On Weyl groups and Artin groups associated to orbifold projective lines

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Summary

- The generalized root system for an orbifold projective line.
  $\Rightarrow$ affine, elliptic, cuspidal root system arise systematically.

1. The generalized root system.
2. Weyl groups and the generalized Coxeter group.
3. The cuspidal Artin group and the fundamental group of the regular orbit space.
4. The cuspidal Artin group and the group of spherical twist functors.

The orbifold projective line $\mathbb{P}^1_{A,\Lambda}$

- Notations
    - $r \in \mathbb{Z}_{\geq 3}$, $A = (a_1, \ldots, a_r)$ such that $a_i \geq 2$.
    - $\mu_A := 2 + \sum_{i=1}^{r}(a_i - 1)$, $\chi_A := 2 + \sum_{i=1}^{r}(1/a_i - 1)$.
    - k: a field such that $\overline{k} = k$ and $\mathrm{char}(k) = 0$.
    - $\Lambda = (\lambda_1, \ldots, \lambda_r)$: $\lambda_i = [\lambda_i^{(1)} : \lambda_i^{(2)}] \in \mathbb{P}^1(k)$ such that $\lambda_i \neq \lambda_j$ for $i \neq j$, $\lambda_1 = \infty$, $\lambda_2 = 0$ and $\lambda_3 = 1$.

1. $S_{A,\Lambda} := k[X_1, \ldots, X_r] \left/ \left(X_i^{a_i} - X_2^{a_2} + \lambda_i X_1^{a_1}; i = 3, \ldots, r\right)\right.$.
2. L_A: an abelian group generated by r-letters $\vec{X}_i$, $i = 1, \ldots, r$ defined as the quotient

$$L_A := \bigoplus_{i=1}^{r} \mathbb{Z}\vec{X}_i \left/ \left(a_i\vec{X}_i - a_j\vec{X}_j; 1 \leq i < j \leq r\right)\right. .$$

Define a stack $\mathbb{P}^1_{A,\Lambda}$ by $\mathbb{P}^1_{A,\Lambda} := \left[\left(\mathrm{Spec}(S_{A,\Lambda}) \backslash \{0\}\right) / \mathrm{Spec}(kL_A)\right]$.

The Generalized Root system (GRS)

1. a free $\mathbb{Z}$-module $K_0(R)$ of finite rank $(=: \mu)$,
2. a symmetric bi-linear form $I_R : K_0(R) \times K_0(R) \longrightarrow \mathbb{Z}$,
3. $\Delta_{re}(R) \subset K_0(R)$ such that:
   - $K_0(R) = \mathbb{Z}\Delta_{re}(R)$,
   - $\forall \alpha \in \Delta_{re}(R)$, $I_R(\alpha, \alpha) = 2$,
   - $\forall \alpha \in \Delta_{re}(R)$, $r_\alpha(\lambda) := \lambda - I_R(\lambda, \alpha)\alpha \in \mathrm{Aut}(K_0(R), I_R)$ makes $\Delta_{re}(R)$ invariant, namely, $r_\alpha(\Delta_{re}(R)) = \Delta_{re}(R)$,
   - for $W(R) := \langle r_\alpha \mid \alpha \in \Delta_{re}(R) \rangle \subset \mathrm{Aut}(K_0(R), I_R)$, there exists $B = \{\alpha_1, \ldots, \alpha_\mu\} \subset \Delta_{re}(R)$ satisfying

$$K_0(R) = \bigoplus_{i=1}^{\mu} \mathbb{Z}\alpha_i, W(R) = \langle r_{\alpha_1}, \ldots, r_{\alpha_\mu} \rangle, \Delta_{re}(R) = W(R)B.$$

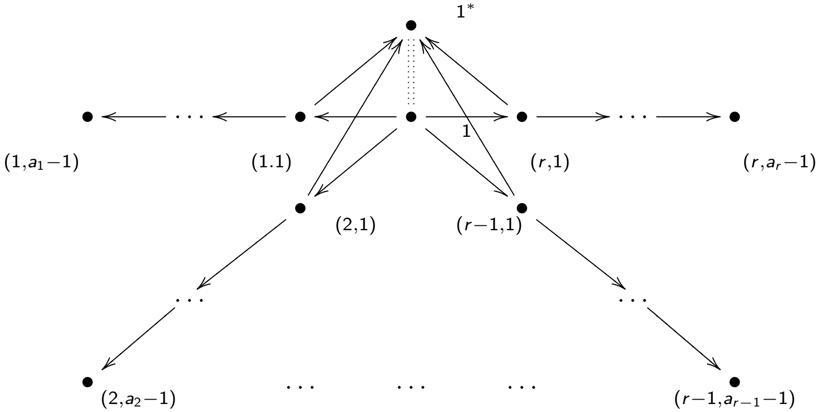
4. an element c_R of $W(R)$, which has the product presentation

$$c_R = r_{\alpha'_1} \cdots r_{\alpha'_\mu}$$

with respect to some root basis $B' = \{\alpha'_1, \ldots, \alpha'_\mu\}$.

The octopus $k\widetilde{\mathbb{T}}_{A,\Lambda}$

The bound quiver algebra of the following quiver:



with relations: $\mathcal{I}_\Lambda := \left\langle \sum_{i=1}^r \lambda_i^{(1)} f_{i,1} f_{i,1^*}, \sum_{i=1}^r \lambda_i^{(2)} f_{i,1} f_{i,1^*} \right\rangle.$

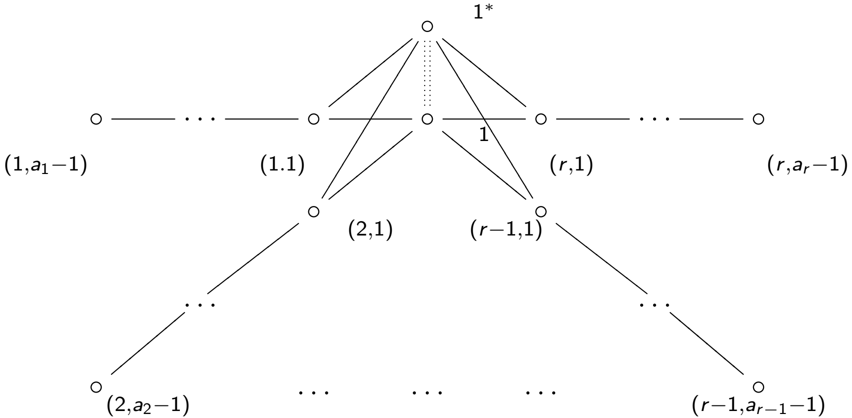
Construction of the GRS $\widetilde{R}_A$ for $\mathbb{P}^1_{A,\Lambda}$

- The GRS as a tri. invariant on an alg. tri. category which has
  - a dg enhancement (to define the mutation),
  - a full strongly exceptional collection (by $\mathcal{D}^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda}) \simeq \mathcal{D}^b(k\widetilde{\mathbb{T}}_{A,\Lambda})$),
  - transitive action of the Artin braid group on the set of isomorphism classes of full exceptional collections (by Meltzer for $\mathcal{D}^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda})$).

- The GRS $\widetilde{R}_A$ for $\mathbb{P}^1_{A,\Lambda}$ consists of
  1. $K_0(\mathcal{D}^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda}))$,
  2. $\chi_{\mathcal{D}^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda})} + {}^t\chi_{\mathcal{D}^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda})}$,
  3. $([E_1], \dots, [E_{\mu_A}])$ for some full exceptional collection,
  4. $c \in \mathrm{Aut}(K_0(\mathcal{D}^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda})))$ induced by $\mathcal{S}[-1] \in \mathrm{Auteq}(\mathcal{D}^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda}))$, where $\mathcal{S}$ is the Serre functor.

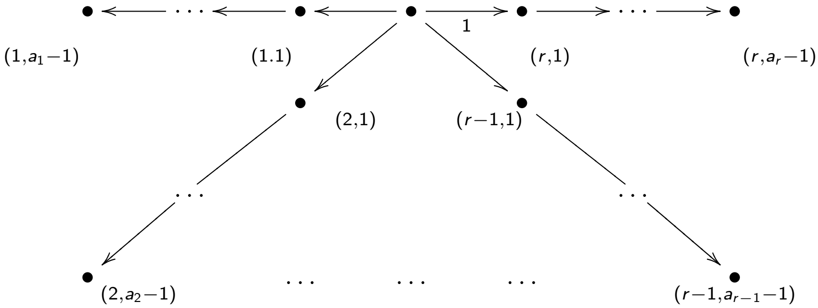
The generalized Coxeter–Dynkin diagram $\widetilde{T}_A$

- The Coxeter–Dynkin diagram for $B := \{\widetilde{\alpha}_v := [S_v]\}$.
- $\{S_v\}$: simple $k\widetilde{\mathbb{T}}_{A,\Lambda}$-modules such that $\chi(P_v, S_{v'}) = \delta_{vv'}$.
- $\{P_v\}$: indecomp. proj. $k\widetilde{\mathbb{T}}_{A,\Lambda}$-modules s. t. $\bigoplus P_v \cong k\widetilde{\mathbb{T}}_{A,\Lambda}$:



The star path algebra $k\mathbb{T}_A$

The path algebra for the following quiver:



- R_A: the GRS associated to $\mathcal{D}^b(k\mathbb{T}_A)$.

The Weyl groups $W(\widetilde{R}_A)$ and $W(R_A)$

Set $\delta := \widetilde{\alpha}_{1^*} - \widetilde{\alpha}_1 \in \mathrm{rad}(I_{\widetilde{R}_A})$. The natural projection map

$$K_0(\widetilde{R}_A) \longrightarrow K_0(\widetilde{R}_A)/\mathbb{Z}\delta \cong K_0(R_A)$$

induces $p : W(\widetilde{R}_A) \twoheadrightarrow W(R_A)$; $p(\widetilde{r}_1) = p(\widetilde{r}_{1^*}) = r_1$, $p(\widetilde{r}_v) = r_v$.

Theorem 1

There is an exact sequence of groups

$$\{1\} \longrightarrow N \longrightarrow W(\widetilde{R}_A) \xrightarrow{p} W(R_A) \longrightarrow \{1\},$$

where $N := \langle r(\widetilde{r}_1\widetilde{r}_{1^*})r \mid r \in W(\widetilde{R}_A)\rangle$. *In particular, we have*

$$W(\widetilde{R}_A) \cong W(R_A) \ltimes (K_0(R_A)/\mathrm{rad}(I_{R_A}))\,.$$

Key facts for Theorem 1

Define the elements as follows:

$$\widetilde{\tau}_1 := \widetilde{r}_1 \widetilde{r}_{1^*},$$

$$\widetilde{\tau}_{(i,1)} := \widetilde{r}_{(i,1)} \widetilde{\tau}_1 \widetilde{r}_{(i,1)} \widetilde{\tau}_1^{-1}, \quad i = 1, \ldots, r,$$

$$\widetilde{\tau}_{(i,j)} := \widetilde{r}_{(i,j)} \widetilde{\tau}_{(i,j-1)} \widetilde{r}_{(i,j)} \widetilde{\tau}_{(i,j-1)}^{-1}, \quad i = 1, \ldots, r, \ j = 2, \ldots, a_i - 1.$$

Proposition 2

$$\widetilde{\tau}_v(\widetilde{\lambda}) = \widetilde{\lambda} - I_{\widetilde{R}_A}(\widetilde{\lambda}, \widetilde{\alpha}_v)\delta, \quad \widetilde{\lambda} \in K_0(\widetilde{R}_A), \quad \forall v \in T_A.$$

In particular, there is a natural surjective group homomorphism

$$\varphi : K_0(R_A) \twoheadrightarrow N, \quad \sum_{v \in T_A} m_v \alpha_v \mapsto \prod_{v \in T_A} \widetilde{\tau}_v^{m_v},$$

which induces $K_0(R_A)/\mathrm{rad}(I_{R_A}) \cong N$.

The generalized Coxeter group $W(\widetilde{T}_A)$

Generators $\{\widetilde{w}_v \mid v \in \widetilde{T}_A\}$

Relations

$$\widetilde{w}_v^2 = 1 \quad \text{for all} \quad v \in \widetilde{T}_A,$$

$$\widetilde{w}_v \widetilde{w}_{v'} = \widetilde{w}_{v'} \widetilde{w}_v \quad \text{if} \quad I_{\widetilde{R}_A}(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = 0,$$

$$\widetilde{w}_v \widetilde{w}_{v'} \widetilde{w}_v = \widetilde{w}_{v'} \widetilde{w}_v \widetilde{w}_{v'} \quad \text{if} \quad I_{\widetilde{R}_A}(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = -1,$$

$$\widetilde{w}_{(i,1)} \sigma_1 \widetilde{w}_{(i,1)} \sigma_1 = \sigma_1 \widetilde{w}_{(i,1)} \sigma_1 \widetilde{w}_{(i,1)},$$

$$\begin{cases} \widetilde{w}_{(i,1)} \sigma_{(j,1)} = \sigma_{(j,1)} \widetilde{w}_{(i,1)} \\ \widetilde{w}_{(j,1)} \sigma_{(i,1)} = \sigma_{(i,1)} \widetilde{w}_{(j,1)} \end{cases} \quad \text{for all} \quad 1 \le i < j \le r,$$

where $\sigma_1 := \widetilde{w}_1 \widetilde{w}_{1^*}$ and $\sigma_{(i,1)} := \widetilde{w}_{(i,1)} \sigma_1 \widetilde{w}_{(i,1)} \sigma_1^{-1}$.

$W(\widetilde{T}_A)$ and $W(R_A) \ltimes K_0(R_A)$

Theorem 3

We have an isomorphism of groups

$$W(\widetilde{T}_A) \cong W(R_A) \ltimes K_0(R_A).$$

In particular, we have

$$W(\widetilde{T}_A) \cong \begin{cases} W(\widetilde{R}_A) & \text{if} \quad \chi_A \neq 0, \\ \widehat{W}(\widetilde{R}_A) & \text{if} \quad \chi_A = 0, \end{cases}$$

where $\widehat{W}(\widetilde{R}_A)$ is the hyperbolic extension of $W(\widetilde{R}_A)$.

Key fact for Theorem 3

Proposition 4

$W(R_A) \ltimes K_0(R_A)$ is described by the following relations:

Generators $\{r_v, \tau_v \mid v \in T_A\}$

Relations

$$r_v^2 = 1 \quad \textit{for all} \quad v \in T_A,$$

$$r_v r_{v'} = r_{v'} r_v \quad \textit{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = 0,$$

$$r_v r_{v'} r_v = r_{v'} r_v r_{v'} \quad \textit{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = -1,$$

$$\tau_v \tau_{v'} = \tau_{v'} \tau_v \quad \textit{for all} \quad v, v' \in T_A,$$

$$r_v \tau_v r_v = \tau_v^{-1} \quad \textit{for all} \quad v \in T_A,$$

$$r_v \tau_{v'} = \tau_{v'} r_v \quad \textit{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = 0,$$

$$r_v \tau_{v'} r_v = \tau_{v'} \tau_v \quad \textit{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = -1.$$

The cuspidal Artin group $G(\widetilde{T}_A)$

Generators $\{\widetilde{g}_v \mid v \in \widetilde{T}_A\}$

Relations

$$\widetilde{g}_v \widetilde{g}_{v'} = \widetilde{g}_{v'} \widetilde{g}_v \quad \text{if} \quad I_{\widetilde{R}_A}(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = 0,$$

$$\widetilde{g}_v \widetilde{g}_{v'} \widetilde{g}_v = \widetilde{g}_{v'} \widetilde{g}_v \widetilde{g}_{v'} \quad \text{if} \quad I_{\widetilde{R}_A}(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = -1,$$

$$\widetilde{g}_{(i,1)} \widetilde{\rho}_1 \widetilde{g}_{(i,1)} \widetilde{\rho}_1 = \widetilde{\rho}_1 \widetilde{g}_{(i,1)} \widetilde{\rho}_1 \widetilde{g}_{(i,1)} \quad \text{for all } i = 1, \ldots, r,$$

$$\begin{cases} \widetilde{g}_{(i,1)} \widetilde{\rho}_{(j,1)} = \widetilde{\rho}_{(j,1)} \widetilde{g}_{(i,1)} \\ \widetilde{g}_{(j,1)} \widetilde{\rho}_{(i,1)} = \widetilde{\rho}_{(i,1)} \widetilde{g}_{(j,1)} \end{cases} \quad \text{for all} \quad 1 \le i < j \le r.$$

where $\widetilde{\rho}_1 := \widetilde{g}_1 \widetilde{g}_{1^*}$ and $\widetilde{\rho}_{(i,1)} := \widetilde{g}_{(i,1)} \widetilde{\rho}_1 \widetilde{g}_{(i,1)} \widetilde{\rho}_1^{-1}$.

The fundamental group of the regular orbit space

- $\mathcal{E}(R_A)$: the complexified Tits cone for R_A.
- $\mathcal{E}(R_A)^{reg} := \mathcal{E}(R_A) \setminus \bigcup_{\alpha \in \Delta_{re}(R_A), n \in \mathbb{Z}} H_{\alpha,n}$.

1. $W(R_A) \ltimes K_0(R_A) \curvearrowright \mathcal{E}(R_A)$: properly discontinuous.
2. $W(R_A) \ltimes K_0(R_A) \curvearrowright \mathcal{E}(R_A)^{reg}$: free.

- $G(\widetilde{R}_A) := \pi_1(\mathcal{E}(R_A)^{reg}/(W(R_A) \ltimes K_0(R_A)), *)$.

Theorem 5

There exists an isomorphism of groups

$$G(\widetilde{T}_A) \cong G(\widetilde{R}_A).$$

Key fact for Theorem 5

Proposition 6 (Van der Lek)

The group $G(\widetilde{R}_A)$ is described by the following relations:

Generators $\{g_v, \rho_v \mid v \in T_A\}$

Relations

$$g_v g_{v'} = g_{v'} g_v \quad \text{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = 0,$$

$$g_v g_{v'} g_v = g_{v'} g_v g_{v'} \quad \text{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = -1,$$

$$\rho_v \rho_{v'} = \rho_{v'} \rho_v \quad \text{for all} \quad v, v' \in T_A,$$

$$g_v \rho_{v'} = \rho_{v'} g_v \quad \text{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = 0,$$

$$g_v \rho_{v'} g_v = \rho_{v'} \rho_v \quad \text{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = -1.$$

The group $\mathrm{Br}(\check{\mathcal{D}}_{A,\Lambda})$ of sherical twist functors

- $\Pi_2(k\widetilde{\mathbb{T}}_{A,\Lambda})$: the 2-Calabi–Yau completion of $k\widetilde{\mathbb{T}}_{A,\Lambda}$.
- $\check{\mathcal{D}}_{A,\Lambda} := \langle S_1, \ldots, S_{\mu_A} \rangle^{tri}_{iso,sum} \subset \mathcal{D}(\Pi_2(\mathcal{A}))$.
- T_{S_v}: the spherical twist functor on $\check{\mathcal{D}}_{A,\Lambda}$ associated to S_v.
- $\mathrm{Br}(\check{\mathcal{D}}_{A,\Lambda}) := \langle T_{S_1}, \ldots, T_{S_{\mu_A}} \rangle \subset \mathrm{Auteq}(\check{\mathcal{D}}_{A,\Lambda})$.

Theorem 7

The correspondence $\widetilde{g}_v \mapsto T_{S_v}$ for $v \in \widetilde{T}_A$ induces

$$G(\widetilde{T}_A) \twoheadrightarrow \mathrm{Br}(\check{\mathcal{D}}_{A,\Lambda}).$$

Key facts for Theorem 7

Proposition 8 (Seidel–Thomas)

1. *For any spherical objects S and S', we have*

$$T_S T_{S'} \cong T_{T_S S'} T_S.$$

2. *If* $\mathbb{R}\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(S', S) \cong k[-1]$, *we have*

$$T_S T_{S'} S \cong S'.$$

Lemma 9

For $1 \leq i < j \leq r$, *we have*

$$\mathbb{R}\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(S_{1^*}, T_{(i,1)} S_1) \cong k[-2], \quad \mathbb{R}\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(T_{(i,1)} S_1, S_{1^*}) \cong k,$$

$$\mathbb{R}\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(S_{(i,1)}, T_1 T_{1^*} S_{(j,1)}) \cong 0, \quad \mathbb{R}\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(T_1 T_{1^*} S_{(j,1)}, S_{(i,1)}) \cong 0.$$

Conjecture

Conjecture

The group homomorphism $G(\widetilde{T}_A) \twoheadrightarrow \mathrm{Br}(\check{\mathcal{D}}_{A,\Lambda})$ *in Theorem 7 should also be injective, and hence isomorphism. In other words, the space of stability condition* $\mathrm{Stab}(\check{\mathcal{D}}_{A,\Lambda})$ *should be simply connected.*