From Calabi–Yau dg Categories to Frobenius manifolds via Primitive Forms: a work in progress

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Kontsevich's Homological Mirror Symmetry

Conjecture 1

There should exist a pair of CY manifolds (X, Y) such that

$$D^b coh(X) \cong D^b Fuk(Y) \tag{1.1}$$

as triangulated categories.
Moreover, this should imply the isomorphism of formal $\mathbb{Z}$-graded Frobenius manifolds

$$H^\bullet(X, \wedge^\bullet \mathcal{T}_X) \cong H^\bullet(Y, \Omega^\bullet_Y). \tag{1.2}$$

LHS: Frob. str. from deformation theory (Barannikov–Kontsevich)
RHS: Frob. str. from Gromov–Witten theory

Problem

HMS leads the following

Problem 2

For a smooth compact Calabi–Yau A_∞-category $\mathcal{A}$, construct a Frobenius structure on $HH^{\bullet+2}(\mathcal{A})$ (the total Hochshild cohomology group of $\mathcal{A}$ shifted by two).

Key observation:

Proposition 3 (Gerstenhaber–Schack, Kontsevich)

Define the n-th Hochschild cohomology group of X by

$$HH^n(X) := Ext^n_{X\times X}(\mathcal{O}_X, \mathcal{O}_X). \tag{1.3}$$

There is an isomorphism of vector spaces

$$HH^n(X) \cong \bigoplus_{n=p+q} H^q(X, \wedge^p \mathcal{T}_X). \tag{1.4}$$

Aim of this talk

Explain an approach to associate Saito structures
(filtered de Rham cohomologies, Gauß–Manin connections and higher residue pairings) and primitive forms
to non-negatively graded smooth compact CY dg algebras.

Remark 4

Frobenius structure = Saito's flat structure.

Remark 5

Recently, Saito structures are also known as

- semi-infinite Hodge structures by Barannikov,
- TE(L)P structures by Hertling,
- nc-Hodge structures by Katzarkov–Kontsevich–Pantev.

Result

It is possible to translate literally Saito's theory of primitive form for weighted homog. polynomials into language of dg algebras up to the following problems to solve:

- we have to prove a kind of formality conjecture,
- we have to construct "naturally" the higher residue pairings,
- we have to prove an isomorphism between the complex of "polyvector fields" and the complex of "differential forms".

Plan

1. Notations and terminologies
   - Differential graded algebras, Calabi–Yau dg algebras
   - Hochschild cohomologies and Hochschild homologies
   - Cartan calculus
2. Saito structures at the generating centers and good sections
   - Formality conjecture
   - Filtered de Rham cohomologies and Gauß–Manin connections
   - Exponents
   - Higher residue pairings
   - Good sections
3. Construction of primitive forms
   - Versal deformations
   - Product $\circ$, unit vector field e and Euler vector field E
   - Deformed Filtered de Rham cohomologies, Gauß–Manin connections and higher residues
   - Primitive forms

dg Algebras

A *differential graded* $\mathbb{C}$*-algebra* (*dg* $\mathbb{C}$*-algebra*) A consists of

- a $\mathbb{Z}$-graded $\mathbb{C}$-algebra $A = \bigoplus_{p\in\mathbb{Z}} A^p$
- a *differential*, a $\mathbb{C}$-linear map $d_A : A \longrightarrow A$ of degree one with $d_A^2 = 0$ satisfying the Leibniz rule:

$$d_A(a_1a_2) = (d_Aa_1)a_2 + (-1)^{\overline{a_1}}a_1(d_Aa_2), \quad a_1 \in A^{\overline{a_1}}, a_2 \in A.$$

A dg $\mathbb{C}$-algebra A is called *non-negatively graded* if $A^p = 0, p < 0$.

Definition 6 (Kontsevich)

Let A be a dg $\mathbb{C}$-algebra. Put $A^e := A^{op} \otimes_{\mathbb{C}} A$.

1. A is called *compact* if A is a perfect dg $\mathbb{C}$-module, equivalently, if $\sum_{p\in\mathbb{Z}} \dim_{\mathbb{C}} H^p(A, d_A) < \infty$.
2. A is called *smooth* if A is a perfect dg A^e-module.

Definition 7 (cf. Keller)

Let A be a dg $\mathbb{C}$-algebra.
A cofibrant replacement of the dg A^e-module $\mathbb{R}\mathcal{H}om_{(A^e)^{op}}(A, A^e)$ is called the *inverse dualizing complex* and is denoted by $A^!$.

Remark 8

$$K_X^{-1} \cong Ext^n_{X\times X}(\mathcal{O}_X, \mathcal{O}_{X\times X}). \tag{2.1}$$

Calabi–Yau dg algebras

Definition 9 (Ginzburg)

Fix an integer $w \in \mathbb{Z}$. A smooth dg $\mathbb{C}$-algebra A is called *Calabi–Yau of dimension w* if there exists a quasi-isomorphism of dg A^e-modules

$$A^! \longrightarrow T^{-w}A. \tag{2.2}$$

Remark 10

For a Calabi–Yau dg $\mathbb{C}$-algebra A of dimension w, we have the following isomorphisms in $D(A^e)$

$$\mathbb{R}\mathcal{H}om_{A^e}(A, A) \cong T^{-w}\mathbb{R}\mathcal{H}om_{A^e}(A^!, A) \cong T^{-w}(A \otimes^{\mathbb{L}}_{A^e} A). \tag{2.3}$$

Hochschild cohomology

Set $C^\bullet(A) = \prod_{n\geq 0} \mathcal{G}r_{\mathbb{C}}((TA)^{\otimes n}, A)$.

- Define a $\mathbb{C}$-linear map $d : C^\bullet(A) \longrightarrow C^{\bullet+1}(A)$ by

$$(df)(Ta_1 \otimes \cdots \otimes Ta_n) := d_A(f(Ta_1 \otimes \cdots \otimes Ta_n))$$
$$-\sum_{i=1}^{n} \pm f(Ta_1 \otimes \cdots \otimes Ta_{i-1} \otimes T(d_A a_i) \otimes Ta_{i+1} \otimes \cdots \otimes Ta_n).$$

- Define a $\mathbb{C}$-linear map $\delta : C^\bullet(A) \longrightarrow C^{\bullet+1}(A)$ by

$$(\delta f)(Ta_1 \otimes \cdots \otimes Ta_{n+1})$$
$$:= \pm f(Ta_1 \otimes \cdots \otimes Ta_n)a_{n+1} \pm a_1 f(Ta_2 \otimes \cdots \otimes Ta_{n+1})$$
$$-\sum_{i=1}^{n} \pm f(Ta_1 \otimes \cdots \otimes Ta_{i-1} \otimes T(a_i a_{i+1}) \otimes Ta_{i+2} \otimes \cdots \otimes Ta_{n+1}).$$

Proposition 11

1. *The $\mathbb{C}$-linear maps d, δ on $C^\bullet(A)$ satisfy*

$$d^2 = 0, \quad \delta^2 = 0, \quad d\delta + \delta d = 0. \tag{2.4}$$

2. *Set $\partial := d + \delta$. We have the following isomorphism in $D(\mathbb{C})$*

$$(C^\bullet(A), \partial) \cong \mathbb{R}\mathcal{H}om_{A^e}(A, A). \tag{2.5}$$

Definition 12

1. The graded $\mathbb{C}$-module $H^\bullet(C^\bullet(A), \partial)$ is denoted by $HH^\bullet(A)$ and called the *Hochschild cohomology* of A.
2. Denote by $\mathcal{T}^\bullet_{poly}(A)$ the graded $\mathbb{C}$-module $H^\bullet(C^\bullet(A), \delta)$.

Remark 13

$\mathcal{T}^\bullet_{poly}(A)$ is an analog of the sheaf $\wedge^\bullet \mathcal{T}_X$ of polyvector fields.

Definition 14

For $f = (f_n) \in C^p(A)$ and $g = (g_n) \in C^q(A)$, define the product $f \circ g = ((f \circ g)_n) \in C^{p+q}(A)$ by

$$(f \circ g)_n(Ta_1 \otimes \cdots \otimes Ta_n) := \sum \pm f_i(Ta_1 \otimes \cdots \otimes Ta_i) g_{n-i}(Ta_{i+1} \otimes \cdots \otimes Ta_n).$$

Definition 15

For $f = (f_n) \in C^p(A)$ and $g = (g_n) \in C^q(A)$, define the *Gerstenhaber bracket* $[f, g]_G \in C^{p+q-1}(A)$ by

$$[f, g]_G := f \circ_{-1} g - (-1)^{(p-1)(q-1)} g \circ_{-1} f, \tag{2.6}$$

where $f \circ_{-1} g = ((f \circ_{-1} g)_n) \in C^{p+q-1}(A)$ is defined as

$$\begin{aligned}(f \circ_{-1} g)_n(Ta_1 \otimes \cdots \otimes Ta_n) \\ := \sum \pm f_{n-j+1}(Ta_1 \otimes \cdots \otimes Ta_{i-1} \\ \otimes T(g_j(Ta_i \otimes \cdots \otimes Ta_{i+j-1})) \otimes Ta_{i+j} \otimes \cdots \otimes Ta_n)\end{aligned}$$

Definition 16

1. Denote by $\mathfrak{m}_A$ the element $(\mathfrak{m}_{A,n}) \in C^2(A)$ defined by $\mathfrak{m}_{A,n} := 0$ if $n \neq 1, 2$ and

$$\mathfrak{m}_{A,1}(Ta_1) := d_A a_1, \ \mathfrak{m}_{A,2}(Ta_1 \otimes Ta_2) := (-1)^{\overline{a_1}} a_1 a_2. \quad (2.7)$$

2. Denote by $\mathfrak{f}_A$ the cohomology class of $\mathfrak{m}_A$ in $T^2_{poly}(A)$.
3. Denote by $\deg_A$ the cohomology class of the element $(\deg_{A,n}) \in C^1(A)$ in $\mathcal{T}^1_{poly}(A)$ defined by

$$\deg_{A,n} := 0 \ (n \neq 1), \quad \deg_{A,1}(Ta_1) := \overline{a_1} \cdot a_1. \quad (2.8)$$

Remark 17

$\mathfrak{f}_A \Longleftrightarrow f(x_1, \ldots, x_n)$: a weighted homogeneous polynomial

$$\deg_A \Longleftrightarrow \sum_{i=1}^{n} w_i x_i \frac{\partial}{\partial x_i}, \quad w_i : \text{the weight of } x_i$$

Proposition 18

1. *The product* $\circ$ *on* $C^\bullet(A)$ *induces structures of graded commutative* $\mathbb{C}$*-algebras on* $\mathcal{T}^\bullet_{poly}(A)$ *and* $HH^\bullet(A)$ *whose unit elements are given by the cohomology classes of* $1_A \in C^0(A)$.
2. *The Gerstenharber bracket* $[-,-]_G$ *induces structures of graded Lie algebras on* $\mathcal{T}^{\bullet+1}_{poly}(A)$ *and* $HH^{\bullet+1}(A)$.
3. *We have*

$$\partial X = [\mathfrak{m}_A, X]_G\,, \quad X \in C^\bullet(A), \tag{2.9}$$

$$dX = [\mathfrak{f}_A, X]_G\,, \quad X \in \mathcal{T}^\bullet_{poly}(A). \tag{2.10}$$

In particular, the "Euler's identity" holds in $\mathcal{T}^\bullet_{poly}(A)$:

$$\mathfrak{f}_A = [\deg_A, \mathfrak{f}_A]_G\,. \tag{2.11}$$

Proposition 19

The tuple $(\mathcal{T}^{\bullet}_{poly}(A), d, \circ, [-,-]_G)$ *is a differential Gerstenhaber algebra. Namely,*

- *the triple* $(\mathcal{T}^{\bullet}_{poly}(A), d, \circ)$ *is a graded commutative dg* $\mathbb{C}$*-algebra,*
- *the triple* $(\mathcal{T}^{\bullet+1}_{poly}(A), d, [-,-]_G)$ *is a dg Lie algebra,*
- *for* $X, Y, Z \in \mathcal{T}^{\bullet}_{poly}(A)$*, we have*

$$[X, Y \circ Z]_G = [X, Y] \circ Z + (-1)^{(\overline{X}+1)\cdot\overline{Y}} Y \circ [X, Z]_G . \quad (2.12)$$

Hochschild homology

Denote by $C_\bullet(A)$ the graded $\mathbb{C}$-module $\coprod_{n\geq 0} A\otimes_{\mathbb{C}}(TA)^{\otimes n}$.

1. Define a $\mathbb{C}$-linear map $d : C_\bullet(A) \longrightarrow C_{\bullet-1}(A)$ by

$$d(a_0\otimes Ta_1\otimes\cdots\otimes Ta_n) := d_A a_0\otimes Ta_1\otimes\cdots\otimes Ta_n$$
$$+\sum \pm a_0\otimes Ta_1\otimes\cdots\otimes Ta_{i-1}\otimes T(d_A a_i)\otimes Ta_{i+1}\otimes\cdots\otimes Ta_n.$$

2. Define a $\mathbb{C}$-linear map $\delta : C_\bullet(A) \longrightarrow C_{\bullet-1}(A)$ by

$$\delta(a_0\otimes Ta_1\otimes\cdots\otimes Ta_n) := \pm a_0 a_1\otimes Ta_2\otimes\cdots\otimes Ta_n$$
$$+\sum \pm a_0\otimes Ta_1\otimes\cdots\otimes Ta_{i-1}\otimes T(a_i a_{i+1})\otimes Ta_{i+2}\otimes\cdots\otimes Ta_n$$
$$\pm\sum \pm a_n a_0\otimes Ta_1\otimes\cdots\otimes Ta_{n-1}.$$

Define a $\mathbb{C}$-linear map $B : C_\bullet(A) \longrightarrow C_{\bullet+1}(A)$ called the *Conne's differential* by

$$\begin{aligned} B(a_0 \otimes Ta_1 \cdots \otimes Ta_n) := \mathrm{id}_A \otimes Ta_0 \otimes \cdots \otimes Ta_n \\ \pm a_0 \otimes T(\mathrm{id}_A) \otimes Ta_1 \otimes \cdots \otimes Ta_n \\ + \sum \pm \mathrm{id}_A \otimes Ta_i \otimes \cdots \otimes Ta_n \otimes Ta_0 \otimes \cdots \otimes Ta_{i-1} \\ + \sum \pm a_i \otimes T(\mathrm{id}_A) \otimes Ta_{i+1} \cdots \otimes Ta_n \otimes Ta_0 \otimes \cdots \otimes Ta_{i-1}. \end{aligned}$$

Proposition 20

1. *The $\mathbb{C}$-linear maps d, δ, B satisfy $d^2 = 0$, $\delta^2 = 0$, $B^2 = 0$ and*
$$d\delta + \delta d = 0,\ dB + Bd = 0,\ \delta B + B\delta = 0. \qquad (2.13)$$
2. *Set $\partial := d + \delta$. We have the following isomorphisms in $D(\mathbb{C})$*
$$(C_\bullet(A), \partial) \cong A \otimes^{\mathbb{L}}_{A^e} A \cong \mathbb{R}\mathcal{H}om_{A^e}(A^!, A). \qquad (2.14)$$

Definition 21

1. The graded $\mathbb{C}$-module $H_\bullet(C_\bullet(A), \partial)$ is denoted by $HH_\bullet(A)$ and called the *Hochschild homology* of A.
2. Denote by $\Omega_\bullet(A)$ the graded $\mathbb{C}$-module $H_\bullet(C_\bullet(A), \delta)$.

Remark 22

$\Omega_\bullet(A)$ is an analog of the sheaf $\Omega^\bullet_X$ of differential forms.

Definition 23

Let f be an element of $\mathcal{G}r_{\mathbb{C}}((TA)^{\otimes p}, A)$.

1. $\mathbb{C}$-linear map $\iota_f : C_\bullet(A) \longrightarrow C_\bullet(A)$ defined by

$$\iota_f(a_0 \otimes Ta_1 \cdots \otimes Ta_n) := \pm f(Ta_{n-p+1} \otimes \cdots \otimes Ta_n) a_0 \otimes Ta_1 \cdots \otimes Ta_{n-p} \tag{2.15}$$

   is called the *contraction*.

2. $\mathbb{C}$-linear map $\mathcal{L}_f : C_\bullet(A) \longrightarrow C_\bullet(A)$ defined by

$$\begin{aligned} \mathcal{L}_f(a_0 \otimes Ta_1 \cdots \otimes Ta_n) :=& \\ \sum \pm a_0 \otimes Tf(Ta_i \otimes \cdots \otimes Ta_{i+p-1}) \otimes Ta_{i+p} \cdots \otimes Ta_n& \\ + \sum \pm f(Ta_{n-p+i+2} \otimes \cdots \otimes Ta_n \otimes Ta_0 \otimes \cdots \otimes Ta_i) \otimes& \\ Ta_{i+1} \cdots \otimes Ta_{n-p+i+1}& \end{aligned} \tag{2.16}$$

   is called the *Lie derivative*.

The following is a direct consequence of a result by Daletski–Gelfand–Tsygan:

Proposition 24 (($\mathcal{T}^{\bullet}_{poly}(A), \Omega_{\bullet}(A)$) is a calculus algebra)

The $\mathbb{C}$-linear maps ι and $\mathcal{L}$ induce morphisms of graded $\mathbb{C}$-modules

$$i : \mathcal{T}^{\bullet}_{poly}(A) \longrightarrow \mathcal{G}r_{\mathbb{C}}(\Omega_{\bullet}(A), \Omega_{\bullet}(A)), \quad X \mapsto i_X \tag{2.17}$$

$$L : \mathcal{T}^{\bullet}_{poly}(A) \longrightarrow T^{-1}\mathcal{G}r_{\mathbb{C}}(\Omega_{\bullet}(A), \Omega_{\bullet}(A)), \quad X \mapsto L_X \tag{2.18}$$

satisfying

$$i_X i_Y = i_{X \circ Y}, \quad [L_X, L_Y] = L_{[X,Y]_G}, \tag{2.19}$$

$$L_X i_Y + (-1)^{\overline{X}} i_Y L_X = L_{X \circ Y}, \quad [i_X, L_Y] = i_{[X,Y]_G}, \tag{2.20}$$

$$[B, i_X] = -L_X, \quad [B, L_X] = 0, \tag{2.21}$$

$$L_{\mathfrak{f}_A} = -d. \tag{2.22}$$

Formality conjecture

Let A be a non-negatively graded smooth dg $\mathbb{C}$-algebra.

Proposition 25 (Dolgushev–Tamarkin–Tsygan)

The pair $(C^\bullet(A), C_\bullet(A))$ *is a homotopy calculus algebra.*

Conjecture 26

The homotopy calculus algebra $(C^\bullet(A), C_\bullet(A))$ *is quasi-isomorphic to the differential calculus algebra* $(\mathcal{T}^\bullet_{poly}(A), \Omega_\bullet(A))$.

Remark 27

Conjecture 26 implies that the dg Lie algebra $(C^{\bullet+1}(A), [\mathfrak{m}_A, -]_G, [-,-]_G)$ is quasi-isomorphic to the dg Lie algebra $(\mathcal{T}^{\bullet+1}_{poly}(A), [\mathfrak{f}_A, -]_G, [-,-]_G)$ as an L_∞-algebra.

From now on, we assume the following:

Assumption 28

The dg $\mathbb{C}$-algebra A is non-negatively graded, smooth, compact and Calabi–Yau of dimension $w \in \mathbb{Z}_{\geq 0}$ satisfying Conjecture 26 and $H^0(A, d_A) = \mathbb{C}[1_A]$.

Under Assumption 28, there are isomorphisms of graded $\mathbb{C}$-modules

$$HH^\bullet(A) \longrightarrow H^\bullet(\mathcal{T}^\bullet_{poly}(A), d), \tag{3.1a}$$

$$HH_\bullet(A) \longrightarrow H_\bullet(\Omega_\bullet(A), d). \tag{3.1b}$$

In particular, an element of $HH_w(A)$ giving the isomorphism

$$A^! \cong T^{-w}A. \tag{3.2}$$

determines a non-zero element $v_1 \in H_w(\Omega_\bullet(A), d)$.
The contraction map

$$\mathcal{T}^\bullet_{poly}(A) \otimes_{\mathbb{C}} \Omega_\bullet(A) \longrightarrow \Omega_\bullet(A), \quad X \otimes v \mapsto i_X v,$$

induces the isomorphism

$$H^p(\mathcal{T}^\bullet_{poly}(A), d) \cong H_{w-p}(\Omega_\bullet(A), d),\ \ p \in \mathbb{Z}, \quad X \mapsto i_X v_1, \tag{3.3}$$

of graded $\mathbb{C}$-modules.

Definition 29

Set

$$Jac(\mathfrak{f}_A) := H^\bullet(\mathcal{T}^\bullet_{poly}(A), d). \tag{3.4}$$

We call the graded $\mathbb{C}$-module $Jac(\mathfrak{f}_A)$ the *Jacobian ring* of A.

Remark 30

Recall that $d = [\mathfrak{f}_A, -]_G$. Note also that the bracket $[-,-]_G$ on $\mathcal{T}^\bullet_{poly}(A)$ is an analog of the Schouten–Nijenhuis bracket.
For a holomorphic function $f : \mathbb{C}^n \longrightarrow \mathbb{C}$, we have an isomorphism $H^\bullet(\wedge^\bullet \mathcal{T}_{\mathbb{C}^n}, [f,-]) \cong Jac(f)$, where $[-,-]$ denotes the Schouten–Nijenhuis bracket.

Proposition 31

The Jacobian ring $Jac(\mathfrak{f}_A)$ is a finite dimensional graded commutative Frobenius $\mathbb{C}$-algebra.
Namely, we have a non-degenerate graded symmetric bilinear form

$$\eta_{\mathfrak{f}_A}^{v_1} : Jac(\mathfrak{f}_A) \otimes_{\mathbb{C}} Jac(\mathfrak{f}_A) \to T^{2w}\mathbb{C}, \quad X \otimes Y \mapsto \eta_{\mathfrak{f}_A}^{v_1}(X, Y), \quad (3.5)$$

such that

$$\eta_{\mathfrak{f}_A}^{v_1}(X \circ Y, Z) = \eta_{\mathfrak{f}_A}^{v_1}(X, Y \circ Z), \quad X, Y, Z \in Jac(\mathfrak{f}_A). \quad (3.6)$$

Remark 32

The bilinear form $\eta_{\mathfrak{f}_A}^{v_1}$ depends on the choice of $v_1^{\otimes 2}$, more precisely, on the choice of isomorphism $(A^e)^! \cong T^{-2w}A^e$ in $D(A^e)$.

Filtered de Rham cohomology

Definition 33

Let u be a formal variable of degree 2.

1. Define a graded $\mathbb{C}((u))$-module $\mathcal{H}_{\mathfrak{f}_A}$ by

$$\mathcal{H}_{\mathfrak{f}_A} := H_\bullet(\Omega_\bullet(A)((u)), d + uB). \tag{3.7}$$

   $\mathcal{H}_{\mathfrak{f}_A}$ is called the *filtered de Rham cohomology* of A.

2. Define graded $\mathbb{C}[[u]]$-modules $\mathcal{H}_{\mathfrak{f}_A}^{(-p)}$ by

$$\mathcal{H}_{\mathfrak{f}_A}^{(-p)} := H_\bullet(\Omega_\bullet(A)[[u]]u^p, d + uB), \;\; p \in \mathbb{Z}. \tag{3.8}$$

3. Define a graded $\mathbb{C}$-module $\Omega_{\mathfrak{f}_A}$ by

$$\Omega_{\mathfrak{f}_A} := H_\bullet(\Omega_\bullet(A), d). \tag{3.9}$$

Remark 34

1. $\mathcal{H}_{\mathfrak{f}_A}$ is an analog of the de Rham cohomology $H^\bullet(X, \mathbb{C})$.
2. $\{\mathcal{H}_{\mathfrak{f}_A}^{(-p)}\}_{p\in\mathbb{Z}}$ is an analog of the Hodge filtration.
3. $\Omega_{\mathfrak{f}_A}$ is an analog of the Hodge cohomology $H^\bullet(X, \Omega_X^\bullet)$.

Degeneration of Hodge to de Rham

Set $\mu_A := \dim_{\mathbb{C}} \Omega_{\mathfrak{f}_A}$.

Proposition 35 (degeneration of Hodge to de Rham)

The graded $\mathbb{C}[[u]]$-modules $\mathcal{H}_{\mathfrak{f}_A}^{(-p)}$ is free of rank μ_A for all $p \in \mathbb{Z}$ and there exists an exact sequence of graded $\mathbb{C}$-modules

$$0 \longrightarrow \mathcal{H}_{\mathfrak{f}_A}^{(-p-1)} \longrightarrow \mathcal{H}_{\mathfrak{f}_A}^{(-p)} \xrightarrow{r^{(-p)}} \Omega_{\mathfrak{f}_A} \longrightarrow 0. \tag{3.10}$$

Key: Kaledin's degeneration of Hodge to de Rham theorem for non-negatively graded smooth compact dg algebras.

It turns out that the $\mathbb{C}[[u]]$-modules $\mathcal{H}_{\mathfrak{f}_A}^{(-p)}$, $p \in \mathbb{Z}$ define an increasing filtration of $\mathcal{H}_{\mathfrak{f}_A}$

$$\cdots \subset \mathcal{H}_{\mathfrak{f}_A}^{(-p-1)} \subset \mathcal{H}_{\mathfrak{f}_A}^{(-p)} \subset \cdots \subset \mathcal{H}_{\mathfrak{f}_A}, \tag{3.11}$$

such that

$$\bigcup_{p \in \mathbb{Z}} \mathcal{H}_{\mathfrak{f}_A}^{(-p)} = \mathcal{H}_{\mathfrak{f}_A} \quad \text{and} \quad \bigcap_{p \in \mathbb{Z}} \mathcal{H}_{\mathfrak{f}_A}^{(-p)} = \{0\} \tag{3.12}$$

so that $\mathcal{H}_{\mathfrak{f}_A}$ is complete with respect to the filtration.
The multiplication of u induces an isomorphism

$$u : \mathcal{H}_{\mathfrak{f}_A}^{(-p)} \cong \mathcal{H}_{\mathfrak{f}_A}^{(-p-1)}. \tag{3.13}$$

Gauß–Manin connection on $\mathcal{H}_{\mathfrak{f}_A}$

Define $\mathcal{T}_{\widehat{\mathbb{A}}^1_u}$ as

$$\mathcal{T}_{\widehat{\mathbb{A}}^1_u} := \mathbb{C}[[u]]\frac{d}{du}. \tag{3.14}$$

Definition 36

Define a connection

$$\nabla : \mathcal{T}_{\widehat{\mathbb{A}}^1_u} \otimes_{\mathbb{C}} \Omega_{\bullet}(A)((u)) \to \Omega_{\bullet}(A)((u)) \tag{3.15}$$

by

$$\nabla_{\frac{d}{du}} := \frac{d}{du} - \frac{1}{u^2} i_{\mathfrak{f}_A}. \tag{3.16}$$

Proposition 37

The connection ∇ *satisfies*

$$\left[\nabla_{u\frac{d}{du}}, d + uB\right] = d + uB. \tag{3.17}$$

Therefore, ∇ *induces a connection on* $\mathcal{H}_{\mathfrak{f}_A}$*. Moreover, we have*

$$\nabla_{u\frac{d}{du}}\left(\mathcal{H}^{(0)}_{\mathfrak{f}_A}\right) \subset \mathcal{H}^{(0)}_{\mathfrak{f}_A}. \tag{3.18}$$

Definition 38

The connection ∇ on $\mathcal{H}_{\mathfrak{f}_A}$ is called the *Gauß–Manin connection*.

Proof

(of Proposition)

$$\left[\nabla_{u\frac{d}{du}}, d + uB\right] = uB - \frac{1}{u}\left[i_{\mathfrak{f}A}, d + uB\right] = uB - L_{\mathfrak{f}A} = d + uB.$$

$$u\frac{d}{du} - \frac{1}{u} i_{\mathfrak{f}A} = u\frac{d}{du} - \frac{1}{u}\left[i_{\deg_A}, L_{\mathfrak{f}A}\right] = u\frac{d}{du} + L_{\deg_A} + \frac{1}{u}\left[d + uB, i_{\deg_A}\right]$$

□

Pairing on the Hochschild homology

We have an isomorphisms of graded $\mathbb{C}$-modules

$$Jac(\mathfrak{f}_A) \cong T^{-w}\Omega_{\mathfrak{f}_A}, \quad X \mapsto i_X v_1. \tag{3.19}$$

Therefore, we can move the $\mathbb{C}$-bilinear form $\eta_{\mathfrak{f}_A}^{v_1}$ on $Jac(\mathfrak{f}_A)$ to the one on $\Omega_{\mathfrak{f}_A}$ which does not depend on v_1.

Definition 39

Define a $\mathbb{C}$-bilinear form $J_{\mathfrak{f}_A} : \Omega_{\mathfrak{f}_A} \otimes_{\mathbb{C}} \Omega_{\mathfrak{f}_A} \to \mathbb{C}$ by

$$J_{\mathfrak{f}_A}(i_X v_1, i_Y v_1) := (-1)^{w\cdot\overline{Y}} \eta_{f_A}^{v_1}(X, Y), \quad X, Y \in Jac(\mathfrak{f}_A). \tag{3.20}$$

Exponents

It follows from the definition of the morphism $\mathcal{L}$ that

$$\mathcal{L}_{\deg_A}(a_0 \otimes Ta_1 \otimes \cdots \otimes Ta_n) := \left(\sum_{i=1}^{n} \overline{a_i} \right) \cdot (a_0 \otimes Ta_1 \otimes \cdots \otimes Ta_n) \tag{3.21}$$

for $a_0 \otimes Ta_1 \otimes \cdots \otimes Ta_n \in C_\bullet(A)$.
This commutes with the operator δ on $C_\bullet(A)$ and hence defines an endomorphism of graded $\mathbb{C}$-modules on $\Omega_\bullet(A)$, which is $L_{\deg_A}$.

Proposition 40

The endomorphism of graded $\mathbb{C}$-modules $L_{\deg_A}$ on $\Omega_\bullet(A)$ induces $\mathbb{C}$-linear endomorphism on $\Omega_{\mathfrak{f}_A}$.

Definition 41

Define graded $\mathbb{C}$-submodules $\Omega_{\mathfrak{f}_A}^{p,q}$ of $\Omega_{\mathfrak{f}_A}$ for $p, q \in \mathbb{Z}$ by

$$\Omega_{\mathfrak{f}_A}^{p,q} := \{\omega \in \Omega_{\mathfrak{f}_A} \mid \overline{\omega} = -p + q,\ L_{\deg_A}\omega = q\omega\}. \tag{3.22}$$

Definition 42

The *Hodge numbers* for A are

$$h^{p,q}(A) := \dim_{\mathbb{C}} \Omega_{\mathfrak{f}_A}^{p,q}, \quad p, q \in \mathbb{Z}.$$

The integer q with $h^{p,q}(A) \neq 0$ is called an *exponent* of A.

Remark 43

Since A is compact, $h^{p,q}(A) < \infty$ for $p, q \in \mathbb{Z}$.

Proposition 44

The Hodge numbers for A satisfy the following properties:

1. $h^{p,q}(A) = 0$ *if* $p < 0$ *or* $q < 0$.
2. $h^{w,0}(A) = 1$.
3. $h^{w-p,q}(A) = h^{p,w-q}(A)$.

Recall here that A is non-negatively graded, $H^0(A, d_A) = \mathbb{C}[1_A]$ and that $J_{\mathfrak{f}_A}$ induces a perfect pairing

$$J_{\mathfrak{f}_A} : \Omega_{\mathfrak{f}_A}^{p,q} \otimes_{\mathbb{C}} \Omega_{\mathfrak{f}_A}^{w-p,w-q} \longrightarrow \mathbb{C}. \tag{3.23}$$

A homogeneous section

Proposition 45

There exists a splitting $s^{(0)} : \Omega_{\mathfrak{f}_A} \longrightarrow \mathcal{H}^{(0)}_{\mathfrak{f}_A}$ of the following exact sequence of graded $\mathbb{C}$-modules

$$0 \longrightarrow \mathcal{H}^{(-1)}_{\mathfrak{f}_A} \longrightarrow \mathcal{H}^{(0)}_{\mathfrak{f}_A} \xrightarrow{r^{(0)}} \Omega_{\mathfrak{f}_A} \longrightarrow 0 \tag{3.24}$$

such that

$$\nabla_{u\frac{d}{du}} \left(s^{(0)} \left(\Omega_{\mathfrak{f}_A} \right) \right) \subset s^{(0)} \left(\Omega_{\mathfrak{f}_A} \right). \tag{3.25}$$

Remark 46

Once such $s^{(0)}$ is given, we have an isomorphism

$$\Omega_{\mathfrak{f}_A}[[u]] \cong \mathcal{H}^{(0)}_{\mathfrak{f}_A}, \quad v \cdot u^p \mapsto s^{(0)}(v) u^p, \tag{3.26}$$

of $\mathbb{C}[[u]]$-modules.

Proof

By definition, $v_1 \in \Omega_{\mathfrak{f}_A}^{w,0}$, namely, it satisfies $L_{\deg_A} v_1 = 0$.

There are elements $v_2, \ldots, v_{\mu_A} \in \Omega_{\mathfrak{f}_A}$ such that $L_{\deg_A} v_i = q_i \cdot v_i$ for some $q_i \in \mathbb{Z}$ and the set $\{v_1, \ldots, v_{\mu_A}\}$ forms a $\mathbb{C}$-basis of $\Omega_{\mathfrak{f}_A}$.

The image $s^{(0)}(\Omega_{\mathfrak{f}_A})$ is inside of $H_\bullet(\Omega_\bullet(A)[u], d + uB)$ since the dg $\mathbb{C}$-algebra A is non-negatively graded.

Since $[L_{\deg_A}, B] = 0$ and $[L_{\deg_A}, d] = d$, we can choose

$$s^{(0)}(v_i) = \sum_{l=0}^{N} \omega_{i,l} u^l,\ \omega_l \in \Omega_\bullet(A) \text{ so that } L_{\deg_A} \omega_{i,l} = (q_i - l) \cdot \omega_{i,l}.$$

Therefore, $\nabla_{u\frac{d}{du}} s^{(0)}(v_i) = q_i \cdot s^{(0)}(v_i)$, $i = 1, \ldots, \mu_A$. □

Higher residue pairings

Definition 47 (higher residue pairings)

Define a $\mathbb{C}$-bilinear form

$$K_{\mathfrak{f}_A} : \mathcal{H}_{\mathfrak{f}_A} \otimes_{\mathbb{C}} \mathcal{H}_{\mathfrak{f}_A} \to \mathbb{C}((u)) \tag{3.27}$$

by setting for $v, v' \in \Omega_{\mathfrak{f}_A}$

$$K_{\mathfrak{f}_A}\left(s^{(0)}(v)u^p, s^{(0)}(v')u^{p'}\right) := (-1)^{p'} \cdot u^{w+p+p'} \cdot J_{\mathfrak{f}_A}\left(v, v'\right). \tag{3.28}$$

For $P \in \mathbb{C}((u))$, define $P^* \in \mathbb{C}((u))$ by $P^*(u) := P(-u)$.

Proposition 48

The bilinear form $K_{\mathfrak{f}_A}$ satisfies the following properties

1. $K_{\mathfrak{f}_A}(\omega_1, \omega_2) = (-1)^{\overline{\omega_1}\cdot\overline{\omega_2}} K_{\mathfrak{f}_A}(\omega_2, \omega_1)^*$.
2. $PK_{\mathfrak{f}_A}(\omega_1, \omega_2) = K_{\mathfrak{f}_A}(P\omega_1, \omega_2) = K_{\mathfrak{f}_A}(\omega_1, P^*\omega_2)$.
3. $K_{\mathfrak{f}_A}(\mathcal{H}^{(0)}_{\mathfrak{f}_A}, \mathcal{H}^{(0)}_{\mathfrak{f}_A}) \subset \mathbb{C}[[u]]u^w$,
4. *We have the following commutative diagram :*

$$\begin{array}{ccc} K_{\mathfrak{f}_A} : \mathcal{H}^{(0)}_{\mathfrak{f}_A} \otimes_{\mathbb{C}} \mathcal{H}^{(0)}_{\mathfrak{f}_A} & \longrightarrow & \mathbb{C}[[u]]u^w \\ \downarrow {\scriptstyle r^{(0)} \otimes r^{(0)}} & & \downarrow {\scriptstyle \mathrm{mod}\ \ \mathbb{C}[[u]]u^{w+1}} \\ u^w J_{\mathfrak{f}_A} : \Omega_{\mathfrak{f}_A} \otimes_{\mathbb{C}} \Omega_{\mathfrak{f}_A} & \longrightarrow & \mathbb{C}u^w. \end{array}$$

5. $u\frac{d}{du}K_{\mathfrak{f}_A}(\omega_1, \omega_2) = K_{\mathfrak{f}_A}(\nabla_{u\frac{d}{du}}\omega_1, \omega_2) + K_{\mathfrak{f}_A}(\omega_1, \nabla_{u\frac{d}{du}}\omega_2)$.

Remark 49

The bilinear form $K_{\mathfrak{f}_A}$ does depend on the choice of $s^{(0)} : \Omega_{\mathfrak{f}_A} \longrightarrow \mathcal{H}^{(0)}_{\mathfrak{f}_A}$.
However, as in the classical case, we expect that $K_{\mathfrak{f}_A}$ should be uniquely determined by properties in the previous proposition.

Existence of a good section

Proposition 50

There exists a $\mathbb{C}$-linear subspace S of $\mathcal{H}_{\mathfrak{f}_A}$ satisfying

$$\mathcal{H}_{\mathfrak{f}_A} = \mathcal{H}_{\mathfrak{f}_A}^{(0)} \oplus S, \tag{3.29}$$

$$u^{-1} S \subset S, \tag{3.30}$$

$$\nabla_{u\frac{d}{du}} S \subset S, \tag{3.31}$$

$$K_{\mathfrak{f}_A}(S, S) \subset \mathbb{C}[u^{-1}]u^{w-2}. \tag{3.32}$$

We can choose as S the subspace $s^{(0)}(\Omega_{\mathfrak{f}_A})[u^{-1}]u^{-1}$.

Versal deformation

We add the following assumption on A.

Assumption 51

The contraction map

$$\mathcal{T}^{\bullet}_{poly}(A) \longrightarrow \Omega_{w-\bullet}(A), \quad X \mapsto i_X v_1,$$

induces an isomorphism of complexes

$$(\mathcal{T}^{\bullet}_{poly}(A), d) \cong (\Omega_{w-\bullet}(A), d).$$

Under Assumption 51, we can define the operator Δ on $\mathcal{T}^{\bullet}_{poly}(A)$:

Definition 52

Define a morphism $\Delta : \mathcal{T}^{\bullet}_{poly}(A) \longrightarrow \mathcal{T}^{\bullet-1}_{poly}(A)$ of graded $\mathbb{C}$-modules by $i_{\Delta(X)} v_1 := B(i_X v_1)$, $X \in \mathcal{T}^{\bullet}_{poly}(A)$.

Proposition 53

The tuple $(\mathcal{T}^{\bullet}_{poly}(A), d, \circ, [-,-]_G, \Delta)$ *is a dGBV algebra. Namely, we have the following equation*:

$$[X, Y]_G = (-1)^{\overline{X}} \Delta(X \circ Y) - (-1)^{\overline{X}} \Delta(X) \circ Y - X \circ \Delta(Y). \quad (4.1)$$

Consider a formal graded manifold M whose structure sheaf $\mathcal{O}_M$ is

$$\mathcal{O}_M := \mathbb{C}[[HH^{\bullet+2}(A)]]. \tag{4.2}$$

Denote by $\mathfrak{m}$ the maximal ideal in $\mathcal{O}_M$.

Let $t_1, \ldots, t_{\mu_A}$ be coordinates dual to the basis $\{v_1, \ldots, v_{\mu_A}\}$ as in the proof of Proposition 45. The tangent sheaf $\mathcal{T}_M$ on M is a graded $\mathcal{O}_M$-free module of derivations on $\mathcal{O}_M$, which satisfies

$$\mathcal{T}_M \cong \bigoplus_{i=1}^{\mu_A} \mathcal{O}_M \frac{\partial}{\partial t_i}. \tag{4.3}$$

Proposition 54 (Existence of a versal deformation of $\mathfrak{f}_A$)

There exists an element $\gamma(t) \in \mathcal{T}^{\bullet}_{poly}(A) \otimes_{\mathbb{C}} \mathfrak{m}$ of degree two satisfying the following properties:

- *The Maurer–Cartan equation is solved by $\gamma(t)$:*

$$d\gamma(t) + \frac{1}{2}\left[\gamma(t), \gamma(t)\right]_G = 0. \tag{4.4}$$

- *For $i = 1, \dots, \mu_A$, the elements*

$$\left.\frac{\partial \gamma(t)}{\partial t_i}\right|_{\mathbf{t}=0} := \left[\frac{\partial \gamma(t)}{\partial t_i}\right] \in \mathcal{T}^{\bullet}_{poly}(A) \otimes_{\mathbb{C}} \left(\mathfrak{m}/\mathfrak{m}^2\right), \tag{4.5}$$

  form a $\mathbb{C}$-basis of $Jac(\mathfrak{f}_A)$.

- *An element $\gamma(t)$ is homogeneous in the sense that*

$$\gamma(t) = \sum_{i=1}^{\mu_A}(1 - q_i) t_i \frac{\partial \gamma(t)}{\partial t_i} + \left[\deg_A, \gamma(t)\right]_G . \tag{4.6}$$

Proof

We can apply Terilla's result since our dGBV algebra $(\mathcal{T}^{\bullet}_{poly}(A), d, \circ, [-,-]_G, \Delta)$ satisfies his "degeneration of the spectral sequence" condition due to the Hodge to de Rham degeneration of our filtered de Rham cohomology $\mathcal{H}_{\mathfrak{f}_A}$.

Remark 55

Calabi–Yau manifolds are unobstructed.

Definition 56

Let $\gamma(t)$ be an element in $\mathcal{T}^{\bullet}_{poly}(A) \otimes_{\mathbb{C}} \mathcal{O}_M$ of degree two given by Proposition 54.

1. Define an element $\mathfrak{F}_A$ of $\mathcal{T}^{\bullet}_{poly}(A) \otimes_{\mathbb{C}} \mathcal{O}_M$ as

$$\mathfrak{F}_A := \mathfrak{f}_A + \gamma(t). \tag{4.7}$$

   It follows that $[\mathfrak{F}_A, \mathfrak{F}_A]_G = 0$ from the Maurer–Cartan equation (4.4).

2. Define an $\mathcal{O}_M$-homomorphism d_γ on $\mathcal{T}^{\bullet}_{poly}(A) \otimes_{\mathbb{C}} \mathcal{O}_M$ as

$$d_\gamma X := [\mathfrak{F}_A, X]_G \,, \quad X \in \mathcal{T}^{\bullet}_{poly}(A) \otimes_{\mathbb{C}} \mathcal{O}_M. \tag{4.8}$$

   It follows that $d_\gamma^2 = 0$ since $[\mathfrak{F}_A, \mathfrak{F}_A]_G = 0$.

Proposition 57

For $X, Y \in \mathcal{T}^{\bullet}_{poly}(A) \otimes_{\mathbb{C}} \mathcal{O}_M$, we have

$$d_\gamma [X, Y]_G = [d_\gamma(X), Y]_G + (-1)^{\overline{X}} [X, d_\gamma(Y)]_G \,, \tag{4.9a}$$

$$d_\gamma(X \circ Y) = d_\gamma(X) \circ Y + (-1)^{\overline{X}} X \circ d_\gamma(Y). \tag{4.9b}$$

Namely, d_γ, $\circ$ and $[-,-]_G$ equip $\mathcal{T}^{\bullet}_{poly}(A) \otimes_{\mathbb{C}} \mathcal{O}_M$ with a structure of differential Gerstenharber algebra.

The product $\circ$ on $\mathcal{T}_M$

Definition 58

Set

$$Jac(\mathfrak{F}_A) := H^\bullet(\mathcal{T}^\bullet_{poly}(A) \otimes_{\mathbb{C}} \mathcal{O}_M, d_\gamma). \tag{4.10}$$

We call the graded $\mathcal{O}_M$-algebra $Jac(\mathfrak{F}_A)$ the *Jacobian ring* of $\mathfrak{F}_A$.

Note that the property $[\mathfrak{F}_A, \mathfrak{F}_A]_G = 0$ implies

$$d_\gamma\left(\mathfrak{F}_A\right) = 0, \tag{4.11}$$

$$d_\gamma\left(\frac{\partial \mathfrak{F}_A}{\partial t_i}\right) = \left[\mathfrak{F}_A, \frac{\partial \mathfrak{F}_A}{\partial t_i}\right]_G = 0, \quad i = 1, \ldots, \mu_A. \tag{4.12}$$

Proposition 59 (Kodaira–Spencer map is an isom.)

The morphism of graded $\mathcal{O}_M$-modules

$$\rho : \mathcal{T}_M \longrightarrow Jac(\mathfrak{F}_A), \quad \frac{\partial}{\partial t_i} \mapsto \left[\frac{\partial \mathfrak{F}_A}{\partial t_i}\right], \ i = 1, \ldots, \mu_A, \tag{4.13}$$

is an isomorphism.

Definition 60

Define a product $\circ : \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \longrightarrow \mathcal{T}_M$ by

$$\rho\left(\frac{\partial}{\partial t_i} \circ \frac{\partial}{\partial t_j}\right) := \rho\left(\frac{\partial}{\partial t_i}\right) \circ \rho\left(\frac{\partial}{\partial t_j}\right) \tag{4.14}$$

$$\Longleftrightarrow \left[\left(\frac{\partial}{\partial t_i} \circ \frac{\partial}{\partial t_j}\right) \mathfrak{F}_A\right] = \left[\frac{\partial \mathfrak{F}_A}{\partial t_i} \circ \frac{\partial \mathfrak{F}_A}{\partial t_j}\right]. \tag{4.15}$$

Vector fields e and E

We can introduce the following two particular elements of $\mathcal{T}_M$.

Definition 61

The element $e := \rho^{-1}([1_A]) \in \mathcal{T}_M$ is called the *unit vector field*.
The element $E := \rho^{-1}([\mathfrak{F}_A]) \in \mathcal{T}_M$ is called the *Euler vector field*.

Proposition 62

We have

$$E = \sum_{i=1}^{\mu_A} (1 - q_i) t_i \frac{\partial}{\partial t_i}, \tag{4.16}$$

and the following "Euler's identity":

$$\mathfrak{F}_A = E\mathfrak{F}_A + [\deg_A, \mathfrak{F}_A]_G\,. \tag{4.17}$$

Definition 63

Define a morphism d_γ of graded $\mathcal{O}_M$-modules on $\Omega_\bullet(A) \otimes_{\mathbb{C}} \mathcal{O}_M$ as

$$d_\gamma := -L_{\mathfrak{F}_A}, \tag{4.18}$$

which is a deformation by γ of the boundary operator d on $\Omega_\bullet(A)$.

Proposition 64

We have

$$d_\gamma^2 = 0. \tag{4.19a}$$

$$[d_\gamma, i_X] = i_{d_\gamma X}, \quad X \in \mathcal{T}^\bullet_{poly}(A) \otimes_{\mathbb{C}} \mathcal{O}_M. \tag{4.19b}$$

$$[B, i_{\mathfrak{F}_A}] = -L_{\mathfrak{F}_A} = d_\gamma. \tag{4.19c}$$

$$[B, d_\gamma] = 0. \tag{4.19d}$$

In particular, d_γ defines a boundary operator on $\Omega_\bullet(A) \otimes_{\mathbb{C}} \mathcal{O}_M$.

Deformed filtered de Rham cohomology $\mathcal{H}_{\mathfrak{F}_A}$

Definition 65

Let u be a formal variable of degree 2.

1. Define a graded $\mathcal{O}_M((u))$-module by

$$\mathcal{H}_{\mathfrak{F}_A} := H_\bullet((\Omega_\bullet(A) \otimes_{\mathbb{C}} \mathcal{O}_M)((u)), d_\gamma + uB). \qquad (4.20)$$

   We call $\mathcal{H}_{\mathfrak{F}_A}$ the *deformed filtered de Rham cohomology*.

2. Define the graded $\mathcal{O}_M[[u]]$-modules

$$\mathcal{H}^{(-p)}_{\mathfrak{F}_A} := H_\bullet((\Omega_\bullet(A) \otimes_{\mathbb{C}} \mathcal{O}_M)[[u]]u^p, d_\gamma + uB), \ p \in \mathbb{Z}. \quad (4.21)$$

3. Define a graded $\mathcal{O}_M$-module $\Omega_{\mathfrak{F}_A}$ by

$$\Omega_{\mathfrak{F}_A} := H_\bullet(\Omega_\bullet(A) \otimes_{\mathbb{C}} \mathcal{O}_M, d_\gamma). \qquad (4.22)$$

Proposition 66

The graded $\mathcal{O}_M[[u]]$-modules $\mathcal{H}_{\mathfrak{F}_A}^{(-p)}$ is free of rank μ_A for all $p \in \mathbb{Z}$ and there is an exact sequence of graded $\mathcal{O}_M$-modules

$$0 \longrightarrow \mathcal{H}_{\mathfrak{F}_A}^{(-p-1)} \longrightarrow \mathcal{H}_{\mathfrak{F}_A}^{(-p)} \longrightarrow \Omega_{\mathfrak{F}_A} \longrightarrow 0. \tag{4.23}$$

The $\mathcal{O}_M[[u]]$-modules $\mathcal{H}_{\mathfrak{F}_A}^{(-p)}$, $p \in \mathbb{Z}$ define an increasing filtration of $\mathcal{H}_{\mathfrak{F}_A}$ such that

$$\bigcup_{p\in\mathbb{Z}} \mathcal{H}_{\mathfrak{F}_A}^{(-p)} = \mathcal{H}_{\mathfrak{F}_A} \quad \text{and} \quad \bigcap_{p\in\mathbb{Z}} \mathcal{H}_{\mathfrak{F}_A}^{(-p)} = \{0\} \tag{4.24}$$

so that $\mathcal{H}_{\mathfrak{F}_A}$ is complete with respect to the filtration.
The multiplication of u induces an isomorphism of $\mathbb{C}$-modules

$$u : \mathcal{H}_{\mathfrak{F}_A}^{(-p)} \cong \mathcal{H}_{\mathfrak{F}_A}^{(-p-1)}. \tag{4.25}$$

Gauß–Manin connection on $\mathcal{H}_{\mathfrak{F}_A}$

Set

$$\mathcal{T}_{\widehat{\mathbb{A}}^1_u \times M} := \mathcal{O}_M[[u]]\frac{d}{du} \oplus \mathcal{O}_M[[u]] \otimes_{\mathcal{O}_M} \mathcal{T}_M. \tag{4.26}$$

Definition 67

Define a connection

$$\nabla^\gamma : \mathcal{T}_{\widehat{\mathbb{A}}^1_u \times M} \otimes_{\mathbb{C}} ((\Omega_\bullet(A) \otimes_{\mathbb{C}} \mathcal{O}_M)((u))) \to (\Omega_\bullet(A) \otimes_{\mathbb{C}} \mathcal{O}_M)((u)) \tag{4.27}$$

by

$$\nabla^\gamma_{\frac{d}{du}} := \frac{d}{du} - \frac{1}{u^2} i_{\mathfrak{F}_A}, \tag{4.28a}$$

$$\nabla^\gamma_{\frac{\partial}{\partial t_i}} := \frac{\partial}{\partial t_i} + \frac{1}{u} i_{\frac{\partial \mathfrak{F}_A}{\partial t_i}}, \quad i = 1, \ldots, \mu_A. \tag{4.28b}$$

Proposition 68

The connection ∇^γ is a flat connection satisfying

$$\left[\nabla^\gamma_{u\frac{d}{du}}, d_\gamma + uB\right] = d_\gamma + uB, \tag{4.29a}$$

$$\left[\nabla^\gamma_{\frac{\partial}{\partial t_i}}, d_\gamma + uB\right] = 0, \quad i = 1, \ldots, \mu_A. \tag{4.29b}$$

Therefore, ∇^γ induces a connection on $\mathcal{H}_{\mathfrak{F}_A}$. Moreover, we have

$$\nabla^\gamma_{\frac{\partial}{\partial t_i}}\left(\mathcal{H}^{(0)}_{\mathfrak{F}_A}\right) \subset \mathcal{H}^{(1)}_{\mathfrak{F}_A}, \quad i = 1, \ldots, \mu_A, \tag{4.30a}$$

$$\nabla^\gamma_{u\frac{d}{du}+E}\left(\mathcal{H}^{(0)}_{\mathfrak{F}_A}\right) \subset \mathcal{H}^{(0)}_{\mathfrak{F}_A}. \tag{4.30b}$$

Fundamental solution to the Gauß–Manin connection

Proposition 69

For all $\omega \in (\Omega_\bullet(A) \otimes_{\mathbb{C}} \mathcal{O}_M/\mathfrak{m}^k)((u))$, *we have*

$$(d_\gamma + uB)\left(e^{-\frac{i_{\gamma(t)}}{u}}\omega\right) = e^{-\frac{i_{\gamma(t)}}{u}}(d + uB)\,\omega. \qquad (4.31)$$

Let $s^{(0)} : \Omega_{\mathfrak{f}_A} \longrightarrow \mathcal{H}^{(0)}_{\mathfrak{f}_A}$ be a good section and let $\{v_1, \ldots, v_{\mu_A}\}$ be elements of $\Omega_{\mathfrak{f}_A}$ as in the proof of Proposition 45.
Set $\zeta_i := s^{(0)}(v_i)$, $i = 1, \ldots, \mu_A$. By the above proposition, it follows that $\left[e^{-\frac{i_{\gamma(t)}}{u}}\zeta_i\right]_k \in \mathcal{H}_{\mathfrak{F}_A,k} := \mathcal{H}_{\mathfrak{F}_A} \otimes_{\mathcal{O}_M} (\mathcal{O}_M/\mathfrak{m}^k)$.

Proposition 70

We have the following equations:

$$\nabla^{\gamma}_{\frac{\partial}{\partial t_j}} \left[e^{-\frac{i_{\gamma(t)}}{u}} \zeta_i \right]_k = 0, \quad i, j = 1, \ldots, \mu_A, \tag{4.32}$$

$$\nabla^{\gamma}_{u\frac{d}{du}} \left[e^{-\frac{i_{\gamma(t)}}{u}} \zeta_i \right]_k = q_i \cdot \left[e^{-\frac{i_{\gamma(t)}}{u}} \zeta_i \right]_k, \quad i = 1, \ldots, \mu_A. \tag{4.33}$$

Therefore, we have the following identification:

$$\mathcal{H}_{\mathfrak{f}_A} \cong \left\{ \omega \in \mathcal{H}_{\mathfrak{F}_A, k} \;\middle|\; \nabla^{\gamma}_{\frac{\partial}{\partial t_i}} \omega = 0,\ i = 1, \ldots, \mu_A \right\}, \tag{4.34}$$

$$\zeta_i \mapsto \left[e^{-\frac{i_{\gamma(t)}}{u}} \zeta_i \right]_k.$$

Higher residue pairing

Corollary 71

The bilinear form $K_{\mathfrak{f}_A}$ induces an $\mathcal{O}_M$-bilinear form

$$K_{\mathfrak{F}_A} : \mathcal{H}_{\mathfrak{F}_A} \otimes_{\mathcal{O}_M} \mathcal{H}_{\mathfrak{F}_A} \longrightarrow \mathcal{O}_M((u))$$

satisfying the axioms of higher residue pairings.

Primitive form

Definition 72

An element $\zeta \in \mathcal{H}^{(0)}_{\mathfrak{F}_A}$ is a *primitive form* for $(\mathcal{H}^{(0)}_{\mathfrak{F}_A}, \nabla, K_{\mathfrak{F}_A})$ if

1. $u\nabla_e \zeta = \zeta$ and ζ induces $\mathcal{O}_M$-isomorphism:
$$\mathcal{T}_M[[u]] \cong \mathcal{H}^{(0)}_{\mathfrak{F}_A}, \quad \sum_{k=0}^{\infty} \delta_k u^k \mapsto \sum_{k=0}^{\infty} u^k (u\nabla_{\delta_k}\zeta).$$
2. $K_{\mathfrak{F}_A}(u\nabla_\delta \zeta, u\nabla_{\delta'}\zeta) \in \mathcal{O}_M \cdot u^w,$ for all $\delta, \delta' \in \mathcal{T}_M$.
3. There exists $r \in \mathbb{C}$ such that $\nabla_{u\frac{d}{du}+E}\zeta = r\zeta$.
4. There exists a connection $\not\nabla$ on $\mathcal{T}_M$ such that
$$u\nabla_X \nabla_Y \zeta = \nabla_{X\circ Y}\zeta + u\nabla_{\not\nabla_X Y}\zeta, \quad X, Y \in \mathcal{T}_M.$$
5. There exists an $\mathcal{O}_M$-endomorphism $N : \mathcal{T}_M \longrightarrow \mathcal{T}_M$ such that
$$u\nabla_{\frac{d}{du}}(u\nabla_X \zeta) = -\nabla_{E\circ X}\zeta + u\nabla_{NX}\zeta, \quad X \in \mathcal{T}_M.$$

In particular, the constant r is called the *minimal exponent*.

The $\mathbb{C}$-linear subspace $S := s^{(0)}(\Omega_{\mathfrak{f}_A})[u^{-1}]u^{-1}$ of $\mathcal{H}_{\mathfrak{f}_A}$ satisfies

$$\mathcal{H}_{\mathfrak{f}_A} = \mathcal{H}^{(0)}_{\mathfrak{f}_A} \oplus S,\ u^{-1}S \subset S,\ \nabla_{u\frac{d}{du}} S \subset S,\ K_{\mathfrak{f}_A}(S,S) \subset \mathbb{C}[u^{-1}]u^{w-2}.$$

Define the element ζ by

$$\zeta := \lim_{k\to\infty} \tau_{\geq 0}\left[e^{-\frac{i_{\gamma(t)}}{u}}\zeta_1\right]_k,$$

where $\tau_{\geq 0} : \mathcal{H}_{\mathfrak{F}_A,k} = \mathcal{H}^{(0)}_{\mathfrak{F}_A,k} \oplus e^{-\frac{i_{\gamma(t)}}{u}} S \longrightarrow \mathcal{H}^{(0)}_{\mathfrak{F}_A,k}$ is the projection. (famous method by M. Saito and Barannikov)

Theorem 73

The element ζ is a (formal) primitive form with the minimal exponent zero for the tuple $(\mathcal{H}^{(0)}_{\mathfrak{F}_A}, \nabla, K_{\mathfrak{F}_A})$.

From primitive forms to Frobenius manifolds

Theorem 74 (cf. Saito–Takahashi)

Let ζ be a primitive form with the minimal exponent zero for the tuple $(\mathcal{H}^{(0)}_{\mathfrak{F}_A}, \nabla, K_{\mathfrak{F}_A})$.
Define an $\mathcal{O}_M$-bilinear form $\eta : \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \longrightarrow \mathcal{O}_M$ by

$$\eta(X, Y) := (-1)^{w \cdot \overline{Y}} K_{\mathfrak{F}_A}(u\nabla_X \zeta, u\nabla_Y \zeta) \cdot u^{-w}. \qquad (4.35)$$

Then, the tuple $(\circ, \eta, e, E)$ gives a formal Frobenius structure on M such that

$$Lie_E(\circ) = \circ, \quad Lie_E(\eta) = (2 - w)\eta.$$

Thank you!