On the Genus Two Free Energies for Semisimple Frobenius Manifolds

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Primitive forms and related subjects
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Based on joint work with Boris Dubrovin and Si-Qi Liu

1. On the genus two free energies for semisimple Frobenius manifolds, *Russian Journal of Mathematical Physics* 19 (2012), 273–298. eprint arXiv: 1205.5990.

and on joint work with Yulong Fu, Si-Qi Liu, Chunhui Zhou

2. Proof of a conjecture on the genus two free energy associated to the A_n singularity, *Journal of Geometry and Physics* 76 (2014), 10–24.

Outline of the talk

1. Introduction

2. Main Results

3. Proofs of the Theorems

4. Checking the Validity of the Conjecture

5. Some Remarks

1. Introduction

Frobenius manifold (Dubrovin): Encodes the properties of the primary free energy $F = F(v^1, \ldots, v^n)$ of a 2d topological field theory

$$\frac{\partial^3 F}{\partial v^1 \partial v^\alpha \partial v^\beta} = \eta_{\alpha\beta} = \text{constant}, \quad (\eta_{\alpha\beta}): \text{ nondegenerate},$$

$$c^\gamma_{\alpha\beta} = \eta^{\gamma\xi} \frac{\partial^3 F}{\partial v^\xi \partial v^\alpha \partial v^\beta} : \quad \begin{array}{c} \text{structure constants of} \\ \text{an associative algebra} \end{array}$$

$$\partial_E F = (3-d)F + \text{quadratic terms in } v^\alpha,$$

$$\text{here the Euler vector } E = \sum_{\alpha=1}^{n} (d_\alpha v^\alpha + r_\alpha) \frac{\partial}{\partial v^\alpha}.$$

They are the WDVV equations of associativity.

Examples of Frobenius manifolds:

- From the genus zero part of a 2d TFT (Gromov-Witten, FJRW), the potential of the Frobenius manifold is given by

$$F(v) = \mathcal{F}_0(t^{1,0}, t^{2,0}, \dots, t^{n,0}, t^{1,1}, t^{2,1}, \dots)|_{t^{\alpha,p\geq 1}=0, t^{\alpha,0}=v^\alpha}.$$

  Here $t^{\alpha,p}, \alpha = 1, \dots, n,\ p \geq 0$ are the coupling constants associated to the n primary fields and their gravitational descendants.

- From the flat structures defined by Kyoji Saito theory of primitive forms for quasi-homogeneous isolated singularities; from the flat structures defined on the orbit space of finite reflection groups by Kyoji Saito, Iamaki Yano and Jiro Sekiguchi, and of the extended affine Weyl groups by Dubrovin, Z..

High genus theory of semisimple Frobenius manifolds:

Developed by Dubrovin-Z. using the approach of integrable systems, and also by Givental using the approach of quantum canonical transformations.

The appearance of integrable hierarchies:

Denote

$$v^{\alpha}(t) = \eta^{\alpha\gamma} \frac{\partial \mathcal{F}_0(t)}{\partial t^{1,0} \partial t^{\gamma,0}}, \quad \alpha = 1, \ldots, n.$$

Here

$$(\eta^{\alpha\beta}) = \left(\frac{\partial^3 \mathcal{F}_0(t)}{\partial t^{1,0} \partial t^{\alpha} \partial t^{\beta}} \right)^{-1}.$$

Redenote $t^{1,0} = x$, then we have the equations

$$\frac{\partial v^{\alpha}}{\partial t^{\beta,q}} = \eta^{\alpha\gamma} \frac{\partial}{\partial x} \left(\frac{\partial^2 \mathcal{F}_0(t)}{\partial t^{\gamma,0} \partial t^{\beta,q}} \right), \quad \alpha, \beta = 1, \ldots, q \geq 0.$$

As observed by Dijkgraaf & Witten (1990) the two-point functions that appear in the r.h.s. of the above equation can be represented as functions of $v^1(t), \ldots, v^n(t)$ (the constitutive relation)

$$\frac{\partial^2 \mathcal{F}_0(t)}{\partial t^{\alpha,0} \partial t^{\beta,q}} = \Omega_{\alpha,0;\beta,q}(v(t)) = \frac{\partial \theta_{\beta,q+1}(v)}{\partial v^\alpha}.$$

A hierarchy of infinite dimensional Hamiltonian systems

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} = P^{\alpha\gamma} \frac{\delta H_{\beta,q}}{\delta v^\gamma}$$

with Hamiltonian operator P and pairwise commutative Hamiltonians given by

$$P^{\alpha\beta} = \eta^{\alpha\beta} \frac{\partial}{\partial x}, \quad H_{\beta,q} = \int \theta_{\beta,q+1}(v(x)) dx.$$

Reconstructing $\mathcal{F}_0(t)$ from Frobenius manifold

We first recover the above mention integrable hierarchy

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} = \eta^{\alpha\gamma}\frac{\partial}{\partial x}\left(\frac{\partial \theta_{\beta,q+1}(v)}{\partial v^\gamma}\right), \quad \alpha,\beta = 1,\ldots,n,\ \ q\geq 0,$$

which is a bihamitonian integrable hierarchy of hydrodynamic type, called the Principal Hierarchy. Then we need to use the particular solution selected by the string equation

$$\sum_{p\geq 1} t^{\alpha,p}\frac{\partial \mathcal{F}_0}{\partial t^{\alpha,p-1}} + \frac{1}{2}\eta_{\alpha\beta}t^{\alpha,0}t^{\beta,0} = \frac{\partial \mathcal{F}_0}{\partial t^{1,0}},$$

in particular, this implies

$$v^\alpha(t)|_{\text{small phase space}} = t^{\alpha,0}, \quad \alpha = 1,\ldots,n.$$

The deformed flat connection of a FM $(M^n; \cdot\,; \langle\ ,\ \rangle;\ e;\ E)$:

$$\tilde{\nabla}_a b = \nabla_a b + z\, a \cdot b$$

Extend it to $M \times \mathbb{C}^*$ by

$$\tilde{\nabla}_{\frac{d}{dz}} b = \partial_z b + E \cdot b - \frac{1}{z} \mu\, b$$

with $\mu = \frac{2-d}{2} - \nabla E$.

The deformed flat coordinates $\tilde{v}_1(v; z), \ldots, \tilde{v}_n(v; z)$ satisfying

$$\tilde{\nabla} d\tilde{v}_\alpha(v; z) = 0, \quad \alpha = 1, \ldots, n.$$

The functions $\theta_{\beta,q}(v)$

The deformed flat coordinates have the form

$$(\tilde v_1(v;z),\dots,\tilde v_n(v;z)) = (\theta_1(v;z),\dots,\theta_n(v;z))z^\mu z^R$$

Here $\theta_1(v;z),\dots,\theta_n(v;z)$ are analytic at $z=0$ with Taylor expansion

$$\theta_\alpha(v;z) = \sum_{p\geq 0}\theta_{\alpha,p}(v)z^p$$

satisfying the normalization conditions

$$\theta_\alpha(v;0) = \eta_{\alpha\beta}v^\beta, \quad \alpha = 1,\dots,n$$
$$\langle\nabla\theta_\alpha(v;-z),\nabla\theta_\beta(v;z)\rangle = \eta_{\alpha\beta}.$$

$\mu, R:$ monodromy data at $z=0$.

Topological solution of the Principal Hierarchy

Particular solution $v^\alpha = v^\alpha(t)$ of the integrable hierarchy that satisfies the string equation is obtained by solving the equations (generalized hodograph transformation)

$$\sum_{q\geq 0} \tilde{t}^{\beta,q}\,\nabla\theta_{\beta,q}(v) = 0, \quad \tilde{t}^{\beta,q} = t^{\beta,q} - \delta_1^\beta \delta_1^q.$$

The genus zero free energy

$$\mathcal{F}_0(t) = \frac{1}{2}\sum_{\alpha,p;\beta,q} \tilde{t}^{\alpha,p}\tilde{t}^{\beta,q}\,\Omega_{\alpha,p;\beta,q}(v)|_{v=v(t)},$$

$$\sum \Omega_{\alpha,p;\beta,q}(v)\,z^p w^q = \frac{< \nabla\theta_\alpha(z), \nabla\theta_\beta(w) > -\eta_{\alpha\beta}}{z+w}.$$

How to reconstruct the higher genera free energies $\mathcal{F}_g(t)$?

We (Dubrovin, Z. 2001) use the properties of the Virasoro symmetries of the Principal Hierarchy to determine $\mathcal{F}_g$.

The first symmetry is the Galilean symmetry:

$$v^\alpha \mapsto v^\alpha + \epsilon \left(\sum_{\beta,q} t^{\beta,q} \frac{\partial v^\alpha}{\partial t^{\beta,q-1}} + \delta_1^\alpha \right) + \mathcal{O}(\epsilon^2),$$

$$\tau \mapsto \tau + \epsilon L_{-1} \tau + \mathcal{O}(\epsilon^2)$$

with the tau function τ and the operator L_{-1} defined by

$$\tau = e^{\mathcal{F}_0(t)}, \quad L_{-1} = \sum_{q\geq 1} t^{\beta,q} \frac{\partial}{\partial t^{\beta,q-1}} + \frac{1}{2} \eta_{\alpha\beta}\, t^{\alpha,0} t^{\beta,0}.$$

Virasoro symmetries acting on the tau function

$$\tau \mapsto \tau + \epsilon \left(a_m^{\alpha,p;\beta,q} \frac{1}{\tau} \frac{\partial \tau}{\partial t^{\alpha,p}} \frac{\partial \tau}{\partial t^{\beta,q}} + b_{m;\alpha,p}^{\beta,q} t^{\alpha,p} \frac{\partial \tau}{\partial t^{\beta,q}} + c_{m;\alpha,p\beta,q} t^{\alpha,p} t^{\beta,q} \tau \right) + \mathcal{O}(\epsilon^2)$$

with the Virasoro operators (for $m \geq -1$)

$$L_m = \varepsilon^2 a_m^{\alpha,p;\beta,q} \frac{\partial^2}{\partial t^{\alpha,p} \partial t^{\beta,q}} + b_{m;\alpha,p}^{\beta,q} t^{\alpha,p} \frac{\partial}{\partial t^{\beta,q}} + \varepsilon^{-2} c_{m;\alpha,p,\beta,q} t^{\alpha,p} t^{\beta,q} + \kappa_0 \, \delta_{m,0}, \quad m \geq -1.$$

Linearization of the Virasoro symmetries

Note that the higher genera free energy $\mathcal{F}_g(t)$ can be represented as functions of the two-point functions

$$\mathcal{F}_g(t) = F_g(v, v_x, v_{xx}, \dots)|_{v=v(t)}$$

We require that the infinitesimal Virasoro symmetries act linearly on the full genera tau function

$$\tau = e^{\varepsilon^{-2}\mathcal{F}_0 + \sum_{g\geq 1} \varepsilon^{2g-2} F_g(v, v_x, v_{xx}, \dots)|_{v=v(t)}}.$$

in the following way

$$\tau \mapsto \tau + \epsilon\, L_m \tau + \mathcal{O}(\epsilon^2), \quad m \geq -1.$$

Here the parameter ε is called the string coupling constant.

The loop equation

The condition of linearization of the Virasoro symmetries is equivalent to a system of equations, called the loop equation, for the functions $F_g, g \geq 1$. For example, when $n = 1$ we have

$$\sum_r \frac{\partial \Delta\mathcal{F}}{\partial v^{(r)}} \partial_x^r \frac{1}{v-\lambda} + \sum_{r\geq 1} \frac{\partial \Delta\mathcal{F}}{\partial v^{(r)}} \sum_{k=1}^{r} \binom{r}{k} \partial_x^{k-1} \frac{1}{\sqrt{v-\lambda}} \partial_x^{r-k+1} \frac{1}{\sqrt{v-\lambda}}$$
$$= \frac{1}{16\,\lambda^2} - \frac{1}{16(v-\lambda)^2} - \frac{\kappa_0}{\lambda^2}$$
$$+ \frac{\epsilon^2}{2} \sum \left[\frac{\partial^2 \Delta\mathcal{F}}{\partial v^{(k)} \partial v^{(l)}} + \frac{\partial \Delta\mathcal{F}}{\partial v^{(k)}} \frac{\partial \Delta\mathcal{F}}{\partial v^{(l)}} \right] \partial_x^{k+1} \frac{1}{\sqrt{v-\lambda}} \partial_x^{l+1} \frac{1}{\sqrt{v-\lambda}}$$
$$- \frac{\epsilon^2}{16} \sum \frac{\partial \Delta\mathcal{F}}{\partial v^{(k)}} \partial_x^{k+2} \frac{1}{(v-\lambda)^2}.$$

Here $\Delta\mathcal{F} = \sum_{g\geq 1} \varepsilon^{2g-2} F_g$.

The ε^0's coefficients of the loop equation give the equation for $\mathcal{F}_1$:

$$\frac{1}{v-\lambda}\frac{\partial F_1}{\partial v} - \frac{3}{2}\frac{v'}{(v-\lambda)^2}\frac{\partial F_1}{\partial v'} = \frac{1}{16\,\lambda^2} - \frac{1}{16(v-\lambda)^2} - \frac{\kappa_0}{\lambda^2}.$$

This implies that

$$\kappa_0 = \frac{1}{16}, \quad F_1 = \frac{1}{24}\log v'.$$

Similarly, the coefficients of ε^2 give the squations for F_2, from which we obtain

$$F_2 = \frac{v^{(4)}}{1152\,v'^2} - \frac{7\,v''v'''}{1920\,v'^3} + \frac{v''^3}{360\,v'^4}.$$

For a general semisimple Frobenius manifold, the ε^0 terms of the loop equation yield the equation

$$\sum_{i=1}^{n} \frac{\partial F_1}{\partial u_i} \frac{1}{u_i - \lambda} - \sum_{i=1}^{n} \frac{\partial F_1}{\partial u_i'} \frac{u_i'}{(u_i - \lambda)^2} + \sum \frac{\partial F_1}{\partial v_x^\gamma} \partial_1 p_\alpha G^{\alpha\beta} \partial_x \partial^\gamma p_\beta$$
$$= -\frac{1}{16} \sum_{i=1}^{n} \frac{1}{(\lambda - u_i)^2} - \frac{1}{2} \sum_{i<j} \frac{V_{ij}^2}{(\lambda - u_i)(\lambda - u_j)} + \frac{1}{4\,\lambda^2} \operatorname{tr}\left(\frac{1}{4} - \hat{\mu}^2\right) - \frac{\kappa_0}{\lambda^2}.$$

Here $u_1, \ldots, u_n$ are the canonical coordinates of the semisimple Frobenius manifold M, which have the property

$$\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} = \delta_{i,j} \frac{\partial}{\partial u_i}.$$

The genus one free energy

$$
\begin{aligned}
F_1(v, v_x) &= \frac{1}{24} \log\det\left(c_{\alpha\beta\gamma}(v) v_x^\gamma\right) + G(v) \\
&= \frac{1}{24} \log\det\left(\frac{\partial^3 \mathcal{F}_0}{\partial t^{1,0} \partial t^{\alpha,0} \partial t^{\beta,0}}\right) + G(v).
\end{aligned}
$$

(Witten-Dijkgraaf and Getzler)

The function $G(v)$ has the form (Dubrovin, Z.)

$$
G(v) = \log \frac{\tau_I(v)}{J^{1/24}(v)}\,.
$$

- For Frobenius manifolds come from ADE singularities, Hertling showed that $G = 0$.
- For Frobenius manifolds come from the extended affine Weyl groups of ADE type, or Gromov-Witten invariants of $\mathbb{P}^1$-orbifolds with positive Euler characteristics. There are at most three orbifold points with multiplicity p, q, r satisfying $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. The equivalence of these two class of FM is established by Milanov & Tseng; Rossi. We have the solutions

$$\begin{array}{ll} (p, q, 1) & \tilde{A}_{p,q} \\ (2, 2, r) & \tilde{D}_{r+2} \\ (2, 3, r) & \tilde{E}_{r+3} \end{array}$$

  We have $G = -\frac{1}{24r} v^n$ (proved for $\tilde{A}_{p,q}$ by Strachan and for $\tilde{D}_{r+2}$ by Liu, Z.)

The genus two free enegy

$$
\begin{aligned}
F_2 = {} & \frac{1}{1152}\frac{u_i^{IV}}{{u_i'}^2\,h_i^2} - \frac{7}{1920}\frac{u_i''\,u_i'''}{{u_i'}^3\,h_i^2} + \frac{1}{360}\frac{{u_i''}^3}{{u_i'}^4\,h_i^2} + \frac{1}{40}\frac{V_{ij}^2\,u_i'''}{u_{ij}\,u_i'\,h_i^2} \\
& + \frac{1}{640}\frac{V_{ij}\,h_j\,u_j'\,u_i'''}{u_{ij}\,{u_i'}^2\,h_i^3} - \frac{19}{2880}\frac{V_{ij}\,u_i'''\,h_j}{u_{ij}\,u_i'\,h_i^3} + \frac{1}{1152}\frac{V_{ij}\,u_i'''\,h_i}{u_{ij}\,u_j'\,h_j^3} + \frac{7}{40}\frac{V_{ij}^2\,V_{ik}^2\,u_i''}{u_{ij}\,u_{ik}\,h_i^2} \\
& - \frac{1}{240}\frac{V_{ij}^2\,V_{ik}\,u_i''\,h_k\,(32\,u_i' - 7\,u_k')}{u_{ij}\,u_{ik}\,u_i'\,h_i^3} + \frac{1}{40}\frac{V_{ij}\,V_{jk}^2\,u_i''\,h_i}{u_{ij}\,u_{jk}\,h_j^3} - \frac{1}{48}\frac{V_{ij}\,V_{jk}^2\,u_j'\,u_i''}{u_{ij}\,u_{jk}\,u_i'\,h_i\,h_j} \\
& - \frac{3}{64}\frac{V_{ij}^2\,u_i''}{u_{ij}^2\,h_i^2} - \frac{11}{480}\frac{V_{ij}^2\,{u_i''}^2}{u_{ij}\,{u_i'}^2\,h_i^2} + \frac{29}{5760}\frac{V_{ij}\,V_{jk}\,u_i''\,h_i\,h_k\,(u_k' - 2\,u_j')}{u_{ij}\,u_{jk}\,u_j'\,h_j^4} \\
& + \frac{1}{384}\frac{V_{ij}\,V_{ik}\,u_i''\,h_k\,(u_i' - u_k')}{u_{ij}\,u_{ik}\,u_j'\,h_j^3} + \frac{1}{1920}\frac{V_{ij}\,V_{ik}\,u_i''\,h_j\,h_k\left(54\,{u_i'}^2 - 25\,u_i'\,u_j' - u_j'\,u_k'\right)}{u_{ij}\,u_{ik}\,{u_i'}^2\,h_i^4} \\
& + \frac{1}{576}\frac{V_{ij}\,V_{jk}\,u_i''\,h_k\,(2\,u_j' - u_k')}{u_{ij}\,u_{jk}\,u_i'\,h_i\,h_j^2} - \frac{1}{5760}\frac{V_{ij}\,V_{jk}\,u_k'\,u_i''\,h_k\,(27\,u_i' + u_k')}{u_{jk}\,u_{ik}\,{u_i'}^2\,h_i^3} \\
& - \frac{19}{1920}\frac{V_{ij}\,V_{jk}\,u_i''\,h_k}{u_{ij}\,u_{ik}\,h_i^3} + \frac{1}{5760}\frac{V_{ij}\,V_{jk}\,h_k\left(27\,u_i'\,u_k' - {u_j'}^2 + 2\,u_j'\,u_k'\right)\,u_i''}{u_{ij}\,u_{jk}\,{u_i'}^2\,h_i^3} \\
& + \frac{1}{288}\frac{V_{ij}\,V_{jk}\,u_i''\,h_i}{u_{jk}\,u_{ik}\,h_k^3} + \frac{1}{384}\frac{V_{ij}\,V_{jk}\,u_i'\,u_i''\,h_i}{u_{ij}\,u_{ik}\,u_k'\,h_k^3} - \frac{1}{576}\frac{V_{ij}\,V_{jk}\,u_k'\,u_i''}{u_{jk}\,u_{ik}\,u_i'\,h_i\,h_k} \\
& - \frac{1}{384}\frac{V_{ik}\,V_{jk}\,u_k'\,u_i''\,h_i}{u_{ik}\,u_{jk}\,u_j'\,h_j^3} + \frac{1}{1920}\frac{V_{ij}\,{u_i''}^2\,h_j\,(11\,u_i' - 5\,u_j')}{u_{ij}\,{u_i'}^3\,h_i^3} - \frac{1}{5760}\frac{V_{ij}\,u_i''\,u_j''\,h_j}{u_{ij}\,{u_i'}^2\,h_i^3} \\
& + \frac{1}{5760}\frac{V_{ij}\,u_i''\,h_j\left(57\,{u_i'}^2 - 27\,u_i'\,u_j' - {u_j'}^2\right)}{u_{ij}^2\,{u_i'}^2\,h_i^3} + \frac{1}{1152}\frac{V_{ij}\,u_i''\,h_i\,(4\,u_j' - 3\,u_i')}{u_{ij}^2\,u_j'\,h_j^3} \\
& - \frac{1}{576}\frac{V_{ij}\,u_j'\,u_i''}{u_{ij}^2\,u_i'\,h_i\,h_j} - \frac{1}{1152}\frac{V_{ij}\,u_i''\,u_j''}{u_{ij}\,u_i'\,u_j'\,h_i\,h_j} + \frac{1}{10}\frac{V_{ij}^2\,V_{ik}^2\,V_{il}^2\,{u_i'}^2}{u_{ij}\,u_{ik}\,u_{il}\,h_i^2} \\
& - \frac{7}{20}\frac{V_{ij}^2\,V_{ik}^2\,V_{il}\,h_l\,{u_i'}^2}{u_{ij}u_{ik}u_{il}\,h_i^3} + \frac{7}{40}\frac{V_{ij}^2\,V_{ik}^2\,V_{il}\,h_l\,u_i'\,u_l'}{u_{ij}\,u_{ik}\,u_{il}\,h_i^3} - \frac{1}{8}\frac{V_{ij}^2\,V_{ik}\,V_{kl}^2\,u_i'\,u_k'}{u_{ij}\,u_{ik}\,u_{kl}\,h_i\,h_k} \\
& + \frac{1}{40}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,h_l\left({u_k'}^2 - 3\,{u_i'}^2 - 2\,u_k'\,u_l'\right)}{u_{ij}\,u_{ik}\,u_{kl}\,h_i^3} + \frac{3}{40}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,u_i'\,u_l'\,h_l}{u_{ij}\,u_{ik}\,u_{il}\,h_i^3} \\
& + \frac{1}{40}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,h_l\left(3\,{u_i'}^2 + {u_l'}^2\right)}{u_{ij}\,u_{kl}\,u_{il}\,h_i^3} + \frac{1}{48}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,h_l\,u_i'\,(2\,u_k' - u_l')}{u_{ij}\,u_{ik}\,u_{kl}\,h_i\,h_k^2} \\
& + \frac{5}{96}\frac{V_{ij}^2\,V_{ik}\,V_{il}\,h_k\,h_l\,\left(4\,{u_i'}^2 - 4\,u_i'\,u_k' + u_k'u_l'\right)}{u_{ij}u_{ik}u_{il}\,h_i^4} - \frac{83}{480}\frac{V_{ij}^2\,V_{ik}^2\,{u_i'}^2}{u_{ij}\,u_{ik}^2\,h_i^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{144}\frac{V_{ij}\,V_{ik}\,V_{jl}\,V_{kl}\,{u_i'}^2}{u_{ik}\,u_{jl}\,u_{il}\,h_i^2}-\frac{1}{144}\frac{V_{ij}\,V_{ik}\,V_{jl}\,V_{kl}\,{u_i'}^2}{u_{ij}\,u_{ik}\,u_{kl}\,h_i^2}-\frac{1}{48}\frac{V_{ij}^2\,V_{ik}\,V_{kl}\,u_i'\,u_l'}{u_{ij}\,u_{kl}\,u_{il}\,h_i\,h_l}\\
&+\frac{29}{1920}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,h_l\left(u_k'\,u_l'-u_i'\,u_k'+2\,{u_i'}^2-2\,u_i'\,u_l'\right)}{u_{ij}\,u_{ik}\,u_{il}\,h_i^4}\\
&-\frac{29}{5760}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,h_l\,u_j'\,(2\,u_k'\,u_l'+2\,u_i'\,u_j'-u_j'\,u_k'-4\,u_i'\,u_l')}{u_{ij}\,u_{ik}\,u_{jl}\,h_i^4\,u_i'}\\
&-\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,h_l\,\,(4\,u_i'\,u_j'-4\,u_i'\,u_l'+u_k'\,u_l')}{u_{ij}\,u_{ik}\,u_{jl}\,h_i^2\,h_j^2}\\
&-\frac{1}{384}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\left(u_i'\,{u_j'}^2-2\,u_j'\,u_i'\,u_l'\right)}{u_{ij}\,u_{ik}\,u_{jl}\,u_k'\,h_k^3}-\frac{29}{5760}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,h_l\,{u_l'}^2\,(2\,u_i'-u_k')}{u_{ik}\,u_{jl}\,u_{il}\,h_i^4\,u_i'}\\
&+\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,{u_i'}^2\,(u_i'-3\,u_l')}{u_{ij}\,u_{ik}\,u_{il}\,u_k'\,h_k^3}-\frac{1}{384}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,u_i'\,{u_l'}^2}{u_{ik}\,u_{jl}\,u_{il}\,u_k'\,h_k^3}\\
&-\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,{u_j'}^2\,(3\,u_l'-2\,u_j')}{u_{ij}\,u_{jl}\,u_{jk}\,u_k'\,h_k^3}-\frac{1}{288}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,u_j'\,(u_j'-2\,u_l')}{u_{ik}\,u_{jl}\,u_{jk}\,h_k^3}\\
&+\frac{1}{576}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,u_k'\,(2\,u_k'-3\,u_l')}{u_{ik}\,u_{jk}\,u_{kl}\,h_k^3}-\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,{u_l'}^3}{u_{jl}\,u_{kl}\,u_{il}\,u_k'\,h_k^3}\\
&+\frac{1}{288}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_l\,{u_l'}^2}{u_{ik}\,u_{jl}\,u_{kl}\,h_k^3}-\frac{1}{576}\frac{V_{ij}\,V_{ik}\,V_{jl}\,h_k\,u_l'\,(u_k'-2\,u_i')}{u_{ik}\,u_{jl}\,u_{il}\,h_i^2\,h_l}\\
&-\frac{7}{1440}\frac{V_{ij}\,V_{ik}\,V_{il}\,h_j\,h_k\,h_l\left(8\,{u_i'}^3-12\,{u_i'}^2\,u_j'-u_j'\,u_k'\,u_l'+6\,u_i'\,u_j'\,u_k'\right)}{u_{ij}\,u_{ik}\,u_{il}\,h_i^5\,u_i'}\\
&-\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,V_{jl}\,u_k'\,u_l'}{u_{ik}\,u_{jl}\,u_{kl}\,h_k\,h_l}-\frac{29}{1152}\frac{V_{ij}\,V_{ik}\,V_{jk}\,{u_i'}^2}{u_{ij}\,u_{ik}^2\,h_i^2}-\frac{53}{1920}\frac{V_{ij}^2\,V_{ik}\,h_k\,u_i'\,u_k'}{u_{ij}\,u_{ik}\,u_{jk}\,h_i^3}\\
&-\frac{1}{320}\frac{V_{ij}^2\,V_{ik}\,h_k\left(3\,{u_i'}^2-8\,{u_k'}^2\right)}{u_{ij}\,u_{ik}^2\,h_i^3}-\frac{V_{ij}^2\,V_{ik}\,u_i'\,h_k}{u_{ij}^2\,u_{jk}\,h_i^3}\left(\frac{27}{640}\,u_k'-\frac{233}{2880}\,u_i'\right)\\
&-\frac{V_{ij}^2\,V_{ik}\,u_i'\,h_k}{u_{ik}^2\,u_{jk}\,h_i^3}\left(\frac{233}{2880}\,u_i'-\frac{67}{960}\,u_k'\right)+\frac{1}{1152}\frac{V_{ij}^2\,V_{ik}\,h_i\,{u_i'}^3}{u_{ij}\,u_{ik}^2\,u_k'\,h_k^3}-\frac{1}{576}\frac{V_{ij}^2\,V_{ik}\,h_i\,{u_i'}^3}{u_{ij}^2\,u_{ik}\,u_k'\,h_k^3}\\
&-\frac{1}{48}\frac{V_{ij}^2\,V_{ik}\,u_i'\,u_k'}{u_{ij}\,u_{ik}^2\,h_i\,h_k}+\frac{233}{1440}\frac{V_{ij}^3\,h_j\,{u_i'}^2}{u_{ij}^3\,h_i^3}-\frac{43}{384}\frac{V_{ij}^3\,h_j\,u_i'\,u_j'}{u_{ij}^3\,h_i^3}-\frac{1}{12}\frac{V_{ij}^3\,u_i'\,u_j'}{u_{ij}^3\,h_i\,h_j}\\
&+\frac{29}{5760}\frac{V_{ij}\,V_{ik}\,h_j\,h_k\left(3\,u_i'\,u_k'+3\,u_j'\,u_k'+6\,u_i'\,u_j'-6\,{u_i'}^2-2\,{u_j'}^2\right)}{u_{ij}^2\,u_{ik}\,h_i^4}\\
&+\frac{29}{5760}\frac{V_{ij}\,V_{ik}\,u_j'\,u_k'\,h_j\,h_k\,(u_k'-6\,u_i')}{u_{ij}\,u_{ik}^2\,u_i'\,h_i^4}+\frac{1}{576}\frac{V_{ij}\,V_{ik}\,u_j'\,h_k\,(2\,u_i'-u_k')}{u_{ij}^2\,u_{ik}\,h_i^2\,h_j}\\
&+\frac{1}{1152}\frac{V_{ij}\,V_{ik}\,u_{ij}\,h_k\left(3\,{u_i'}^2\,u_k'-3\,u_i'\,{u_k'}^2+{u_k'}^3-{u_i'}^3\right)}{u_{ik}^2\,u_{jk}^2\,u_j'\,h_j^3}\\
&+\frac{1}{576}\frac{V_{ij}\,V_{ik}\,u_{ik}\,h_k\left(-{u_i'}^3+3\,{u_j'}^2\,u_k'-4\,u_i'\,u_j'\,u_k'+2\,{u_i'}^2\,u_j'-2\,{u_j'}^3\right)}{u_{ij}^2\,u_{jk}^2\,u_j'\,h_j^3}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{384}\,\frac{V_{ij}\,V_{ik}\,h_k\left(-u_i'\,{u_k'}^2+{u_i'}^3-6\,{u_j'}^2\,u_k'\right)}{u_{ij}\,u_{jk}^2\,u_j'\,h_j^3}+\frac{1}{384}\,\frac{V_{ij}\,V_{ik}\,h_k\,{u_i'}^2\,u_k'}{u_{ij}^2\,u_{jk}\,u_j'\,h_j^3}\\
&+\frac{1}{288}\,\frac{V_{ij}\,V_{ik}\,h_k\left(4\,u_i'\,u_k'+{u_k'}^2-2\,{u_i'}^2+3\,{u_j'}^2\right)}{u_{ij}\,u_{jk}^2\,h_j^3}-\frac{1}{576}\,\frac{V_{ij}\,V_{ik}\,u_j'\,u_k'}{u_{ik}\,u_{jk}^2\,h_j\,h_k}\\
&+\frac{1}{384}\,\frac{V_{ij}\,V_{ik}\,h_k\left(2\,u_i'\,{u_k'}^2-{u_i'}^2\,u_k'-{u_k'}^3\right)}{u_{ik}\,u_{jk}^2\,u_j'\,h_j^3}+\frac{1}{288}\,\frac{V_{ij}\,V_{ik}\,h_k\left({u_k'}^2-2\,u_i'\,u_k'+{u_i'}^2\right)}{u_{ik}\,u_{jk}^2\,h_j^3}\\
&+\frac{1}{1152}\,\frac{V_{ij}^2\,u_i'\left(37\,u_i'\,u_j'\,h_j^2+10\,u_i'\,u_j'\,h_i^2-3\,{u_i'}^2\,h_i^2+11\,{u_j'}^2\,h_j^2\right)}{u_{ij}^3\,u_j'\,h_i^2\,h_j^2}\\
&-\frac{1}{576}\,\frac{V_{ij}\,h_j\left(4\,{u_i'}^3+4\,u_i'\,{u_j'}^2-6\,{u_i'}^2\,u_j'-{u_j'}^3\right)}{u_{ij}^3\,u_i'\,h_i^3}+\frac{1}{576}\,\frac{V_{ij}\,u_i'\,u_j'}{u_{ij}^3\,h_i\,h_j}.
\end{aligned}
$$

The loop equation can be solved recursively to yield a unique solution $F_g(v, v_x, \ldots, v^{3g-2}), g \geq 1$ (up to the addition of constants). If the genus zero from energy $\mathcal{F}_0$ is constructed from the topological solution of the Principal Hierarchy, then tau function (partition function, total descendant potential)

$$\tau(t) = e^{\varepsilon^{-2}\mathcal{F}_0(t) + \sum_{g\geq 2} \varepsilon^{2g-2} F_g(v, v_x, \ldots, v^{(3g-2)})|_{v=v(t)}}.$$

satisfies the Virasoro constraints

$$L_m|_{t^{1,1}\to t^{1,1}-1}\,\tau(t) = 0, \quad m \geq -1.$$

Note that the validity of Virasoro constraints for Gromov-Witten invariants of Fano varieties was formulated and conjectured by Eguchi, Hori, Xiong in 1997.

Motivation of our work:

To have a better understanding of the high genus free energies, attempt to represent the genus g free energy in terms of the flat coordinates, or in other words, in terms of the genus zero correlation functions, like in the genus one case

$$\begin{aligned} F_1(v, v_x) &= \frac{1}{24} \log \det \left(c_{\alpha\beta\gamma}(v) v_x^\gamma \right) + G(v) \\ &= \frac{1}{24} \log \det \left(\frac{\partial^3 \mathcal{F}_0}{\partial t^{1,0} \partial t^{\alpha,0} \partial t^{\beta,0}} \right) + G(v). \end{aligned}$$

Such a representation of F_2 for A_2 topological minimal model was given by Eguchi, Yamada, Yang, 1994.

Elements needed to represent the genus two free energy

Let $F = F(v^1, \ldots, v^n)$ be the potential of the Frobenius manifold, $v^1, \ldots, v^n$ are the flat coordinates, in theses coordinates, the flat metric

$$\eta_{\alpha\beta} = \frac{\partial^3 F(v)}{\partial v^1 \partial v^\alpha \partial v^\beta} = \text{constant}.$$

The canonical coordinates $u_1, \ldots, u_n$ are defined so that the multiplication table defined on the tangent spaces is given by

$$\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}$$

and the Euler vector field has the form

$$E = \sum u_i \frac{\partial}{\partial u_i}.$$

The rotation coefficients and the Lamé coefficients

In the canonical coordinates the flat metric takes the diagonal form

$$\sum_{i=1}^{n} \eta_{ii}(u) du_i^2.$$

The rotation and Lamé coefficients

$$\gamma_{ij} = \frac{1}{h_i}\frac{\partial h_j}{\partial u_i}, \quad h_i = \sqrt{\eta_{ii}}, \quad i = 1, \ldots, n.$$

They satisfy the equations

$$\frac{\partial \gamma_{ij}}{\partial u_k} = \gamma_{ik}\gamma_{kj}, \ \ i, j, k \text{ distinct},$$
$$\frac{\partial \gamma_{ij}}{\partial u_i} = \frac{\sum_{k=1}^{n} (u_j - u_k)\gamma_{ik}\gamma_{kj} - \gamma_{ij}}{u_i - u_j}.$$

The genus two free energy are represented in terms of

$$\gamma_{ij},\ h_i,\ u_i,\ u_i^{(k)} = \partial_x^k u_i, \quad k = 1, 2, 3, 4.$$

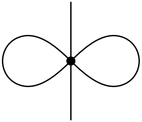
Note that $V_{ij} = (u_j - u_i)\gamma_{ij}$ appear in the formula of F_2.

2. Main Results

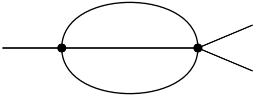
Theorem 1.
Let M be a semisimple Frobenius manifold of dimension n, and F_2 be the genus two free energy for M, then we have

$$F_2(v) = \sum_{p=1}^{16} c_p\, Q_p + G^{(2)}(u, u_x, u_{xx}).$$

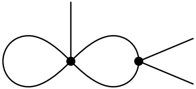
Here each term Q_p corresponds to a dual graph of stable curves (we will explain their meaning later)



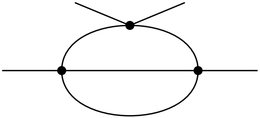
Q_1



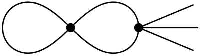
Q_2



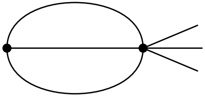
Q_3



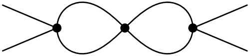
Q_4



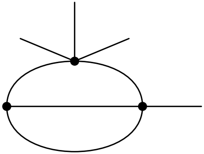
Q_5



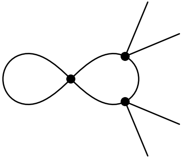
Q_6



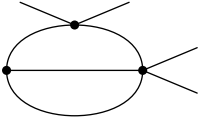
Q_7



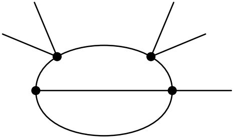
Q_8



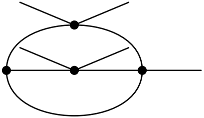
Q_9



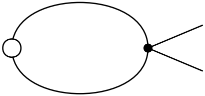
Q_{10}



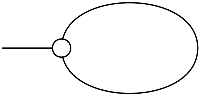
Q_{11}



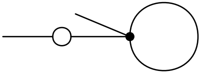
Q_{12}



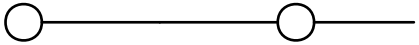
Q_{13}



Q_{14}



Q_{15}



Q_{16}

The constants c_p are fixed up to one arbitrary constant parameter.

$$c_1 = 0, \quad c_2 = -\frac{1}{960}, \quad c_3 = \frac{1}{5760}, \quad c_4 = \frac{1}{1152},$$
$$c_{13} = -\frac{1}{60}, \quad c_{14} = \frac{1}{48}, \quad c_{15} = -\frac{7}{240}, \quad c_{16} = \frac{7}{10}.$$

The function $G^{(2)}(u, u_x, u_{xx})$ is called the genus two G-function, which has the following form

$$G^{(2)}(u, u_x, u_{xx}) = \sum_{i=1}^{n} G_i^{(2)}(u, u_x)u_{xx}^i + \sum_{i \neq j} G_{ij}^{(2)}(u)\frac{(u_x^j)^3}{u_x^i} + \sum_{i,j} P_{ij}^{(2)}(u)u_x^i u_x^j.$$

Here the functions $G_i^{(2)}, G_{ij}^{(2)}$ are fixed uniquely once the above mentioned parameter is fixed.

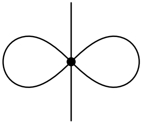
Meaning of the graphs

We introduce a matrix

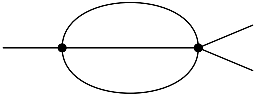
$$M_{\alpha\beta} = \frac{\partial^3 \mathcal{F}_0}{\partial t^{1,0}\, \partial t^{\alpha,0}\, \partial t^{\beta,0}},$$

and denote its inverse by $(M^{-1})^{\alpha\beta}$. Then we have

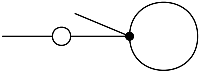
$$Q_1 = \frac{\partial^6 \mathcal{F}_0}{\partial t^{1,0}\, \partial t^{1,0}\, \partial t^{\alpha,0}\, \partial t^{\alpha',0}\, \partial t^{\beta,0}\, \partial t^{\beta',0}} (M^{-1})^{\alpha\alpha'} (M^{-1})^{\beta\beta'}$$



Q_1



Q_2



Q_{15}

$$Q_2 = \frac{\partial^4 \mathcal{F}_0}{\partial t^{1,0}\,\partial t^{\alpha,0}\,\partial t^{\beta,0}\,\partial t^{\gamma,0}}\,(M^{-1})^{\alpha\alpha'}\,(M^{-1})^{\beta\beta'}\,(M^{-1})^{\gamma\gamma'}$$
$$\frac{\partial^5 \mathcal{F}_0}{\partial t^{1,0}\,\partial t^{1,0}\,\partial t^{\alpha',0}\,\partial t^{\beta',0}\,\partial t^{\gamma',0}},$$

$$Q_{15} = \frac{\partial^4 \mathcal{F}_0}{\partial t^{1,0}\,\partial t^{\alpha,0}\,\partial t^{\alpha',0}\,\partial t^{\beta,0}}(M^{-1})^{\alpha\alpha'}(M^{-1})^{\beta\beta'}\,\frac{\partial^2 \mathcal{F}_1}{\partial t^{1,0}\,\partial t^{\beta',0}}.$$

Characterization of the 16 graphs

- Each graph is the dual graph of a stable curve of arithmetic genus two. This condition also implies that the graph is planar, and the valence and genus of its vertices satisfy

$$2g(v_i) - 2 + n(v_i) > 0, \quad \sum_{i=1}^{m} g(v_i) + B_1(Q) = 2.$$

- Cutting of an edge connecting two genus zero vertices does not destroy the connectivity of the graph. A graph with this property is called to be one-particle irreducible (1PI) in physics literature.

- The number of edges and the number of legs are equal to $N_v(Q) + B_1(Q) - 1$. This property is equivalent to the Euler formula for the graph

$$N_e(Q) - N_v(Q) + 1 = B_1(Q)$$

and the condition that the function associated to Q must have degree two with respect to the jet variables $\partial_x^p v^\alpha$, i.e.

$$\sum_{i=1}^{m} (2g(v_i) - 2 + n(v_i)) - N_e(Q) = 2\,.$$

- There is at most one vertex with valence $n(v_i) = 3 - 2g(v_i)$ in the graph. Moreover, if the graph contains only one genus one vertex, then the valence of each of its vertices v_i satisfies $n(v_i) > 3 - 2g(v_i)$.

Conjecture.

If M is a Frobenius manifold defined on the orbit space of a Coxeter group of ADE type or on that of an extended affine Weyl group of ADE type, then

$$G^{(2)}(u, u_x, u_{xx}) = 0.$$

For A_2 (Eguchi, Yamada, Yang, 1994).

$$F_2 = \frac{1}{1152}\, Q_1 - \frac{1}{360}\, Q_2 - \frac{1}{1152}\, Q_3 + \frac{1}{360}\, Q_4.$$

The first class of Frobenius manifolds is isomorphic to the ones defined on the space of miniversal deformations of simple singularities of ADE type. These Frobenius manifolds can also be obtained from the genus zero Fan-Jarvis-Ruan-Witten invariants theory for ADE singularities.

The second class of Frobenius manifolds is isomorphic to the ones defined on the space of certain tri-polynomials, and they can also be obtained from the genus zero Gromov-Witten invariants theory for $\mathbb{P}^1$-orbifolds of ADE type, as shown by Milanov & Tseng, Rossi.

Reason for the appearance of an arbitrary parameter

Theorem 2.

If M is a Frobenius manifold defined on the orbit space of a Coxeter group or on that of an extended affine Weyl group, then we have the following identity

$$\begin{aligned}(Q_1 - Q_6) + 2(Q_7 - Q_5) + 3(Q_8 - Q_2) \\ + 4(Q_9 - Q_3) + 6(Q_4 + Q_{10} - Q_{11} - Q_{12}) = 0.\end{aligned}$$

3. Proofs of the Theorems

Proof of Theorem 1

We only need to represent the above 16 graphs in terms of the functions

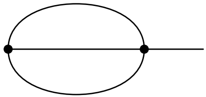
$$\gamma_{ij}(u),\ h_i(u),\ u_i,\ \partial_x^k u_i,\ i,j=1,\dots,n;\ k=1,2,3,4,5,6$$

and compare the expression

$$\sum_{p=1}^{16} c_p\, Q_p$$

with the formula of the genus two free energy F_2 obtained by solving the loop equation.

For example, we have



$$= \frac{\partial^3 \mathcal{F}_0}{\partial t^{\alpha,0}\,\partial t^{\beta,0}\,\partial t^{\gamma,0}} (M^{-1})^{\alpha\alpha'} (M^{-1})^{\beta\beta'} (M^{-1})^{\gamma\gamma'}$$

$$\times \frac{\partial^4 \mathcal{F}_0}{\partial t^{1,0}\,\partial t^{\alpha',0}\,\partial t^{\beta',0}\,\partial t^{\gamma',0}}$$

$$= \sum_{1\le i<j\le n} \gamma_{ij} \frac{(h_i^4\, u_{i,x} - h_j^4\, u_{j,x})(u_{j,x} - u_{i,x})}{h_i^3 h_j^3 u_{i,x} u_{j,x}} + \sum_{i=1}^{n} \frac{u_{i,xx}}{h_i^2 u_{i,x}^2}.$$

Recall that $u_i, 1 \leq i \leq n$ are the canonical coordinates of the Frobenius manifold

$$\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i},$$

in these canonical coordinates the flat metric on the Frobenius manifold has the diagonal form

$$\sum_{i=1}^{n} \eta_{ii}(u) du_i^2.$$

γ_{ij} are the rotation coefficients of the diagonal metric

$$\gamma_{ij} = \frac{1}{h_i} \frac{\partial h_j}{\partial u_i}, \quad \text{and} \quad h_i = \sqrt{\eta_{ii}}\,.$$

by using the fact that

$$\frac{\partial^2 \mathcal{F}_0}{\partial t^{\alpha,p}\partial t^{\beta,q}} = \Omega_{\alpha,p;\beta,q}(v(t)), \quad \frac{\partial^3 \mathcal{F}_0(t)}{\partial t^{\alpha,0}\partial t^{\beta,0}\partial t^{\gamma,0}} = c^{\xi}_{\alpha\beta}(v(t))M_{\xi\gamma},$$

where

$$M_{\xi\gamma} = c_{\xi\gamma\rho}(v(t))v^{\rho}_{x},$$

the Principal Hierarchy and the equations

$$\frac{\partial\gamma_{ij}}{\partial u_k} = \gamma_{ik}\gamma_{kj}, \ \ i,j,k \text{ distinct},$$

$$\frac{\partial\gamma_{ij}}{\partial u_i} = \frac{\sum_{k=1}^{n}(u_j - u_k)\gamma_{ik}\gamma_{kj} - \gamma_{ij}}{u_i - u_j}.$$

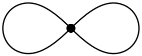
Proof of Theorem 2

Lemma.

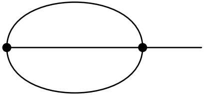
We have the following identity

$$\begin{aligned}&(Q_1 - Q_6) + 2(Q_7 - Q_5) + 3(Q_8 - Q_2)\\ &\qquad + 4(Q_9 - Q_3) + 6(Q_4 + Q_{10} - Q_{11} - Q_{12})\\ =&\partial_x^2 (O_1 - O_2)\,.\end{aligned}$$

with



O_1



O_2

Lemma.

For any semisimple Frobenius manifold, the following identity holds true

$$O_1 - O_2 = \eta^{\alpha\xi}\eta^{\beta\zeta}\frac{\partial^4 F(v^1,\dots,v^n)}{\partial v^\xi \partial v^\zeta \partial v^\alpha \partial v^\beta}\bigg|_{v^\gamma = v^\gamma(t)} = c^{\alpha\beta}_{\alpha\beta}(v)|_{v^\gamma = v^\gamma(t)}.$$

Proposition.

For any Frobenius manifold defined on the orbit space of a Coxeter group or on that of a extended affine Weyl group, the difference $O_1 - O_2$ *is equal to zero or to a constant respectively.*

Proof We note that

$$\deg c_{\alpha\beta}^{\alpha\beta}(v) = \begin{cases} d-1<0, & \text{Coxeter groups;} \\ d-1=0, & \text{extended affine Weyl groups.} \end{cases}$$

On the other hand, for a Coxeter group the potential $F(v)$ of the Frobenius manifold is a polynomial of $v^1, \dots, v^n$ with

$$0<d<1, \quad \deg v^\alpha > 0,$$

and for an extended affine Weyl group the potential $F(v)$ of the Frobenius manifold is a polynomial of $v^1, \dots, v^{n-1}, e^{v^n}$ with

$$d=1, \quad \deg v^\alpha > 0 \, (1 \le \alpha \le n-1), \quad \deg e^{v^n} > 0.$$

Thus we proved the proposition. □

4. Checking the Validity of the Conjecture

Frobenius manifolds associated to simple singularities

$f = f(z^1, \ldots, z^m)$: a polynomial which has an isolated critical point at $0 \in \mathbb{C}^m$ of ADE type with Milnor number n.

$F : \mathbb{C}^m \times B \to \mathbb{C},\ (z, t) \mapsto F(z, t)$ a miniversal deformation of f, where B is an open set in $\mathbb{C}^n$.

There is a semisimple Frobenius manifold structure on the base space $B \setminus C$ outside of the caustic $C \subset B$ with the flat metric

$$\langle \partial', \partial'' \rangle_t = -\operatorname{res}_{z=\infty} \frac{(\partial' F(z,t))(\partial'' F(z,t))\ dz^1 \wedge \cdots \wedge dz^m}{\partial_{z^1} F \cdots \partial_{z^m} F}$$

for any $\partial',\ \partial'' \in T_t B$.

Canonical coordinates

For a given $t \in B \setminus C$, the equations

$$\partial_{z^\alpha} F = 0, \ \alpha = 1, \dots, m$$

has n solutions $z^{(i)}(t) = (z^{(i),1}, \dots, z^{(i),m})$ $(i = 1, \dots, n)$. The canonical coordinates u^i on $B \setminus C$ are given by

$$u_i(t) = F(z^{(i)}(t), t), \quad i = 1, \dots, n.$$

Denote

$$h_{\alpha\beta}(z,t) = \partial_{z^\alpha}\partial_{z^\beta} F(z,t), \quad H(z,t) = \det(h_{\alpha\beta}(z,t))$$

and

$$(h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}.$$

The flat metric in canonical coordinates

$$\sum_{i=1}^{n} \eta_{ii}(du_i)^2, \quad \text{with } \eta_{ii} = \frac{1}{H(z^{(i)}(t), t)}.$$

The rotation and Lamé coefficients

$$\gamma_{ki} = \frac{\partial_i\, h_k}{h_i} = \frac{h_k}{h_i}\,\Gamma_{ki}, \quad h_i = \sqrt{\eta_{ii}}$$

and

$$\Gamma_{ki} := \Gamma^k_{ki} = \frac{\partial_{u_i}\eta_{kk}}{2\,\eta_{kk}} = -\frac{1}{2}\partial_{z^\alpha}\left(h^{\alpha\beta}(z,t)\,\partial_{u_i}\partial_{z^\beta}F(z,t)\right)|_{z=z^{(k)}}.$$

Example: A_n singularity

$$f(z) = z^{n+1}, \quad F(z,t) = z^{n+1} + t^1 z^{n-1} + \cdots + t^n.$$

Lemma.

$$\Gamma_{ki}(t) := \Gamma_{ki}^k = \frac{1}{(z^{(k)}(t) - z^{(i)}(t))^2 F''(z^{(i)}(t), t)}.$$

We use the critical points $z^{(1)}, \ldots, z^{(n)}$ and an additional parameter $z^{(0)}$ to represent $F(z,t)$ as

$$F(z,t) = \lambda(z) = \int_0^z (n+1)\prod_{k=1}^{n}(\xi - z^{(k)})d\xi + z^{(0)}$$

Note that $z^{(1)}, \ldots z^{(n)}$ are not independent, they satisfy

$$z^{(n)} = -\sum_{k=1}^{n-1} z^{(k)}.$$

Then we have

$$u^i = \lambda(z^{(i)}), \quad h_i = \psi_{i,1} = \frac{1}{\sqrt{\lambda''(z^{(i)})}}, \quad \gamma_{ij} = \frac{h_i\, h_j}{(z^{(i)} - z^{(j)})^2}.$$

Proof of the validity of the Conjecture for A_n singularity

Substituting these expressions into the formula for the difference

$$G^{(2)}(u, u_x, u_{xx}) = F_2 - \sum_{p=1}^{16} c_p Q_p,$$

it becomes a rational function of $z^{(0)}, \ldots, z^{(n-1)}$. We prove the vanish of this rational function, and thus prove the validity of the above conjecture for A_n singularities.

Example: D_n singularity

In this case, $m = 2$. Denote $x = z^1, y = z^2$, then

$$f(z) = x^{n-1} + xy^2,$$
$$F(z,t) = x^{n-1} + xy^2 + t^1 x^{n-2} + \cdots + t^{n-1} + t^n y.$$

Denote the critical points of F by $z^{(i)} = (x_i, y_i)$, and introduce the function

$$\lambda(x,t) = x^{n-1} + t^1 x^{n-2} + \cdots + t^{n-1} - \frac{(t^n)^2}{4x}$$

Lemma.

$$\Gamma_{ki}(t) := \Gamma_{ki}^k = \frac{x_k + x_i}{2x_i(x_k - x_i)^2 \lambda''(x_i)}$$

Verifying the validity of the Conjecture for D_n singularity

Represent $\lambda(x,t)$ in terms of $x_1,\ldots,x_{n-1}$ and x_0 in the form

$$\lambda(x)=\int_0^x (n-1)\xi^{-2}\prod_{k=1}^{n}(\xi-x_i)d\xi+x_0$$

Here $\frac{1}{x_n}=-\sum_{k=1}^{n-1}\frac{1}{x_k}$. Then we have

$$u_i=\lambda(x_i),\quad h_i=\frac{1}{\sqrt{2x_i\lambda''(x_i)}},\quad \gamma_{ij}=\frac{(x_i+x_j)h_ih_j}{(x_i-x_j)^2}.$$

By using these data, one can also verify the Conjecture for $n\leq 10$.

Example: Frobenius manifold structures associated to extended affine Weyl group of ADE type

Isomorphic to the ones defined on the space of tri-polynomials of type (p, q, r) (for certain specific values)

$$F(z,t) = -z^1 z^2 z^3 + P_1(z_1) + P_2(z_2) + P_3(z_3),$$

$$P_1(z_1) = \sum_{i=1}^{p-1} t_i z_1^i + z_1^p, \quad P_2(z_2) = \sum_{i=1}^{q-1} t_{p-1+i} z_2^i + z_2^p,$$

$$P_3(z_3) = \sum_{i=0}^{r} t_{p+q-1+i} z_3^i,$$

where $t = (t^1, \ldots, t^n) \in B$, and B is an open set in $\mathbb{C}^{n-1} \times \mathbb{C}^*$, i.e. $t^n \neq 0$. The computation of the rotation coefficients is similar to that of simple singularities.

Proof of the conjecture for ADE singularities by Xiaobo Liu & Xin Wang

Using the properties of semisimple Frobenius manifolds and the following properties of FJRW invariants of ADE singularities:

1. $\langle \phi_\alpha \phi^\alpha \phi_\beta \phi^\beta \phi_{\alpha_1} \dots \phi_{\alpha_k} \rangle_0 = 0$;
2. $\langle \phi_{\alpha_1} \dots \phi_{\alpha_k} \rangle_1 = 0$;
3. $\langle \phi_{\alpha_1} \dots \phi_{\alpha_k} \rangle_2 = 0, \quad \langle \tau_1(\phi_{\alpha_1}) \phi_{\alpha_2} \dots \phi_{\alpha_k} \rangle_2 = 0$.

The first property is equivalent to the property $c_{\alpha\beta}^{\alpha\beta} = \mathrm{const}$, and the second property is equivalent to the vanishing of the genus one G-function.

A conceptual proof of the conjecture is in progress

Using the analyticity property of the potential of the Frobenius manifolds and their genus one G-function, and the nonnegative property of the degrees of the flat coordinates, and the property of the charge $d \leq 1$. To complete the proof we need to prove the analyticity of the genus two G function.

5. Some Remarks

Works to be done

- To elucidate the geometrical and physical meaning of the genus two G-function, and the conditions of the vanishing of this function.
- Possible generalization to higher genera.

Thanks