytical approximations to these running parameters using the Jacobi method
- We show that for large \( \theta_{13} \), the running of \( \theta_{23} \) and \( \delta_{CP} \) can be neglected, simplifying the probability expression
- We need to rotate only \( \theta_{12} \) and \( \theta_{13} \)


Agarwalla, Kao, Takeuchi, JHEP 1404, 047 (2014)
Use the expressions for the vacuum oscillation probabilities as it is, but make the following replacements:

\[
\theta_{12} \rightarrow \theta_{12}', \quad \theta_{13} \rightarrow \theta_{13}', \quad \delta m_{jk}^2 \rightarrow \lambda_j - \lambda_k
\]

where

\[
\tan 2\theta_{12}' = \frac{(\delta m_{21}^2 / c_{13}^2) \sin 2\theta_{12}}{(\delta m_{21}^2 / c_{13}^2) \cos 2\theta_{12} - a}, \quad \tan 2\theta_{13}' = \frac{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \sin 2\theta_{13}}{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \cos 2\theta_{13} - a},
\]

\[
\lambda_1 = \lambda_+ \quad \lambda_+ = \frac{(\delta m_{21}^2 + ac_{13}^2) \pm \sqrt{(\delta m_{21}^2 - ac_{13}^2)^2 + 4a^2 c_{13}^2 s_{12}^2 \delta m_{21}^2}}{2}
\]

\[
\lambda_2 = \lambda_\pm \quad \lambda_\pm = \frac{\left[ \lambda_+ + (\delta m_{31}^2 + as_{13}^2) \right] \pm \sqrt{\left[ \lambda_+ - (\delta m_{31}^2 + as_{13}^2) \right]^2 + 4a^2 s_{12}^2 c_{13}^2 s_{13}^2}}{2}
\]

upper (lower) sign

for NH (IH)

Approximation works when \(\theta_{13} = O(\varepsilon)\), where \(\varepsilon = \sqrt{\delta m_{21}^2 / |\delta m_{31}^2|} = 0.17\)

Agarwalla, Kao, Takeuchi, JHEP 1404, 047 (2014)
\[ a = \frac{\alpha}{|\delta m_{31}^2|} = \varepsilon^{-\beta}, \quad \varepsilon = \sqrt{\frac{\delta m_{21}^2}{\delta m_{31}^2}} \approx 0.17 \]

\( \theta_{12}' \) (NH)

\( \theta_{13}' \) (IH)

\( a \)-dependence of effective mixing angles

Agarwalla, Kao, Takeuchi, JHEP 1404, 047 (2014)
a-dependence of effective mass-squared differences

Normal Hierarchy

Inverted Hierarchy

Agarwalla, Kao, Takeuchi, JHEP 1404, 047 (2014)
Accuracy of Our Method and Comparison with Existing Literature

$L = 1300 \text{ km}, \delta = 0$, Normal Hierarchy

\[ P(\nu_e \to \nu_\mu) \]

- **Exact**
- **This work**
- **Cervera et al.** [hep-ph/0002108]
- **Asano–Minakata** [arXiv:1103.4387]
- **Freund** [hep-ph/0103300]

Agarwalla, Kao, Takeuchi, JHEP 1404, 047 (2014)
Accuracy of Our Method and Comparison with Existing Literature

$L = 810 \text{ km}, \delta = 0$, Normal Hierarchy

$P(\nu_e \rightarrow \nu_\mu)$

- **Exact**
- **This work**
- **Cervera et al.**
- **Asano–Minakata**
- **Freund**

$E \text{ (Gev)}$

Agarwalla, Kao, Takeuchi, JHEP 1404, 047 (2014)
Accuracy of Our Method and Comparison with Existing Literature

$L=295 \text{ km}, \delta=0$, Normal Hierarchy

$P(v_e \rightarrow v_\mu)$

- **Exact**
- **This work**
- **Cervera et al.**
- **Asano-Minakata**
- **Freund**

$E \text{ (GeV)}$

Agarwalla, Kao, Takeuchi, JHEP 1404, 047 (2014)

S. K. Agarwalla, Prospects of Neutrino Physics, Kavli IPMU, Japan, 12th April, 2019
Other analytical expressions suffer in accuracy due to their reliance on expansion in $\theta_{13}$, or in simplicity when higher order terms in $\theta_{13}$ included.

Our method gives accurate probability for all channels, baselines, and energies.
Compact Perturbative Expressions For Neutrino Oscillations in Matter

Peter B. Denton\textsuperscript{a,b} Hisakazu Minakata\textsuperscript{c,d} Stephen J. Parke\textsuperscript{a}

\textsuperscript{a}Theoretical Physics Department, Fermi National Accelerator Laboratory, P. O. Box 500, Batavia, IL 60510, USA
\textsuperscript{b}Physics \& Astronomy Department, Vanderbilt University, PMB 401807, 2301 Vanderbilt Place, Nashville, TN 37235, USA
\textsuperscript{c}Instituto de Física, Universidade de São Paulo, C. P. 66.318, 05315-970 São Paulo, Brazil
\textsuperscript{d}Department of Physics, Yachay Tech University, San Miguel de Urcuquí, 100119 Ecuador

E-mail: peterbd1@gmail.com, hminakata@yachaytech.edu.ec, parko@fnal.gov

ABSTRACT: We further develop and extend a recent perturbative framework for neutrino oscillations in uniform matter density so that the resulting oscillation probabilities are accurate for the complete matter potential versus baseline divided by neutrino energy plane. This extension also gives the exact oscillation probabilities in vacuum for all values of baseline divided by neutrino energy. The expansion parameter used is related to the ratio of the solar to the atmospheric $\Delta m^2$ scales but with a unique choice of the atmospheric $\Delta m^2$ such that certain first-order effects are taken into account in the zeroth-order Hamiltonian. Using a mixing matrix formulation, this framework has the exceptional feature that the neutrino oscillation probability in matter has the same structure as in vacuum, to all orders in the expansion parameter. It also contains all orders in the matter potential and $\sin \theta_{13}$. It facilitates immediate physical interpretation of the analytic results, and makes the expressions for the neutrino oscillation probabilities extremely compact and very accurate even at zeroth order in our perturbative expansion. The first and second order results are also given which improve the precision by approximately two or more orders of magnitude per perturbative order.

Similar treatment by Ioannisian \& Pokorski, arXiv:1801.10488
Neutrino oscillation probabilities through the looking glass

Gabriela Barenboim$^{a,1}$, Peter B. Denton$^{b,2}$, Stephen J. Parke$^{c,3}$, Christoph A. Ternes$^{d,4}$

$^a$Departament de Física Teórica and IFIC, Universitat de València-CSIC, E-46100, Burjassot, Spain
$^b$Physics Department, Brookhaven National Laboratory, Upton, New York 11973, USA
$^c$Theoretical Physics Department, Fermi National Accelerator Laboratory, P. O. Box 500, Batavia, IL 60510, USA
$^d$Institut de Física Corpuscular (CSIC-Universitat de València), Parc Científic de la UV, C/ Catedrático José Beltrán, 2, E-46980 Paterna (València), Spain

Abstract

In this paper we review different expansions for neutrino oscillation probabilities in matter in the context of long-baseline neutrino experiments. We examine the accuracy and computational efficiency of different exact and approximate expressions. We find that many of the expressions used in the literature are not precise enough for the next generation of long-baseline experiments, but several of them are while maintaining comparable simplicity. The results of this paper can be used as guidance to both phenomenologists and experimentalists when implementing the various oscillation expressions into their analysis tools.

Keywords: Neutrino physics, Neutrino oscillations in matter
In order to qualify as an expansion parameter, the authors require that the probability recovers the exact (to all orders) expression as that parameter goes to zero. That is, $x$ is an expansion parameter if and only if

$$\lim_{x \to 0} P_{\text{approx}}(x) = P_{\text{exact}}(x = 0)$$

Berenboim, Denton, Parke, Ternes, arXiv:1902.00517
DUNE Sensitivity to Reject Vacuum Solution

\[ \Delta \chi^2 \]

\[ \delta_{CP} \text{ (true)} \]

For $\delta_{CP} = -90$ degree
Relative 1\(\sigma\) precision in $\rho_{avg}$ is $\sim 15\%$
\[ \rho_{\text{avg}} \rightarrow \alpha_{\text{SF}} \times \rho_{\text{avg}} \]

Vacuum: \( \alpha_{\text{SF}} = 0 \)

Standard Profile: \( \alpha_{\text{SF}} = 1 \)

Super-K
arXiv:1710.09126

Solar+KamLAND
arXiv:1507.05287

Standard \( \alpha_{\text{SF}} = 1 \)
disfavored by
Solar + KamLAND
data due to tension
between them in
measuring \( \Delta m^2_{21} \)

SKA, Soumya C., Masoom Singh, in preparation

S. K. Agarwalla, Prospects of Neutrino Physics, Kavli IPMU, Japan, 12th April, 2019
Degeneracies in $\rho_{\text{avg}} - \delta_{\text{CP}}$ Plane for DUNE

SKA, Soumya C., Masoom Singh, in preparation

S. K. Agarwalla, Prospects of Neutrino Physics, Kavli IPMU, Japan, 12th April, 2019
Degeneracies in $\rho_{\text{avg}} - \sin^2\theta_{23}$ Plane for DUNE

SKA, Soumya C., Masoom Singh, in preparation
Concluding Remarks

• Earth’s Matter Effect plays an important role in present and future long-baseline neutrino oscillation experiments

• Precise understanding of mixing angles and mass-squared differences in matter is important to explain the results

• Approximate analytical expressions of oscillation probability can help to understand various parameter degeneracies

• Future goal is to have a robust test of three-flavor paradigm in presence of Earth Matter

• Future large-scale oscillation facilities should measure Earth Matter Density and explore possible degeneracies among line-averaged constant density, CP Phase, and $\theta_{23}$

Thank you
Hierarchy – $\delta_{CP}$ degeneracy in $\nu_\mu \rightarrow \nu_e$ oscillation channel

For $\nu$: Max: NH, -90° and Min: IH, 90°

Favorable combinations
NH, LHP (-180° to 0°) and IH, UHP (0° to 180°)

Degeneracy pattern different between T2K & NOvA

DUNE: Large Earth matter effects
Clear separation between NH and IH

Octant – $\delta_{CP}$ degeneracy in $\nu_\mu \rightarrow \nu_e$ oscillation channel

L=810km, $\sin^2 \theta_{13} = 0.089$, NH

For neutrino:
- Maximum: HO, $-90^\circ$
- Minimum: LO, $90^\circ$

LO: $\sin^2 \theta_{23} = 0.41$
HO: $\sin^2 \theta_{23} = 0.59$

For anti-neutrino:
- Maximum: HO, $90^\circ$
- Minimum: LO, $-90^\circ$

Unfavorable CP values for neutrino are favorable for anti-neutrino & vice-versa

Agarwalla, Prakash, Sankar, arXiv: 1301.2574
Oscillation Data and Neutrino Mixing Schemes

Sum Rule: \[
\cos \delta_{\mathrm{CP}} = \frac{\tan \theta_{23}}{\sin 2\theta_{12} \sin \theta_{13}} \left[ \cos 2\theta_{12}' + (\sin^2 \theta_{12} - \cos^2 \theta_{12}') (1 - \cot^2 \theta_{23} \sin^2 \theta_{13}) \right]
\]

**DUNE + T2HK**

<table>
<thead>
<tr>
<th>Symmetry form</th>
<th>(\theta_{12}^{\nu} [^\circ])</th>
<th>(\cos \delta_{\mathrm{CP}})</th>
<th>(\delta_{\mathrm{CP}} [^\circ])</th>
</tr>
</thead>
<tbody>
<tr>
<td>BM</td>
<td>(\arcsin(1/\sqrt{3}) \approx 35)</td>
<td>unphysical</td>
<td>unphysical</td>
</tr>
<tr>
<td>TBM</td>
<td>(\arcsin(1/\sqrt{3}) \approx 35)</td>
<td>(-0.16)</td>
<td>99 (\vee) 261</td>
</tr>
<tr>
<td>GRA</td>
<td>(\arctan(1/\phi) \approx 32)</td>
<td>(0.21)</td>
<td>78 (\vee) 282</td>
</tr>
<tr>
<td>GRB</td>
<td>(\arccos(\phi/2) = 36)</td>
<td>(-0.24)</td>
<td>104 (\vee) 256</td>
</tr>
<tr>
<td>HG</td>
<td>30</td>
<td>(0.39)</td>
<td>67 (\vee) 293</td>
</tr>
</tbody>
</table>

\[ U_{\text{PMNS}} = U_{e} U_{\nu} \]

golden ratio: \(\phi = (1 + \sqrt{5})/2\)

black dot: current best-fit value of \(\delta_{\mathrm{CP}} = 248^\circ\) which means

\[ \sin^2 \theta_{12}' = 0.364 (\Delta \chi^2 = 0) \]

Agarwalla, Chatterjee, Petcov, Titov, arXiv:1711.02107

the coloured dots corresponding to the values of \(\sin^2 \theta_{12}'\) which characterise the GRB (violet), TBM (red), GRA (blue) and HG (green) symmetry forms.
Mass Hierarchy Discovery with T2K and NOνA

Adding data from T2K and NOνA is useful to kill the intrinsic degeneracies

T2K:  Total p.o.t.: $7.8 \times 10^{21}$

NOνA:  Total p.o.t.: $3.6 \times 10^{21}$

CP asymmetry $\propto 1/\sin2\theta_{13}$

Large $\theta_{13}$ increases statistics but reduces asymmetry

Systematics are important

---

If $\theta_{23} < 41^\circ$ or $\theta_{23} > 50^\circ$, we can resolve the octant issue at $2\sigma$ irrespective of $\delta_{\text{CP}}$.

If $\theta_{23} < 39^\circ$ or $\theta_{23} > 52^\circ$, we can resolve the octant issue at $3\sigma$ irrespective of $\delta_{\text{CP}}$.

**Important message:** T2K must run in anti-neutrino mode in future.
Diagonalization of the Effective Hamiltonian

\[ H_a = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger + \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \tilde{U} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tilde{U}^\dagger, \]

\[ H'_a = Q^\dagger U^\dagger H_a U Q \]

\[ = Q^\dagger \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} + \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U \right\} Q \]

\[ = \begin{bmatrix} ac_{12}^2 c_{13}^2 & ac_{12} s_{12} c_{13}^2 & ac_{12} c_{13} s_{13} \\ ac_{12} s_{12} c_{13}^2 & as_{12}^2 c_{13}^2 + \delta m_{21}^2 & as_{12} c_{13} s_{13} \\ ac_{12} c_{13} s_{13} & as_{12} c_{13} s_{13} & as_{13}^2 + \delta m_{31}^2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{bmatrix}. \]
Jacobi Method (1846)

- Carl Gustav Jacob Jacobi (1804-1851)
1st Rotation

\[
H'_a = \begin{bmatrix}
ac_{12}^2c_{13}^2 & ac_{12}s_{12}c_{13}^2 & ac_{12}c_{13}s_{13} \\
ac_{12}s_{12}c_{13}^2 & as_{12}^2c_{13}^2 + \delta m_{21}^2 & as_{12}c_{13}s_{13} \\
ac_{12}c_{13}s_{13} & as_{12}c_{13}s_{13} & as_{13}^2 + \delta m_{31}^2
\end{bmatrix}
\]

\[
V = \begin{bmatrix}
c_\varphi & s_\varphi & 0 \\
-s_\varphi & c_\varphi & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \tan 2\varphi = \frac{a\sin 2\theta_{12}}{(\delta m_{21}^2 / c_{13}^2) - a\cos 2\theta_{12}}
\]

\[
H''_a = V^+ H'_a V = \begin{bmatrix}
\lambda'_- & 0 & ac'_{12}c_{13}s_{13} \\
0 & \lambda'_+ & as'_{12}c_{13}s_{13} \\
ac'_{12}c_{13}s_{13} & as'_{12}c_{13}s_{13} & as_{13}^2 + \delta m_{31}^2
\end{bmatrix},
\]

\[
\theta'_{12} = \theta_{12} + \varphi, \quad \lambda'_\pm = \frac{(\delta m_{21}^2 + ac_{13}^2) \pm \sqrt{(\delta m_{21}^2 - ac_{13}^2)^2 + 4ac_{13}^2s_{12}^2\delta m_{21}^2}}{2}
\]
$\tan 2\theta'_{12} = \tan 2(\theta_{12} + \varphi) = \frac{(\delta m_{21}^2 / c_{13}^2)\sin 2\theta_{12}}{(\delta m_{21}^2 / c_{13}^2)\cos 2\theta_{12} - a}$
2\textsuperscript{nd} Rotation

\[
H''_a = \begin{bmatrix}
\lambda' & 0 & ac'_{12}c_{13}s_{13} \\
0 & \lambda' & as'_{12}c_{13}s_{13} \\
ac'_{12}c_{13}s_{13} & as'_{12}c_{13}s_{13} & \lambda^2 + \delta m^2_{31}
\end{bmatrix}
\]

\[
W = \begin{bmatrix}
1 & 0 & 0 \\
0 & c_{\phi} & s_{\phi} \\
0 & -s_{\phi} & c_{\phi}
\end{bmatrix}, \quad \tan 2\phi = \frac{as'_{12} \sin 2\theta_{13}}{\delta m^2_{31} + as^2_{13} - \lambda'^2} \approx \frac{a \sin 2\theta_{13}}{(\delta m^2_{31} - \delta m^2_{31}s_{12}^2) - a \cos 2\theta_{13}}
\]

\[
H''_a = W^\dagger H''_a W = \begin{bmatrix}
\lambda' & -ac'_{12}c_{13}s_{13}c_{\phi} & ac'_{12}c_{13}s_{13}c_{\phi} \\
-ac'_{12}c_{13}s_{13}s_{\phi} & \lambda'' & 0 \\
ac'_{12}c_{13}s_{13}c_{\phi} & 0 & \lambda''
\end{bmatrix},
\]

\[
\lambda'' = \left[ \lambda' + (\delta m^2_{31} + as^2_{13}) \right] \pm \sqrt{\left[ \lambda' - (\delta m^2_{31} + as^2_{13}) \right]^2 + 4a^2 s^2_{12}c^2_{13}s^2_{13}}
\]

\[
\delta m^2_{31} = s^2_{12}c^2_{13}s^2_{13}
\]

\[
W^\dagger H''_a W = \begin{bmatrix}
\lambda' & -ac'_{12}c_{13}s_{13}c_{\phi} & ac'_{12}c_{13}s_{13}c_{\phi} \\
-ac'_{12}c_{13}s_{13}s_{\phi} & \lambda'' & 0 \\
ac'_{12}c_{13}s_{13}c_{\phi} & 0 & \lambda''
\end{bmatrix}
\]
Effective Mixing Matrix

\[
\tilde{U} = UQVW
\]

\[
= R_{23}(\theta_{23}, 0) R_{13}(\theta_{13}, \delta) R_{12}(\theta_{12}, 0) Q R_{12}(\varphi, 0) R_{23}(\phi, 0)
\]

\[
= R_{23}(\theta_{23}, 0) Q R_{13}(\theta_{13}, 0) R_{12}(\theta_{12}, 0) R_{12}(\varphi, 0) R_{23}(\phi, 0)
\]

\[
= R_{23}(\theta_{23}, 0) Q R_{13}(\theta_{13}, 0) R_{12}(\theta_{12} + \varphi, 0) R_{23}(\phi, 0)
\]

\[
= R_{23}(\theta_{23}, 0) Q R_{13}(\theta_{13}, 0) R_{12}(\theta'_{12}, 0) R_{23}(\phi, 0)
\]

\[
\approx R_{23}(\theta_{23}, 0) Q R_{13}(\theta_{13}, 0) R_{13}(\phi, 0) R_{12}(\theta'_{12}, 0)
\]

\[
= R_{23}(\theta_{23}, 0) Q R_{13}(\theta_{13} + \phi, 0) R_{12}(\theta'_{12}, 0)
\]

\[
= R_{23}(\theta_{23}, 0) Q R_{13}(\theta'_{13}, 0) R_{12}(\theta'_{12}, 0)
\]

\[
= R_{23}(\theta_{23}, 0) R_{13}(\theta'_{13}, \delta) R_{12}(\theta'_{12}, 0) Q
\]

\[
\theta'_{13} = \theta_{13} + \phi
\]
Effective Mixing Matrix

\[
R_{12}(\theta_{12}',0) R_{23}(\phi,0)
\]

\[
= \begin{bmatrix}
  c'_{12} & s'_{12} & 0 \\
  -s'_{12} & c'_{12} & 0 \\
  0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & c_\phi & s_\phi \\
  0 & -s_\phi & c_\phi \\
\end{bmatrix}
\approx \begin{bmatrix}
  0 & 1 & 0 \\
  -1 & 0 & 0 \\
  0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & c_\phi & s_\phi \\
  0 & -s_\phi & c_\phi \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  c_\phi & 0 & s_\phi \\
  0 & 1 & 0 \\
  -s_\phi & 0 & c_\phi \\
\end{bmatrix}
\begin{bmatrix}
  0 & 1 & 0 \\
  -1 & 0 & 0 \\
  0 & 0 & 1 \\
\end{bmatrix}
\approx \begin{bmatrix}
  c_\phi & 0 & s_\phi \\
  0 & 1 & 0 \\
  -s_\phi & 0 & c_\phi \\
\end{bmatrix}
\begin{bmatrix}
  c'_{12} & s'_{12} & 0 \\
  -s'_{12} & c'_{12} & 0 \\
  0 & 0 & 1 \\
\end{bmatrix}
\]

\[
= R_{13}(\phi,0) R_{12}(\theta_{12}',0)
\]
Effective Mixing Angles

\[ \tan 2\theta'_{12} = \frac{(\delta m_{21}^2 / c_{13}^2) \sin 2\theta_{12}}{(\delta m_{21}^2 / c_{13}^2) \cos 2\theta_{12} - a}, \]

\[ \tan 2\theta'_{13} = \frac{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \sin 2\theta_{13}}{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \cos 2\theta_{13} - a}, \]
Other analytical expressions suffer in accuracy due to their reliance on expansion in $\theta_{13}$, or in simplicity when higher order terms in $\theta_{13}$ included.

Our method gives accurate probability for all channels, baselines, and energies.
Comparison between Constant and Varying Earth Density Profile

Agarwalla, Kao, Saha, Takeuchi, JHEP 1511 (2015) 035
Three Flavor Effects in $\nu_\mu \rightarrow \nu_e$ oscillation probability

\[
P(\nu_\mu \rightarrow \nu_e) - P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) =
\]
\[
\frac{16A}{\Delta m^2_{31}} \sin^2 \left( \frac{\Delta m^2_{31} L}{4E} \right) c^2_{13} s^2_{13} s^2_{23} (1 - 2s^2_{13})
\]
\[
- \frac{2AL}{E} \sin \left( \frac{\Delta m^2_{31} L}{4E} \right) c^2_{13} s^2_{13} s^2_{23} (1 - 2s^2_{13})
\]
\[
- 8 \frac{\Delta m^2_{21} L}{2E} \sin^2 \left( \frac{\Delta m^2_{31} L}{4E} \right) \sin \delta s_{13} c_{13} c_{23} s_{23} c_{12} s_{12}
\]

\[\text{Here, } A = 2 \sqrt{2} G_F n_e E = 7.6 \times 10^{-5} \text{eV}^2 \cdot \frac{\rho}{\text{g cm}^{-3}} \cdot \frac{E}{\text{GeV}}\]
How to proceed?

First possibility:

Choose small $L (~ 200 \text{ km})$, so that matter effects are small.

But, we want to work at oscillation maximum:

$$\frac{\Delta m^2_{31}L}{4E} \sim \frac{\pi}{2} \Rightarrow E_\nu < 1 \text{ GeV}$$

Since, $\sigma \propto E_\nu$: we need a high flux at oscillation maximum.

Off-axis beam: narrow range of neutrino energies.

This is the working principle of Hyper-Kamiokande.
Second possibility:

Take large $L (> 1000 \text{ km})$

Estimate the matter effects, and settle the issue of Mass Hierarchy

But, we still want to work at oscillation maximum:

$\frac{\Delta m_{31}^2 L}{4E} \sim \frac{\pi}{2} \Rightarrow E_\nu > 2 \text{ GeV}$

Unfold CP-violation from matter effects through energy dependence

On-axis beam: wide range of neutrino energies

This is the working principle of DUNE
Information on $\theta_{23}$ comes from: a) atmospheric neutrinos and b) accelerator neutrinos

In two-flavor scenario:

\[ P_{\mu\mu} = 1 - \sin^2 2\theta_{\text{eff}} \sin^2 \left( \frac{\Delta m_{\text{eff}}^2 L}{4E} \right) \]

For accelerator neutrinos: relate effective 2-flavor parameters with 3-flavor parameters:

\[ \Delta m_{\text{eff}}^2 = \Delta m_{31}^2 - \Delta m_{21}^2 (\cos^2 \theta_{12} - \cos \delta_{\text{CP}} \sin \theta_{13} \sin 2\theta_{12} \tan \theta_{23}) \]

where \[ \frac{|U_{\mu 3}|^2}{|U_{\tau 3}|^2} = \tan^2 \theta_{23} \]


Combining bean and atmospheric data in MINOS, we have:


\[ \sin^2 2\theta_{\text{eff}} = 0.95^{+0.035}_{-0.036} (10.71 \times 10^{21} \text{ p.o.t}) \]

\[ \sin^2 2\bar{\theta}_{\text{eff}} = 0.97^{+0.03}_{-0.08} (3.36 \times 10^{21} \text{ p.o.t}) \]

Atmospheric data, dominated by Super-Kamiokande, still prefers maximal value of \( \sin^2 2\theta_{\text{eff}} = 1 \) \( (\geq 0.94 \text{ (90\% C.L.)}) \)

Talk by Y. Itow in Neutrino 2012 conference, Kyoto, Japan
All the three global fits indicate for non-maximal 2-3 mixing!

In $\nu_\mu$ survival probability, the dominant term is mainly sensitive to $\sin^2 2\theta_{23}$!

If $\sin^2 2\theta_{23}$ differs from 1 (as indicated by recent data), we get two solutions for $\theta_{23}$:
one in lower octant (LO: $\theta_{23} < 45$ degree), other in higher octant (HO: $\theta_{23} > 45$ degree)

In other words, if $(0.5 - \sin^2 \theta_{23})$ is $+ve$ (-ve) then $\theta_{23}$ belongs to LO (HO)

This is known as the octant ambiguity of $\theta_{23}$!

$\nu_\mu$ to $\nu_e$ oscillation data can break this degeneracy!

The preferred value would depend on the choice of the neutrino mass hierarchy!

Fogli and Lisi, hep-ph/9604415

<table>
<thead>
<tr>
<th></th>
<th>Forero etal</th>
<th>Fogli etal</th>
<th>Gonzalez-Garcia etal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin^2 \theta_{23}$ (NH)</td>
<td>$0.427^{+0.034}<em>{-0.027} \oplus 0.613^{+0.022}</em>{-0.040}$</td>
<td>$0.386^{+0.024}_{-0.021}$</td>
<td>$0.41^{+0.037}<em>{-0.025} \oplus 0.59^{+0.021}</em>{-0.022}$</td>
</tr>
<tr>
<td>3σ range</td>
<td>0.36 → 0.68</td>
<td>0.331 → 0.637</td>
<td>0.34 → 0.67</td>
</tr>
<tr>
<td>$\sin^2 \theta_{23}$ (IH)</td>
<td>$0.600^{+0.026}_{-0.031}$</td>
<td>$0.392^{+0.039}_{-0.022}$</td>
<td></td>
</tr>
<tr>
<td>3σ range</td>
<td>0.37 → 0.67</td>
<td>0.335 → 0.663</td>
<td></td>
</tr>
</tbody>
</table>

Relative 1σ precision of 11%
Octant – $\delta_{CP}$ degeneracy in $\nu_\mu \rightarrow \nu_e$ oscillation channel

$$P_{\mu e} = \beta_1 \sin^2 \theta_{23} + \beta_2 \cos(\hat{\Delta} + \delta_{CP}) + \beta_3 \cos^2 \theta_{23}$$ (upto second order in $\alpha = \Delta_{21}/\Delta_{31}$ and $\sin 2 \theta_{13}$)

$$\beta_1 = \sin^2 2 \theta_{13} \frac{\sin^2 \hat{\Delta} (1 - \hat{A})}{(1 - \hat{A})^2}, \quad \beta_3 = \alpha^2 \sin^2 2 \theta_{12} \cos^2 \theta_{13} \frac{\sin^2 \hat{\Delta} \hat{A}}{\hat{A}^2}$$

$$\beta_2 = \alpha \cos \theta_{13} \sin 2 \theta_{12} \sin 2 \theta_{13} \sin 2 \theta_{23} \frac{\sin \hat{\Delta} \hat{A} \sin \hat{\Delta} (1 - \hat{A})}{\hat{A} \hat{A} (1 - \hat{A})}$$

$$A(\text{eV}^2) = 0.76 \times 10^{-4} \rho \ (\text{g/cc}) E(\text{GeV}) \quad \hat{\Delta} = \Delta_{31} L/4 E, \quad \hat{A} = A/\Delta_{31}$$


We demand that: $$P_{\mu e}(\text{LO, } \delta_{CP}^{\text{LO}}) = P_{\mu e}(\text{HO, } \delta_{CP}^{\text{HO}})$$

Above condition gives us: $$\cos(\hat{\Delta} + \delta_{CP}^{\text{LO}}) - \cos(\hat{\Delta} + \delta_{CP}^{\text{HO}}) = \frac{\beta_1 - \beta_3}{\beta_2} (\sin^2 \theta_{23}^{\text{HO}} - \sin^2 \theta_{23}^{\text{LO}})$$

For $L=810 \ \text{km} \ & \ E=2 \ \text{GeV}$, we get for NH and neutrino: $$\cos(\hat{\Delta} + \delta_{CP}^{\text{LO}}) - \cos(\hat{\Delta} + \delta_{CP}^{\text{HO}}) = 1.7$$

$$P_{\mu e}(\text{LO, } -116^\circ \leq \delta_{CP} \leq -26^\circ) \ \text{is degenerate with} \quad P_{\mu e}(\text{HO, } 64^\circ \leq \delta_{CP} \leq 161^\circ)$$

Agarwalla, Prakash, Uma Sankar, arXiv:1301.2574
Octant – $\delta_{CP}$ degeneracy in $\nu_\mu \rightarrow \nu_e$ oscillation channel

At $2 \text{ GeV}$, $P_{\mu e}(\text{LO}, -116^\circ \leq \delta_{CP} \leq -26^\circ)$ is degenerate with $P_{\mu e}(\text{HO}, 64^\circ \leq \delta_{CP} \leq 161^\circ)$

As an example, $P_{\mu e}(\text{LO}, \delta_{CP} = -90^\circ)$ is degenerate with $P_{\mu e}(\text{HO}, \delta_{CP} \approx 66^\circ)$
For neutrino:
favorable combinations:
Max: HO, -90°
Min: LO, 90°

For anti-neutrino:
favorable combinations:
Max: HO, 90°
Min: LO, -90°

Unfavorable CP values for neutrino are favorable for anti-neutrino and vice-versa!

Agarwalla, Prakash, Uma Sankar, arXiv:1301.2574
Octant – $\delta_{CP}$ degeneracy in LBNE and LBNO

Agarwalla, Prakash, Sankar, arXiv:1301.2574 [hep-ph]; see also the talk by T. Nakadaira in this workshop.

neutrino vs. anti-neutrino events for various octant-hierarchy combinations, ellipses due to varying $\delta_{\text{CP}}$!

If $\delta_{\text{CP}} = -90^\circ (90^\circ)$, the asymmetry between $\nu$ and anti-$\nu$ events is largest for NH (IH)

For NOvA & T2K, the ellipses for the two hierarchies overlap whereas the ellipses of LO are well separated from those of HO, the same is true for T2K as well!

Octant discovery: balanced neutrino & anti-neutrino runs needed in each experiment!
Balanced neutrino & anti-neutrino runs from T2K are mandatory if HO turns out to be the right octant!

Allowed regions in test $\sin^2 \theta_{23}$ - true $\delta_{CP}$ plane

HO-IH true, IH test, $2\sigma$

Balanced neutrino & anti-neutrino runs from T2K are mandatory if HO turns out to be the right octant!

A 2\(\sigma\) resolution of the octant, for all combinations of neutrino parameters, becomes possible if we add the balanced neutrino and anti-neutrino runs from T2K (2.5 years \(\nu\) + 2.5 years anti-\(\nu\)) and NOvA (3 years \(\nu\) + 3 years of anti-\(\nu\)).

**Important message:** T2K must run in anti-neutrino mode in future!
Octant discovery in $\theta_{23}$ (true) – $\delta_{CP}$ (true) plane with T2K & NOvA

If $\theta_{23} < 41^\circ$ or $\theta_{23} > 50^\circ$, we can resolve the octant issue at 2$\sigma$ irrespective $\delta_{CP}$

If $\theta_{23} < 39^\circ$ or $\theta_{23} > 52^\circ$, we can resolve the octant issue at 3$\sigma$ irrespective $\delta_{CP}$