Analytic construction of the Quantum invariants of singularity (FJRW invariant), Gauged Witten equation and analytic construction of LG B model

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Part I: Analytic construction of FJRW invariants

(with Jarvis and Ruan)

Definition 0.1

A quasi-homogeneous polynomial $W(x_1, \cdots, x_N)$ satisfies

$$W(\lambda_1^{q_1} x_1, \cdots, \lambda_N^{q_N} x_N) = \lambda W(x_1, \cdots, x_N),$$

where q_i are the weights of x_i.

W is called nondegenerate if

(1) W contains no monomial of the form $x_i x_j$ for $i \neq j$ and

(2) the hypersurface defined by W is non-singular in projective space.

Theorem 0.2 (K. Saito)

If W is a non-degenerate quasi-homogeneous polynomial, then the weights $q_i \leq \frac{1}{2}$ and are uniquely determined.

Theorem 0.3 (Fan-Jarvis-Ruan, CPAM 2008)

Let $W \in \mathbb{C}[x_1, \ldots, x_N]$ be a non-degenerate, quasi-homogeneous polynomial with weights $q_i := \mathrm{wt}(x_i) < 1$ for each variable $x_i, i = 1, \ldots, N$. Then for any t-tuple $(u_1, \ldots, u_N) \in \mathbb{C}^N$ we have

$$|u_i| \leq C \left(\sum_{i=1}^{N} \left| \frac{\partial W}{\partial x_i}(u_1, \ldots, u_N) \right| + 1 \right)^{\delta_i},$$

where $\delta_i = \frac{q_i}{\min_j(1-q_j)}$ and the constant C depends only on W. If $q_i \leq 1/2$ for all $i \in \{1, \ldots, N\}$, then $\delta_i \leq 1$ for all $i \in \{1, \ldots, N\}$. If $q_i < 1/2$ for all $i \in \{1, \ldots, N\}$, then $\delta_i < 1$ for all $i \in \{1, \ldots, N\}$.

Proof.

Use algebraic geometry and matrix analysis. □

Lemma 0.4

If W is non-degenerate, then the group

$$G_W := \{(\alpha_1, \ldots, \alpha_N) \in (\mathbb{C}^*)^N \mid W(\alpha_1 x_1, \ldots, \alpha_N x_N) = W(x_1, \ldots, x_N)\}$$

of diagonal symmetries of W is finite.

Definition 0.5

We write each element $\gamma \in G_W$ (uniquely) as

$$\gamma = (\exp(2\pi i \Theta_1^\gamma), \ldots, \exp(2\pi i \Theta_N^\gamma)),$$

with $\Theta_i^\gamma \in [0, 1) \cap \mathbb{Q}$.

$$J := (\exp(2\pi i q_1), \ldots, \exp(2\pi i q_N)),$$

The cyclic group $\langle J \rangle$ will play important roles.

Definition 0.6

Orbicurve $(\mathscr{C}, p_1, \cdots, p_k)$: possible orbifold structure at p_i or nodal points; A uniformizing system is given by $z \to z^{m_i}$; Local group $G_{p_i} \cong Z_{m_i}$.
If $\mathscr{L}$ is a orbifold line bundle and (z,s) is the local coordinates of the uniformizing system near p_i, the action of G_{p_i} is $(z,s) \to (\exp(2\pi i/m_i)z, \exp(2\pi iv/m_i)s)$.
We can naturally define the group action at the nodal points.

Definition 0.7

A W-curve $\mathfrak{C} = (\mathscr{C}, p_1, \cdots, p_k, \mathscr{L}_1, \cdots, \mathscr{L}_N, \varphi_1, \cdots, \varphi_s)$ is a genus g orbicurve $\mathscr{C}$, having k marked points and with the W-structure $(\mathscr{L}_1, \cdots, \mathscr{L}_N, \varphi_1, \cdots, \varphi_s)$. A W structure means that the orbifold line bundles $\mathscr{L}_1, \cdots, \mathscr{L}_N$ should satisfy the isomorphisms:

$$\varphi_i : W(\mathscr{L}_1, \cdots, \mathscr{L}_N) \to K_{log} = K_{\mathscr{C}} \otimes (\mathscr{O}(p_1)) \otimes \cdots (\mathscr{O}(p_k)), \forall i = 1, \cdots, s.$$

- **broad line bundles and narrow line bundles**: If the orbifold action of a line bundle $\mathscr{L}_i$ at a marked (or nodal) marked point p is trivial, i.e., $\Theta_i^\gamma = 0$, then $\mathscr{L}_i$ is called a broad line bundles at p; Otherwise it is called the narrow line bundle at p.
- If all the line bundles at a marked (or a nodal) point p are narrow line bundles, then p is called a narrow point; otherwise it is called a broad marked (or nodal) point.
- Let $\gamma \in G$ be the generator of the local group at p, then it has the action to $\mathbb{C}^N$. Define $W_\gamma = W|_{\mathbb{C}^N_\gamma}$.
- (explain here why one needs "orbifold" structure and do a summary)

3. Moduli space of W curves

- $\overline{\mathscr{M}}_{g,k} := \{(\mathscr{C}, p_1, \cdots, p_k)\}$: moduli space of genus g curves with k marked points.
- $\overline{\mathscr{W}}_{g,k}(\gamma) := \{(\mathfrak{C})\}$: moduli space of W-curves $\mathfrak{C}$.It is a stratified space

Natural maps

$$st : \overline{\mathscr{W}}_{g,k}(\gamma) \to \overline{\mathscr{M}}_{g,k}\ \text{, (forgetting map)}\theta : \overline{\mathscr{W}}_{g,k+1,W}(\gamma, J) \to \overline{\mathscr{W}}_{g,k}(\gamma)$$

and other cutting-gluing operations.
taughtological ring in $H^*(\overline{\mathscr{W}}_{g,k}(\gamma))$. ψ_i, κ_i classes and etc.

Theorem 0.8

For any non-degenerate, quasi-homogeneous polynomial W, the stack $\overline{\mathscr{W}}_{g,k}$ is a smooth, compact orbifold (Deligne-Mumford stack) with projective coarse moduli. In particular, the morphism $st : \overline{\mathscr{W}}_{g,k} \to \overline{\mathscr{M}}_{g,k}$ is flat, proper and quasi-finite (but not representable).

Rigidified W curve

$\overline{\mathscr{W}}_{g,k}(\gamma)$ is not an appropriate working space for analysis, We need consider the moduli space of the rigidified W curves, $\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\gamma)$. It is a branched covering space of $\overline{\mathscr{W}}_{g,k}(\gamma)$.

Definition 0.9

A rigidification ψ at a marked point p is a local trivialization of the orbifold structure such that it preserves the W-structure, i.e., the diagram commutes

$$\begin{tikzcd} j_p^*\left(\bigoplus_{m=1}^{N} \mathscr{L}_m\right) \arrow[r, "\psi"] \arrow[d, "\phi_\ell \circ W_\ell"'] & {[\mathbb{C}^N/G_p]} \arrow[d, "W_\ell"] \\ j_p^*(K_{log}) \arrow[r, "residue"] & \mathbb{C} \end{tikzcd} \tag{1}$$

where the residue map takes $\frac{dz}{z}$ to 1.

Definition 0.10

A rigidified W-curve is defined as $\mathfrak{C} := \{\mathscr{C}, p_1, \cdots, p_k, \mathscr{L}_1, \mathscr{L}_N, \varphi_1, \cdots, \varphi_s, \psi_1, \cdots, \psi_k\}$, where ψ_i are rigidification at marked point p_i. The moduli space of the rigidified W-curves is the equivalence of those $\mathfrak{C}$, and is denoted by $\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\gamma)$.

4. Witten map and its perturbation

Metric choices: Choose cylindric metric near a marked or nodal point, i.e., let $|\frac{dz}{z}| = 1$. This metric will induce metrics of line bundles $\mathscr{L}_i$ by W-structure. Let $\mathfrak{C} = (\mathscr{C}, \mathscr{L}, \Psi)$ be a rigidified W-curve with the cylindrical metric. We can define a metric-preserving map:

$$\tilde{I}_1 : \Omega(\tilde{\Sigma}, \bar{\mathscr{L}}_j^{-1} \otimes \Lambda^{0,1}) \to \Omega(\tilde{\Sigma}, \mathscr{L}_j \otimes \Lambda^{0,1}).$$

Now the Witten equation on orbifolds is defined as

$$\widetilde{WM}(\mathfrak{C}, \mathbf{u}) := \bar{\partial} u_i + \tilde{I}_1 \left(\overline{\frac{\partial W}{\partial u_i}} \right) = 0, \forall i = 1, \cdots, N.$$

- Stratified Orbifold Fréchet bundles B^0 and $B^{0,1}$ over $\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\boldsymbol{\gamma})$: The fiber spaces at $\mathfrak{C}$ are:

$$C^\infty(\mathscr{C},\mathscr{L}_j) := \{(u_{j,\nu}) \in \oplus_\nu C^\infty(\mathscr{C}_\nu,\mathscr{L}_j) | u_{j,\nu}(p_\nu) = u_{j,\mu}(p_\mu),\ \text{if}\ \pi_\nu(p_\nu) = \pi_\mu(p_\mu)\}.$$

$$C^\infty(\mathscr{C},\mathscr{L}_j \otimes \Lambda^{0,1}) := \{(u_{j,\nu}) \in \oplus_\nu C^\infty(\mathscr{C}_\nu,\mathscr{L}_j \otimes \Lambda^{0,1})\}.$$

Now the Witten map $\widetilde{WM}$ is viewed as a section from B^0 to $\pi^* B^{0,1}$.
Some difficulties:

- In uniformizing system, the Witten map is G-equivariant.
- W is a highly degenerate Hamiltonian function
- Can't perturb $\widetilde{WM}$ directly.

Perturbed Witten map as multisection

We can modify the Witten map near each broad marked or nodal point such that it looks like

$$\bar{\partial}u_i + \tilde{I}_1\left(\overline{\frac{\partial W + W_{0,\gamma}}{\partial u_i}}\right) = 0, \text{if } u_i \text{ is broad variable.}$$

Where $W_{0,\gamma}$ is a linear perturbation of W_γ such that $W_{0,\gamma} + W_\gamma$ is a holomorphic morse function. The perturbation depends on the parameter $\mathbf{b} = (b_1, \cdots, b_N)$.
Since the perturbation will break the G-equivariance, the perturbed Witten section $WI : B^0 \to B^{1,0}$ is a **multisection**!.

- The perturbation parameters for the two components connecting at a nodal point p should satisfy some compatibility condition.
- Define the perturbed Witten map globally over $\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}$ by partition of unity.

5. Nonlinear analysis for the perturbed Witten equation

Note: This system is a non-conformal nonlinear Cauchy-Riemann system. So the analysis here is **different** to that for 4d Yang-Mills equation and pseudo-holomorphic curves. It is also different from the Seiberg-Witten equation, whose solutions have C^0-norm estimate.
Near the marked points, it is same as the trajectory equation in Floer's theory, but in the interior it is like the semilinear elliptic equations.

- **Interior estimate**: $\|\mathbf{u}\|_{C^k} \leq C$.(Use the crucial inequality in Lemma 0.3)
- **Convergence at marked or nodal points**: $\mathbf{u} = (u_1, \cdots, u_N) \to \kappa_i$ as $z \to p$, where κ_i is one of the critical points of $W_\gamma + W_{0,\gamma}$.
- **Exponential convergence**

6. Soliton space

Consider the equation on $(\mathbb{R}^1 \times S^1, -\infty, 0, +\infty, \gamma, *, \gamma^{-1}, \psi)$. (In cylinder co-ordinates $\zeta = s + i\theta$)

$$\frac{\bar{\partial} u_i}{\partial \bar{\xi}} - 2\overline{\frac{\partial (W + W_{0,\gamma})}{\partial u_i}} = 0.$$

Definition 0.11

The nontrivial solution is called **Soliton solution**, and if the solution is also independent of the angle, then it is called BPS-Soliton.

Notice: The BPS soliton is S^1-invariant solution (bring the cone structure near the possible boundary).

If $\mathbf{u}$ is a solution, then

$$(W_\gamma + W_{0,\gamma})(\kappa^-) - (W_\gamma + W_{0,\gamma})(\kappa^+) = 2\int_{-\infty}^{+\infty}\int_{S^1}\sum_i \left|\frac{\partial(W + W_{0,\gamma})}{\partial u_i}\right|^2.$$

Conclusions:

Only if we choose the perturbation parameter $\mathbf{b}$ for γ such that for two different critical points κ^+ and κ^- for $W_\gamma + W_{0,\gamma}$, the following holds

$$\text{Im}(W_\gamma + W_{0,\gamma})(\kappa^i) = \text{Im}(W_\gamma + W_{0,\gamma})(\kappa^j).$$

there exists soliton solutions.

Definition 0.12

If $\mathbf{b}$ is chosen such that for all $\gamma \in G$, $W_\gamma + W_{0,\gamma}$ is a holomorphic morse function, the perturbation is called regular; for regular $\mathbf{b}$, if for any γ the above equality does not hold, the perturbation is called strongly regular.

Theorem 0.13

The regular but not strongly regular parameters $\mathbf{b}$ *consists of a generic set in finite real codimension 1 hypersurfaces which separate* $\mathbb{C}^N$ *into chambers.*

Compactness Loss: If the moduli space $\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\gamma,\varkappa)$ is strongly regular perturbed, then there is no loss of compactness. If it is regular perturbed but not strongly regular perturbed, then there is **compactness loss phenomena** due to the existence of **soliton solutions.** In the latter case, we need add the "**soliton W sections**" into our moduli space. So We need to consider a larger space $\overline{\mathscr{W}}^{\mathrm{rig,s}}_{g,k}(\gamma,\varkappa)$.

- $\overline{\mathscr{W}}^{\mathrm{rig,s}}_{g,k}(\gamma,\varkappa)$
  is a stratified space stratified by the decorated dual graphs.
- We can define the Gromov-Hausdorff topology such that $\overline{\mathscr{W}}^{\mathrm{rig,s}}_{g,k}(\gamma,\varkappa)$ is a compact Hausdorff space.

- If $\overline{\mathscr{W}}^{\mathrm{rig,s}}_{g,k}(\boldsymbol{\gamma},\boldsymbol{\varkappa}) = \overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\boldsymbol{\gamma},\boldsymbol{\varkappa})$ (strongly regular perturbed), then it is a space "without boundary".
- If the perturbation is regular but not strongly regular perturbed, then $\overline{\mathscr{W}}^{\mathrm{rig,s}}_{g,k}(\boldsymbol{\gamma},\boldsymbol{\varkappa})$ is a space "with boundary" and the boundary is related to BPS soliton. Solitons but not BPS solitons appear in the interior of the moduli space(with finite automorphism group).

8. Fredholm theory and the virtual dimension

For any $\mathfrak{C} \in \overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\boldsymbol{\gamma})$ we have the Witten map:

$$WI_{\mathfrak{C}} : L^p_1(\mathscr{C}, \mathscr{L}_1 \times \cdots \times \mathscr{L}_N) \to L^p(\mathscr{C}, \mathscr{L}_i \otimes \Lambda^{0,1}),$$

which has the following form:

$$WI_{\mathfrak{C}}(\mathbf{u}) = (\bar{\partial}_{\mathscr{C}} u_1 + \tilde{I}_1\left(\overline{\frac{\partial(W + W_{0,\beta})}{\partial u_1}}\right), \cdots, \bar{\partial}_{\mathscr{C}} u_N + \tilde{I}_1\left(\overline{\frac{\partial(W + W_{0,\beta})}{\partial u_N}}\right)).$$

Here the perturbation term $W_{0,\beta}$ has the form $\varpi(\zeta)\beta_i W_{0,\gamma}$ which is determined by the combinatorial type of $\mathscr{C}$ and the group element γ and the cut-off section β_i.

We will always set $p > 2$ in our discussion.
We have the linearized operator $D_{\mathfrak{C},\mathbf{u}}WI$ of $WI_{\mathfrak{C}}$ at $\mathbf{u}$:

$$D_{\mathfrak{C},\mathbf{u}}WI(\boldsymbol{\phi}) := D_{\mathfrak{C},\mathbf{u}}WI(\phi_1, \cdots, \phi_N) :=$$
$$\left(\bar{\partial}_{\mathscr{C}}\phi_1 + \sum_j \tilde{I}_1\left(\overline{\frac{\partial^2(W+W_{0,\beta})}{\partial u_1 \partial u_j}\phi_j}\right), \cdots, \bar{\partial}_{\mathscr{C}}\phi_N + \sum_j \tilde{I}_1\left(\overline{\frac{\partial^2(W+W_{0,\beta})}{\partial u_N \partial u_j}\phi_j}\right)\right). \tag{2}$$

$D_{\mathfrak{C},\mathbf{u}}WI$ is a map from $L^p_1(\mathscr{C}, \mathscr{L}_1 \times \cdots \times \mathscr{L}_N)$ to $L^p(\mathscr{C}, \mathscr{L}_1 \otimes \Lambda^{0,1}) \times \cdots L^p(\mathscr{C}, \mathscr{L}_N \otimes \Lambda^{0,1})$.

- The broad marked points and the narrow marked points have different contribution to the index !

Theorem 0.14

Let $(\mathfrak{C}, \mathbf{u}) \in \overline{\mathscr{W}}^{\mathrm{rig,s}}_{g,k,w}(\boldsymbol{\gamma}, \boldsymbol{\varkappa})$ *and assume that* $\mathfrak{C}$ *is connected. Then its linearized operator* $D_{\mathfrak{C},\mathbf{u}} WI : L^p_1(\mathscr{C}, \mathscr{L}_1 \times \cdots \times \mathscr{L}_N) \to L^p(\mathscr{C}, \mathscr{L}_1 \otimes \Lambda^{0,1}) \times \cdots \times L^p(\mathscr{C}, \mathscr{L}_N \otimes \Lambda^{0,1})$ *is a real linear Fredholm operator of index* $2\hat{c}_W(1-g) - \sum_\tau 2\iota(\gamma_\tau) - \sum_{\tau=1}^k N_{\gamma_\tau}$*, where* $\hat{c}_W = \sum_i (1 - 2q_i)$*,* $\iota(\gamma_\tau) = \sum_i (\Theta_i^{\gamma_\tau} - q_i)$ *and* $N_{\gamma_\tau} = \dim \mathbb{C}^N_{\gamma_\tau}$ *(if* $\mathbb{C}^N_{\gamma_\tau} = \{0\}$*, we set* $N_{\gamma_\tau} = 0$*).*

Corollary 0.15

Let $(\mathbf{u}_{j_1,j_2}, \gamma) \in S_\gamma(\kappa^{j_1}, \kappa^{j_2})$*. Then the linearized operator* $D_{\mathbf{u}_{j_1,j_2}}(WI)$ *is a real linear Fredholm operator of index* 0 *on* $\mathbb{R} \times S^1$*.*

9. Orientable Kuranishi structure and virtual fundamental cycles: Fukaya-Ono's machinery (developed later by Fukaya-Oh-Ono-Ota)

- Remark: many other virtual constructions by Li-Tian, Ruan, Siebert,Hofer Chen....
- **Construction of the interior Kuranishi neighborhood**: $(U_\sigma, E_\sigma, s_\sigma, \Psi_\sigma)$ By modified Implicit functional theorem and a priori estimates for the solutions. (No transversality)
- **Construction of the Kuranishi nbhd. on boundary**: Study the BPS soliton carefully including computation of the obstruction bundles (2 cases: Tree and Loop cases).
- **Gluing** Glue the K-nbhd from the lower strata and choose the suitable obstruction bundle on the overlaps.
- **orientation** Show that the Kuranishi structure is orientable and coherent, i.e., the orientation should respect the gluing operation. It is much the same to the treatment of Floer's Hamiltonian trajectory.

- By Fukaya-Ono's machinery, we can get the virtual fundamental cycle $[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\boldsymbol{\gamma},\boldsymbol{\varkappa})]^{vir} \in H_*(\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\boldsymbol{\gamma},\boldsymbol{\varkappa}))$ if the perturbation is strongly regular. Its (real) dimension is $6g-6+2k-2D-\sum_{i=1}^{k} N_{\gamma_i} = 2((\hat{c}_W-3)(1-g)+k-\sum_{\tau=1}^{k}\iota(\gamma_\tau))-\sum_{i=1}^{k} N_{\gamma_i}$.
- We can define the boundary cycles $[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\Gamma)]^{vir}$ w.r.t. the decorated dual graph Γ.
- If two perturbation parameter $\mathbf{b}$ and $\mathbf{b}'$ are in the same chamber of $\mathbb{C}^N$, then the corresponding virtual cycles are the same. Hence the virutal cycles actually depends on the vanishing cycles (or Lefschetz thimbles) of $W_\gamma + W_{0,\gamma}$.

Quantum Picard-Lefschetz theory:Wall-crossing formula

If take a path $\mathbf{b}(\lambda)$ go through the wall of the chamber, then the virtual cycles will transform in the same way as the corresponding vanishing cycles (Lefschetz thimbls) change in the classical Picard-Lefschetz theory.
Let $(b_1(\lambda), \cdots, b_{N_{\tilde{\gamma}}}(\lambda)), \lambda \in [-1, 1]$ be a generic crossing path in $\mathbb{C}^{N_{\tilde{\gamma}}}$.
Let $\{\kappa^1(\pm), \cdots, \kappa^i(\pm), \kappa^{i+1}(\pm), \cdots, \kappa^{\mu_{N_{\tilde{\gamma}}}}(\pm)\}$ be the set of ordered critical points at $\lambda = \pm 1$. We can assume that $\kappa^j(\pm) = \kappa^j$ is fixed for $j \neq i$, $\kappa^i(\pm) = \kappa^i(\lambda = \pm 1)$ and $\mathrm{Im}(\alpha^i(\lambda = 0)) = \mathrm{Im}(\alpha^{i+1})$.
If the perturbation satisfies $\mathrm{Re}\alpha^i(\lambda) < \mathrm{Re}\alpha^{i+1}$, we have the left-transformation:

$$[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^j(+))]^{vir} = [\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^j(-))]^{vir}, \ \forall j \neq i, i+1 \tag{3}$$

$$[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^i(+))]^{vir} = [\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^{i+1}(-))]^{vir} + R_{i,i+1} \cdot [\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^i(-))]^{vir} \tag{4}$$

$$[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^{i+1}(+))]^{vir} = [\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^i(-))]^{vir}, \tag{5}$$

where $R_{i,i+1}$ is the intersection number defined as above.

Wall-crossing formula

If the perturbation satisfies Re$\alpha^i(\lambda) >$ Reα^{i+1}, we have the right-transformation:

$$[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^j(+))]^{vir} = [\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^j(-))]^{vir}, \ \forall j \neq i, i+1 \tag{6}$$

$$[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^i(+))]^{vir} = [\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^{i+1}(-))]^{vir}, \tag{7}$$

$$\begin{aligned}[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^{i+1}(+))]^{vir} &= [\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^i(-))]^{vir} + \\ R_{i,i+1} \cdot [\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^{i+1}(-))]^{vir} &\end{aligned} \tag{8}$$

Definition of the virtual cycle $[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\boldsymbol{\gamma})]^{vir}$

Fix $\boldsymbol{\gamma} = \{\gamma_1, \cdots, \gamma_k\}$ and choose the moduli space $\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\boldsymbol{\gamma}, \boldsymbol{\varkappa})$ to be strongly regular. For each $\gamma \in G$, choose the basis $\{S_j^-(\gamma), j = 1, \cdots, \mu_\gamma\}$ in $H_{N_\gamma}(\mathbb{C}^N_\gamma, (W\gamma + W_{0,\gamma})^{-\infty}, \mathbb{Q})$ corresponding to the critical points of $W_\gamma + W_{0,\gamma}$ and the dual basis $\{S_j(\gamma), j = 1, \cdots, \mu_\gamma\}$ in $H_{N_\gamma}(\mathbb{C}^N_\gamma, (W_\gamma + W_{0,\gamma})^{\infty}, \mathbb{Q})$. Then each combination $\left(S^-_{j_1}(\gamma_1), \cdots, S^-_{j_k}(\gamma_k)\right)$ corresponds to the combination of k critical points, $\boldsymbol{\varkappa}_{j_1\cdots j_k} := \left(\kappa^-_{j_1}(\gamma_1), \cdots, \kappa^-_{j_k}(\gamma_k)\right)$. We obtain the virtual cycle $[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\Gamma; \boldsymbol{\gamma}, \boldsymbol{\varkappa}_{j_1\cdots j_k})]^{vir}$
$=: [\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\Gamma; \boldsymbol{\gamma}, S^-_{j_1}(\gamma_1), \cdots, S^-_{j_k}(\gamma_k))]^{vir}$. Now we fix a strongly regular parameter (b_i^0); the Gauss-Manin connection provides the isomorphisms

$$GM_{(b_i^0)} : H_{N_\gamma}(\mathbb{C}^N_\gamma, (W\gamma + W_{0,\gamma})^{\pm\infty}, \mathbb{Q}) \to H_{N_\gamma}(\mathbb{C}^N_\gamma, (W_\gamma)^{\pm\infty}, \mathbb{Q})$$

Using the isomorphisms we can identify $H_{N_\gamma}(\mathbb{C}^N_\gamma, (W\gamma + W_{0,\gamma})^{\pm\infty}, \mathbb{Q})$ with $H_{N_\gamma}(\mathbb{C}^N_\gamma, (W_\gamma)^{\pm\infty}, \mathbb{Q})$.

Define

$$\left[\overline{\mathscr{W}}^{\text{rig}}_{W}(\Gamma)\right]^{vir} := \sum_{j_1,\cdots,j_k}\left(\left[\overline{\mathscr{W}}^{\text{rig}}_{g,k,W}(\Gamma;\boldsymbol{\gamma},\boldsymbol{\varkappa}_{j_1\cdots j_k})\right]^{vir}\otimes\prod_{i=1}^{k}S_{j_i}(\gamma_i)\right) \tag{9}$$

$$\in H_*(\overline{\mathscr{W}}^{\text{rig}}_{g,k}(\Gamma))\otimes\prod_{\tau\in T(\Gamma)}H_{N_{\gamma_\tau}}(\mathbb{C}^N_{\gamma_\tau},W^{\infty}_{\gamma_\tau},\mathbb{Q}) \tag{10}$$

By the Wall-crossing formula, We have

Proposition 0.16

The virtual cycle $\left[\overline{\mathscr{W}}^{\text{rig}}_{W}(\Gamma)\right]^{vir}$ *is independent of the choice of the basis* $\{S_{j_i}(\gamma_i)\}$ *of* $H_{N_\gamma}(\mathbb{C}^N_\gamma,(W_\gamma)^{\pm\infty},\mathbb{Q})$ *at each marked point* p_i.

Since the parallel transport induced by the Gauss-Manin connection preserves the inner product of the homology bundle, the above proposition justifies the definition of the virtual cycle $\left[\overline{\mathscr{W}}^{\text{rig}}_{W}(\Gamma)\right]^{vir}$.

Definition of the virtual cycle $[\overline{\mathscr{W}}_{g,k}(\gamma)]^{vir}$

Define

$$[\overline{\mathscr{W}}_W(\Gamma)]^{vir} := \frac{1}{\deg so_\Gamma}(so_\Gamma)_*[\overline{\mathscr{W}}^{\mathrm{rig}}_W(\Gamma)]^{vir},$$

where

$$so_\Gamma : \overline{\mathscr{W}}^{\mathrm{rig}}_W(\Gamma) \to \overline{\mathscr{W}}_W(\Gamma)$$

is the soften map. In particular, one has

$$[\overline{\mathscr{W}}_W(\gamma)]^{vir} := \frac{1}{\deg so}(so)_*[\overline{\mathscr{W}}^{\mathrm{rig}}_W(\gamma)]^{vir}.$$

Theorem 0.17

The virtual cycle $[\overline{\mathscr{W}}_{g,k}(\gamma)]^{vir}$ satisfies the CohFT axioms.

12. Some words about algebraic construction

- Algebraic construction of virtual cycle for purely narrow case. Contribution by many mathematicians:
- For broad case, it is still mysterious. People must understand how to use Hironaka's resolution of singularity and understand the totally real structure appeared in the picture.
- Real structure should be understood as the $\mathbb{Z}_2$ symmetry of algebraic structure, after modulo such $\mathbb{Z}_2$ equivalence, can one get the right index formula and algebraic construction.
- A challenge for algebraic geometers, but doable.
- Geometry and topology of $\overline{\mathscr{W}}_{g,k}$ stack is a very interesting object to be studied.

Part II: Gauged Linear Sigma Model via gauged Witten equation

(with Jarvis and Ruan)

1. Symplectic quotient (GIT) and W-structure

- $V \cong \mathbb{C}^n$, $\mathbb{C}^*_{\mathbb{R}}$ action $(z_1, \cdots, z_n) \mapsto (\lambda^{c_1} z_1, \cdots, \lambda^{c_n} z_n)$
- $G \subset GL(V, \mathbb{C})$ reductive algebraic group such that G and $\mathbb{C}^*_{\mathbb{R}}$ is compatible, i.e., they satisfy
  1. $\mathbb{C}^*_{\mathbb{R}} \subset N(G), G \subset N(\mathbb{C}^*_{\mathbb{R}})$;
  2. $G \cap \mathbb{C}^*_{\mathbb{R}} = \langle J \rangle$.
- gauge group $\Gamma = G \cdot \mathbb{C}^*_{\mathbb{R}}$ and canonical homomorphism $\zeta : \Gamma \mapsto \mathbb{C}^*$ by

$$g \cdot (\lambda^{c_1}, \cdots, \lambda^{c_n}) \mapsto \lambda^d.$$

- Superpotential $W : V \mapsto \mathbb{C}$ is of degree d w.r.t. $\mathbb{C}^*_{\mathbb{R}}$-action and invariant under the action of G. Then W is well-defined on the symplectic quotient $V_\tau = V//_\tau \Gamma$, where $\tau \in \eta := Lie(\Gamma)$ lying on possible different phase separated by the critical value of the moment map $\mu : V \mapsto \eta$.
- W-structure on a orbicurve $\mathscr{C}$ is a Γ-principle bundle $\mathscr{P} : \mathscr{C} \to B\Gamma$ such that (1) this map to the classifying stack is representale;(2)$\exists$ isomorphism $\epsilon : \zeta_*(\mathscr{P}) \cong P(\omega_{log,\mathscr{C}})$.

2. Gauged Witten equation and interior compactness

- Take a representation $\rho : \Gamma \to V$. We have the associated bundle $\mathscr{E} = \mathscr{P} \times_\Gamma V$ and the dual bundle $\mathscr{E}^*$. Fix a hermitian metric on E which provide an isomorphism $I : \mathscr{E}^* \to \mathscr{E}$.
- Since $W : V \to \mathbb{C}$ is Γ-equivariant and G-invariant, it provides a bundle homomorphism $W : \mathscr{E} \to \omega_{log,\mathscr{C}}$. the differential $d_u W(u)$ along the section u is a linear map $d_u W(u) : \mathscr{E} \to \omega_{log,\mathscr{C}}$, i.e., $d_u W(u) \in \mathscr{E}^* \otimes \omega_{log,\mathscr{C}}$, $\Longrightarrow I(d_u W(u)) \in \mathscr{E} \otimes \omega_{log,\mathscr{C}}$. Hence we have the first equation of the Gauged Witten equation:

$$\bar{\partial}_A u + I(d_u W(u)) = 0,$$

where A is any connection.
- On the other hand, Given any Γ-connection A on $\mathscr{E}$, the curvature F_A is the η-valued 2-form and we have the connection equation:

$$*F_A = \mu(u)$$

Hence we have the Gauged Witten equation:

$$\begin{cases} \bar{\partial}_A u + I(d_u W(u)) = 0 \\ *F_A = \mu(u) \end{cases}$$

- It is different to the symplectic vortex equation.
- GWE is invariant under the gauge group provided by the structure group Γ.
- (Interior compactness theorem) Assume that $\Gamma \subset U(n)$ and $\mu(u)$ has at most polynomial growth. Under an extra gauge condition $d^*A = 0$, all the solution (A, u) of the gauged Witten equation satisfying the gauge condition have uniform C^m bound in the interior of $\mathscr{C}$ away from the orbifold points.

Proof.

A brief proof. Using the technique from FJRW theory, from equation (1) one obtain the estimate(local):

$$||u||_{W^{1,2}} \leq C||A||_{L^2} + C$$

By the second and the third equation, essentially using the Hodge decomposition($d + d^*$), one obtain

$$||A||_{W^{1,2}} \leq C(||\mu(u)||_{L^2} + ||A \wedge A||_{L^2} + ||A||_{L^2}).$$

Now by Sobolev embedding theorem, one get the $W^{1,2}$ norm estimate of A and then u. By bootstrapping technique, one obtain any C^m norm estimate. □

3. Virtual cycle for the narrow sector

In this case, all the marked points are narrow. then $d_u W(u)$ will provide extra vanishing information such that the second term of the first equation of GWE lying on $\mathscr{E} \otimes \omega_{\mathscr{C}}$. Now by Witten lemma, this equation is decouple into

$$\bar{\partial}_A u = 0, d_u W(u) \equiv 0,$$

which shows that u is A-holomorphic section whose image is in the critical locus of W. The second equation of GWE become the Hitchin system, which by Hitchin-Kobayashi correspondence, should correspond to the stable bundle structure on $\mathscr{C}$. Hence in narrow case, we are in the situation of quasi-map,cosection, and etc. by Ciocan-Fontanine and B, Kim, B. Kim, Kiem-Li, Li-Zhang, Li-Li-Zhang,...

Theorem 0.18

The virtual cycle for the narrow sector can be constructed algebraically.

Part III: LG B model: study by analytic method

(by H. Fan)

1. Aim: Using P. D. E. technique to understand LG B model, in particular Saito's flat structure for singularity.
2. This work was based on the pioneer work of Cecotti, Cecotti-Vafa (around 1990) and Losev (1998).
3. Other method: i) K. Saito's original way by singularity theory ii) Li-Li-Saito's method using polyvector field.
4. Paper: Schrödinger equation, deformation theory and tt^* geometry, arxiv.1107.1290, 114 pages
5. I was informed by Si Li in this conference that S. Klimek and A. Lesniewski has the following paper related to partial of my work:
   Local Rings of Singularities and N= 2 Supersymmetric Quantum Mechanics, Comm. Math. Phys. 136,327-344 (1991)

1.Differential operators

- Initial data: (M, g) non-compact complete K ahler manifold with bounded geometry. f: holomorphic function on M.
- twisted operator: $\bar{\partial}_f = \bar{\partial} + \partial f \wedge, \partial_f = \partial + \overline{\partial f} \wedge, \bar{\partial}_f^\dagger, \partial_f^\dagger, \Delta_f = \bar{\partial}_f \bar{\partial}_f^\dagger + \bar{\partial}_f^\dagger \bar{\partial}_f$...
- Locally, $\Delta_f = \Delta_{\bar{\partial}} + L_f + |\partial f|^2$ is a matrix-valued Schrödinger operator on M. L_f depends on $\nabla \partial f$ linearly.
- Strongly tame condition of (M, g, f): $\forall C > 0$, $|\partial f|^2 - C|\nabla \partial f| \to \infty$, as $d(z, z_0) \to \infty$.
- Examples: $(\mathbb{C}^n, \frac{i}{2} \sum_l dz^l \wedge dz^{\bar{l}}, W), ((\mathbb{C}^*)^n, \frac{i}{2} \sum_l \frac{dz^l}{z^l} \wedge \frac{dz^{\bar{l}}}{z^{\bar{l}}}, f)$, where W is non-degenerate quasi-homogeneous polynomial and f is non-degenerate and convinient Laurent polynomial. They are strongly tame!. E.g. $f = z_1 + \cdots + z_n + \frac{1}{z_1 \cdots z_n}$.

2. Spectrum theorem, variation of Hodge structure and stability theorem

Theorem 0.19

If (M, g, f) is strongly tame, then Δ_f has purely discrete spectrum.

Theorem 0.20

Hodge theorem, Hard Lefschetz theorem hold and we obtain the Dolbeaut isomorphism between the space of L^2 harmonic forms and the L^2 $\bar{\partial}_f$ cohomology.

Theorem 0.21

If (M, g, f) is strongly tame and M is stein manifold, then the $L^2 - \bar{\partial}_f$ cohomology is isomorphic to the space of Lefschetz thimble of f, and then isomorphic to the Milnor ring of f

We can define and study the "strong deformation" $\tau f_t, (\tau, t) \in \mathbb{C}^* \times S \subset \mathbb{C}^* \times \mathbb{C}^m$.

Proposition 0.22

1. *Strong deformation of a non-deg. quasi-homo. pol. W contains all marginal and relevant deformation*
2. *strong deformation of a non-deg. and convenient Laurent pol. contains all of its subdiagram deformation.*
3. *universal deformation of all singularities with central charge $\hat{C}_W \leq 1$.*

We go ahead:

- after strong deformation, we have $\bar{\partial}_{\tau f_t}, \Delta_{\tau f_t}, \ldots . \Longrightarrow$ Hodge bundle $\mathscr{H} \to \mathbb{C}^* \times S$
- Stability theorem: the Green function $G_{\tau,t}$ of $\Delta_{\tau f_t}$ depends smoothly on τ, t.
- We can endow geometrical structure to $\mathscr{H}$: hermitian metric h, metric connection ∇, Higgs field $C, \bar{C}$ and flat connection $\mathscr{D} = \nabla + C + \bar{C} \Longrightarrow$ deformation of holomorphic bundle $\mathscr{H}$ after gauge transformation and a family of Chern connections $\nabla^G + \bar{\partial}$.

- Solve $\mathscr{D} \cdot s = 0$ via the Hodge theory, we obtain the period matrix: the fundamental matrix $\Pi = (\Pi_1, \cdots, \Pi_\mu)$.
- the first column vector of Π, denote by Π_1 has the form

$$\Pi_1 = \begin{pmatrix} \tau^{\frac{n}{2}-1} \int_{\Gamma_1} e^{\tau f_t + \overline{\tau f_t}} dz^1 \wedge \cdots dz^n \\ \vdots \\ \tau^{\frac{n}{2}-1} \int_{\Gamma_\mu} e^{\tau f_t + \overline{\tau f_t}} dz^1 \wedge \cdots dz^n \end{pmatrix}$$

  We call it "primitive vector", because we have $\Pi_j = \partial_j \Pi_1$ for $j = 2, \cdots, \mu$.
- If $\dim S = m = \mu$, then we have an isomorphism $\hat{\Pi} : TS \to \mathscr{H}$ by

$$X \to \iota_X(B) \cdot \Pi_1.$$

- Then we pull back all the structure from $\mathscr{H}$ to TS, and let $\bar{\tau} \to 0$, then we obtain the Frobenius manifold structure on the holomorphic tangent bundle TS.

Remark 0.23

1. I believe $\widehat{\Pi_1}|_{\bar\tau} = 0$ corresponds to Saito's primitive form
2. If we do not let $\bar\tau \to 0$, then $\widehat{\Pi_1}$ is only C^∞, and in this case, the pull-back metric has a flat torsion free Chern connection defined on TS, i.e., the Kähler flat metric.
3. the hermitian metric h on $\mathscr{H}$ is just the residue pairing plus the real structure, and I believe the pull-back pairing is just Saito's higher residue pairing if we expand τ.
4. In another word, Saito's higher residue pairing is part of a Kähler-Ricci flat metric, i.e., a Calabi-Yau metric.
5. If we consider the o.d.e w.r.t the coupling constant τ, then we get a isomonodromic deformtion, which is related to Riemann-Hilbert problem.

3. Recent progress and potential applications

- Generalize to the operator $\bar{\partial}_\theta$ twisted with a closed holomorphic 1-form θ and generalize the "strong tame condition" to the tame condition:

$$\lim_{d\to\infty}(|\theta|^2 - C|\nabla\theta|) > 0, \text{for some } C > 0.$$

- Spectrum theorem: under the tame condition about (M, g, θ), the continue spectrum of Δ_θ exists but has a positive lower bound.$\Longrightarrow$ Hodge theory holds and all argument can almost go through.
- potential application 1: If consider system $(\mathbb{C}^n, \frac{i}{2}\sum_l dz^l \wedge dz^{\bar{l}}, W)$, then one can recover Saito's flat structure about singularity.
- potential pplication 2: if consider system $((\mathbb{C}^*)^n, \frac{i}{2}\sum_l \frac{dz^l}{z^l} \wedge \frac{dz^{\bar{l}}}{z^{\bar{l}}}, f)$, then one should get the result of GKZ system.
- potential application 3: If consider θ be the derivative of the log function of the hyperplane arrangement in affine space, then one should get KZ system. (The log system and its cohomology has been considered by K. Saito also)
- potential pplication 4: may apply to some topological insulator in physics.

4. Heat kernal expansion and analytic torsion of singularity

Cooperated with Hao Fang (iowa Univ.), we obtained the following results: Let W be a quasi-homo. pol on $\mathbb{C}^n$ satisfies $q_M - q_m < 1/3$, then

1. The heat kernel expansion of $e^{-t\Delta_f}$ exists when $t \to 0$
2. Index formula holds for $\bar{\partial}_f$ such that the milnor number can be expressed as a Gaussian type integral over $\mathbb{C}^n$.
3. The$p$-th Zeta function of Δ_f and the p-th analytic torsion was defined.
4. The first zeta function vanishes.
5. The second torsion is nontrivial and the torsion of z^2 is $\zeta_R'(-1)$. This is just the BCOV type torsion for LG model.
6. The proof of transgression formula is in progress

Thanks!