Some Aspects of Modeling Quantum Behaviour with Classical Simulations

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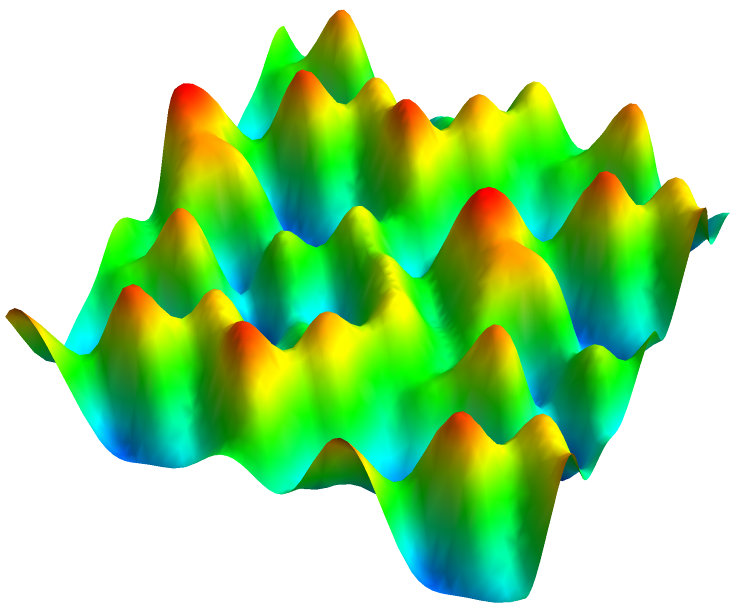
May 22 2019

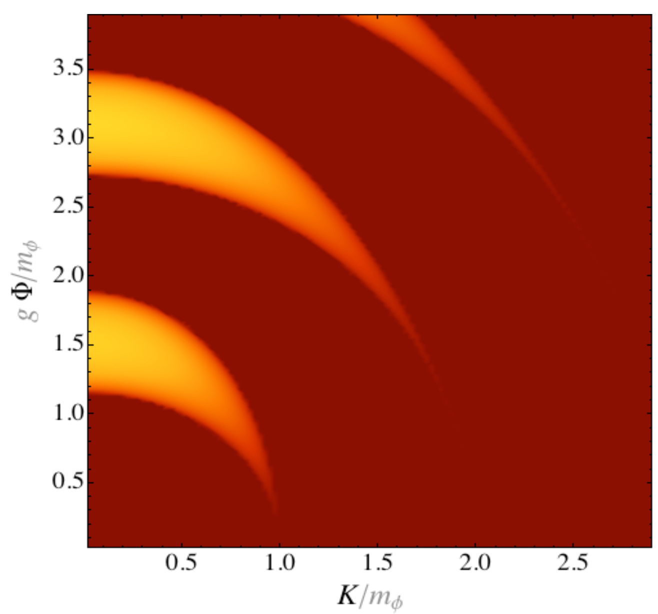
Part 1: Axion Dark Matter



- Hertzberg, Tegmark, Wilczek 0807.1726;
- Guth, Hertzberg, Prescod-Weinstein 1412.5930;
- Hertzberg 1609.01342

Macroscopic spreading of wave-function

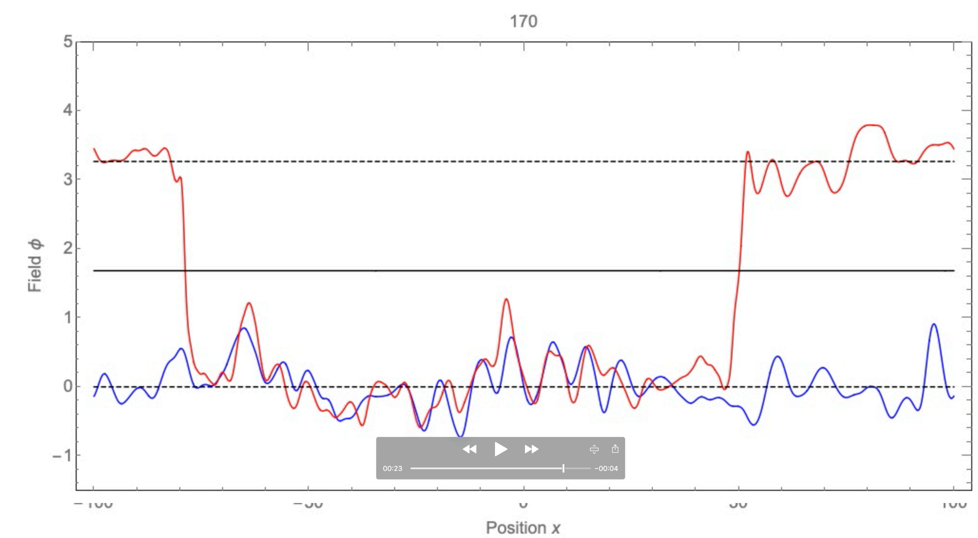
Part 2: Post-Inflation



- Hertzberg 1003.3459;
- Amin, Easther, Finkel, Flauger, Hertzberg 1106.335;
- Amin, Hertzberg, Kaiser, Karouby 1410.3808;
- Hertzberg, Schiappacasse 1805.00430

Beginning with low occupancy modes

Part 3: Tunneling



- Hertzberg, Yamada 1904.08565;

Usually thought to be forbidden classically

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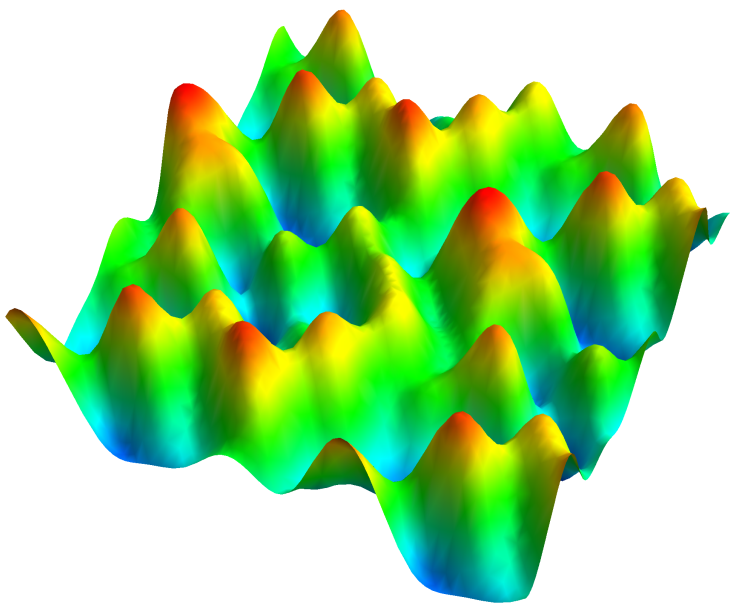

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Macroscopic spreading of wave-function

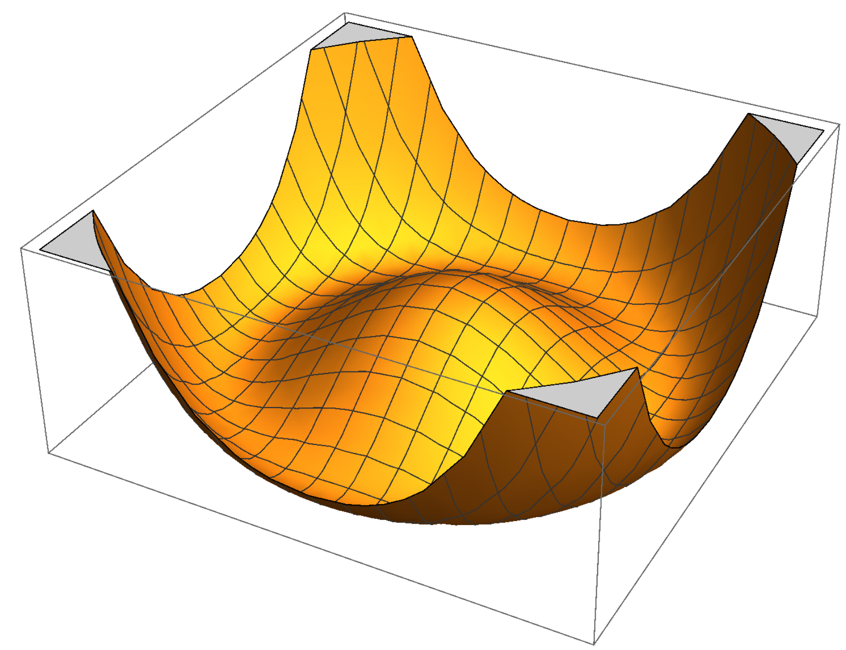
QCD-Axion Reminder

$$\Delta\mathcal{L}_{qcd} \sim \theta\, \mathbf{E}^a \cdot \mathbf{B}^a$$

$$|\theta| \lesssim 10^{-10}$$

(Peccei-Quinn, Weinberg, Wilczek)

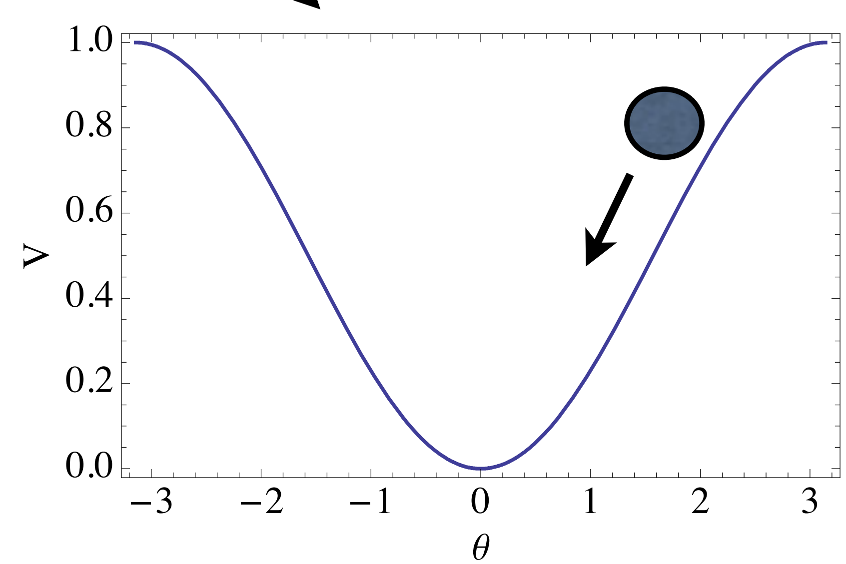
$$\theta \to \phi / f_a$$



$$\Delta\mathcal{L}_a \sim \frac{\phi}{f_a}\mathbf{E}^a \cdot \mathbf{B}^a + \frac{1}{2}(\partial\phi)^2 - V(\phi)$$

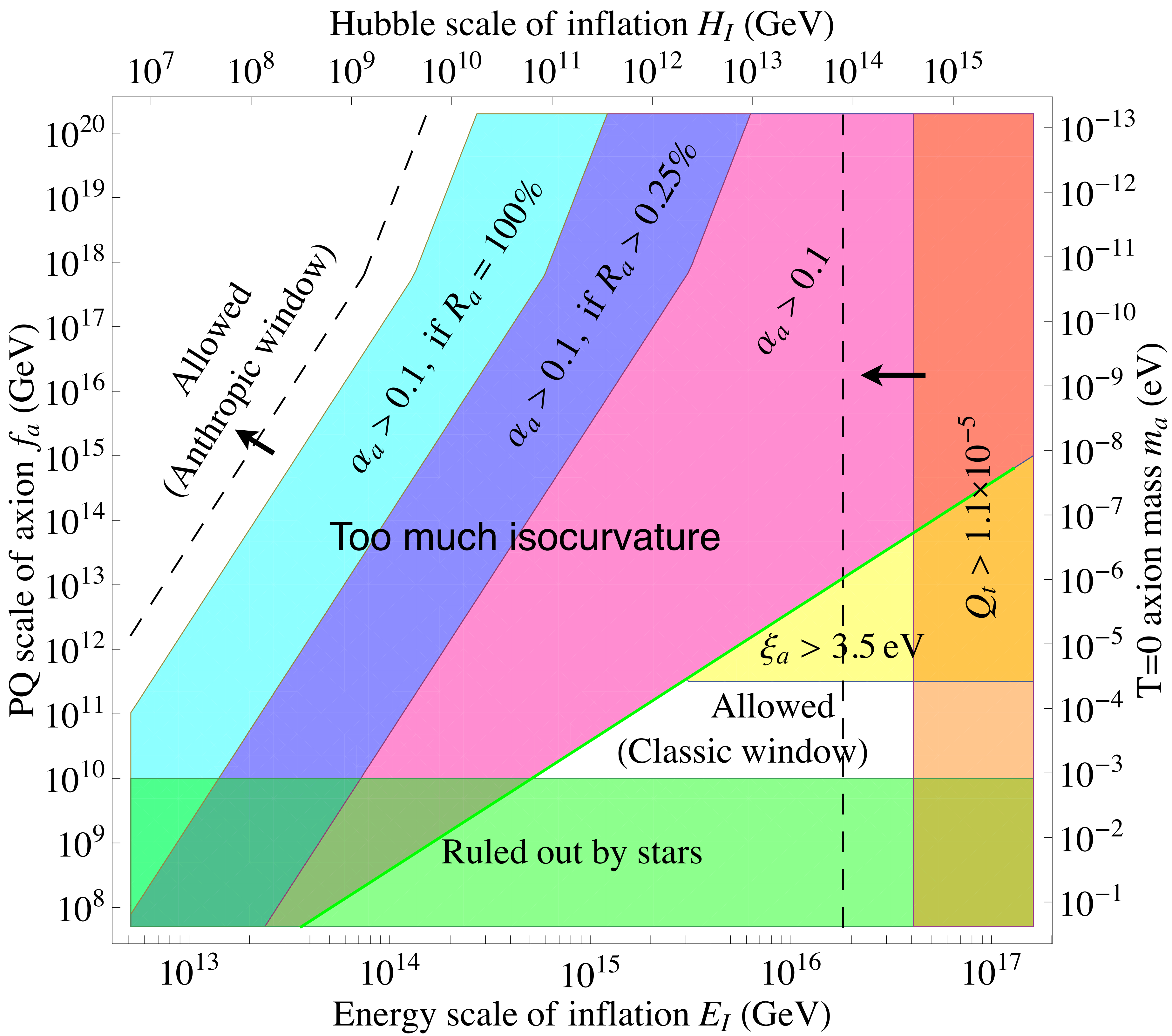
Abundance

$$\Omega_a \approx \langle \theta_i^2 \rangle \left(\frac{f_a}{10^{12}\mathrm{GeV}} \right)^{7/6} < 0.25$$



Related issues for string-axions,
light bosonic DM

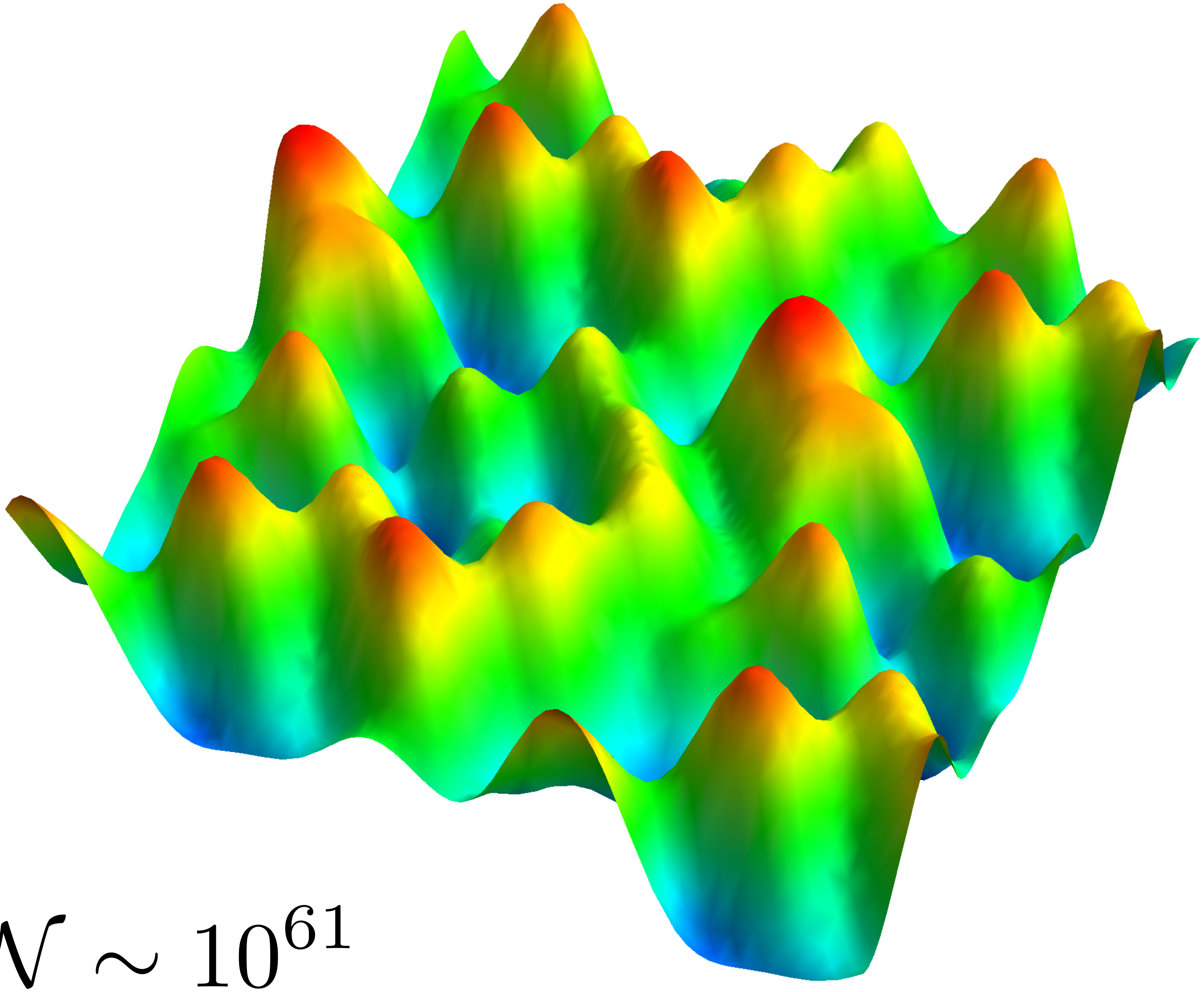
QCD-Axion Allowed Windows



Hertzberg, Tegmark, Wilczek 0807.1726

Focus on Classic Window

In Classic Window; Axion Initial Distribution



$$\mathcal{N} \sim 10^{61}$$

Consider Non-Relativistic Behavior

$$\phi(\mathbf{x},t) = \frac{1}{\sqrt{2\,m}}\left(e^{-imt}\psi(\mathbf{x},t) + e^{imt}\psi^*(\mathbf{x},t)\right)$$

Hamiltonian $\hat{H} = \hat{H}_{\mathbf{kin}} + \hat{H}_{\mathbf{int}} + \hat{H}_{\mathbf{grav}}$

$$\hat{H}_{\mathbf{kin}} = \int d^3x \frac{1}{2m} \nabla\hat{\psi}^\dagger \cdot \nabla\hat{\psi}$$

$$\hat{H}_{\mathbf{int}} = \int d^3x \frac{\lambda}{16m^2} \hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi}$$

$$\hat{H}_{\mathbf{grav}} = -\frac{Gm^2}{2}\int d^3x \int d^3x' \frac{\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x}')\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

Number Density $\hat{n}(\mathbf{x}) = \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})$

(For rigorous treatment: Namjoo, Guth, Kaiser 2017)

Approach to Equilibrium?

Equation of Motion

$$i\,\dot{\psi} = -\frac{1}{2m}\nabla^2\psi + \frac{\lambda}{8m^2}|\psi|^2\psi - Gm^2\psi\int d^3x'\,\frac{|\psi(\mathbf{x}')|^2}{|\mathbf{x}-\mathbf{x}'|}$$

Interaction Rate of Modes

$$\Gamma_k \equiv \frac{\dot{\mathcal{N}}_k}{\mathcal{N}_k}$$

$$\Gamma_k \sim \frac{\lambda\, n_{ave}}{8m^2} \quad \propto \frac{1}{a^3}$$

$$\Gamma_k \sim \frac{8\pi G\, m^2\, n_{\rm ave}}{k^2} \quad \propto \frac{1}{a}$$

$\Gamma_k > H$ Early universe

$\Gamma_k > H$ Late universe

Equilibrium with high occupancy suggests BEC

Axion BEC Literature

- Sikivie, Yang (2009)
- Erken, Sikivie, Tam, Yang (2011)
- Chavanis (2012)
- Banik, Sikivie (2013)
- Davidson, Elmer (2013)
- Saikawa, Yamaguchi (2013)
- Noumi, Saikawa, Sato, Yamaguchi (2014)
- Vega, Sanchez (2014)
- Li, Rindler-Daller, Shapiro (2014)
- Berges, Haeckel (2014)
- Banik, Christopherson, Sikivie, Todarello (2015)
- Davidson (2015)
- .....

Classical Description of BEC Phase Transition

Free Theory

$$F[\psi] = \int \frac{d^3k}{(2\pi)^3} \left[\frac{k^2}{2m} - \mu(T) \right] |\psi_k|^2$$

Number

$$\langle N \rangle = \frac{\int \mathcal{D}\psi \, N[\psi] \, \exp\left(-F[\psi]/T\right)}{\int \mathcal{D}\psi \, \exp\left(-F[\psi]/T\right)}$$

Density

$$n_{\rm th} = \int \frac{d^3k}{(2\pi)^3} \frac{T}{\frac{k^2}{2m} - \mu(T)}$$

Critical Temperature

$$T_{\rm crit} = \frac{\pi^2 \, n_{\rm tot}}{m \, k_{\rm UV}}$$

Guth, Hertzberg, Prescod-Weinstein 1412.5930

Classical vs Quantum with Interactions

What About Interactions?

Fundamental claim of Sikivie, Todarello, 1607.00949

On time scales $t > \tau = 1/\Gamma$ the classical description breaks down, requiring the full quantum theory, which is the only way to see thermalization

Toy Model

Second Quantized Language

$$\hat{H} = \sum_i \omega_i \, \hat{a}_i^\dagger \hat{a}_i + \frac{1}{4} \sum_{ijkl} \Lambda_{ij}^{kl} \, \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l \, ,$$

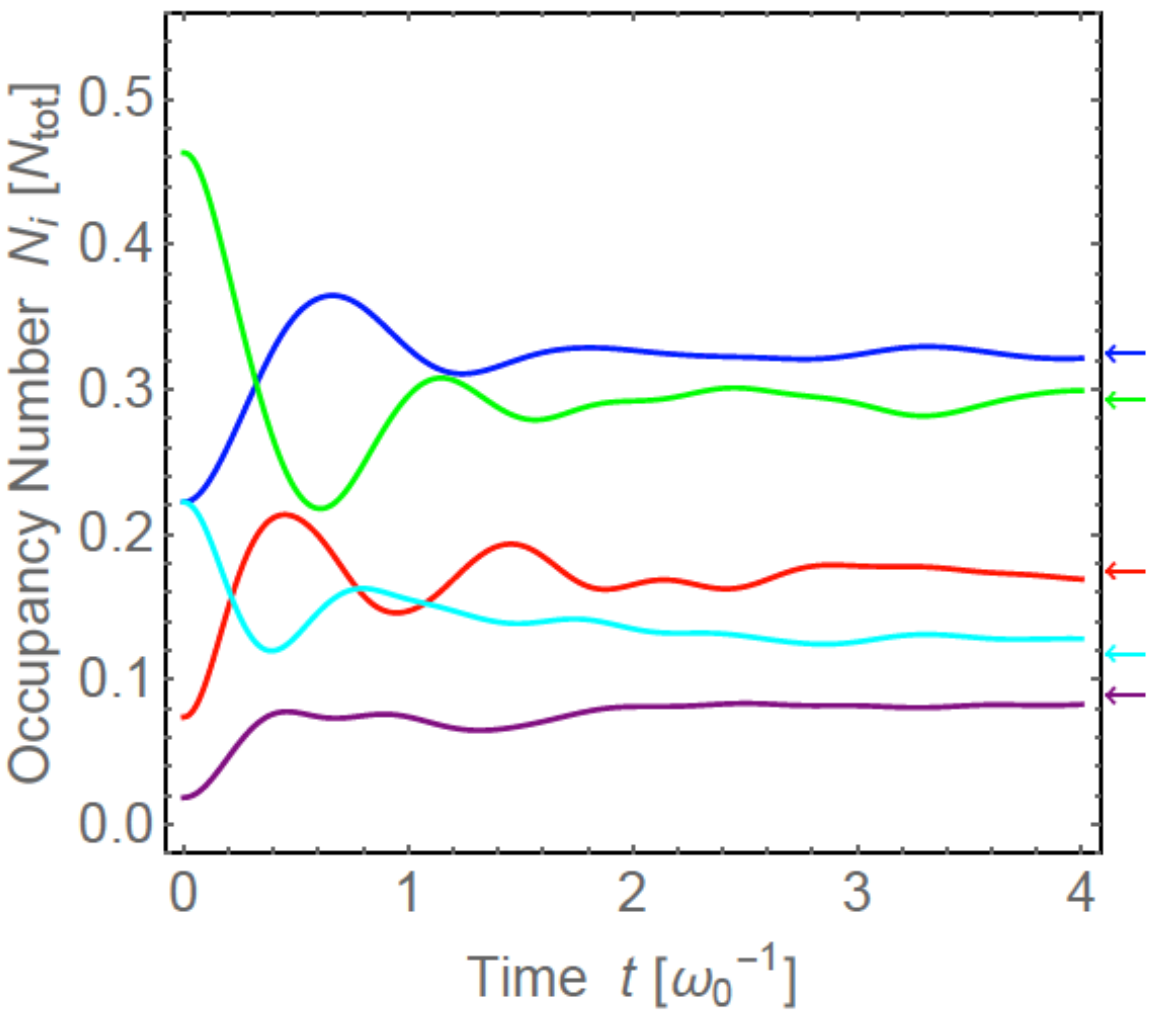
Consider just 5 oscillators for simplicity

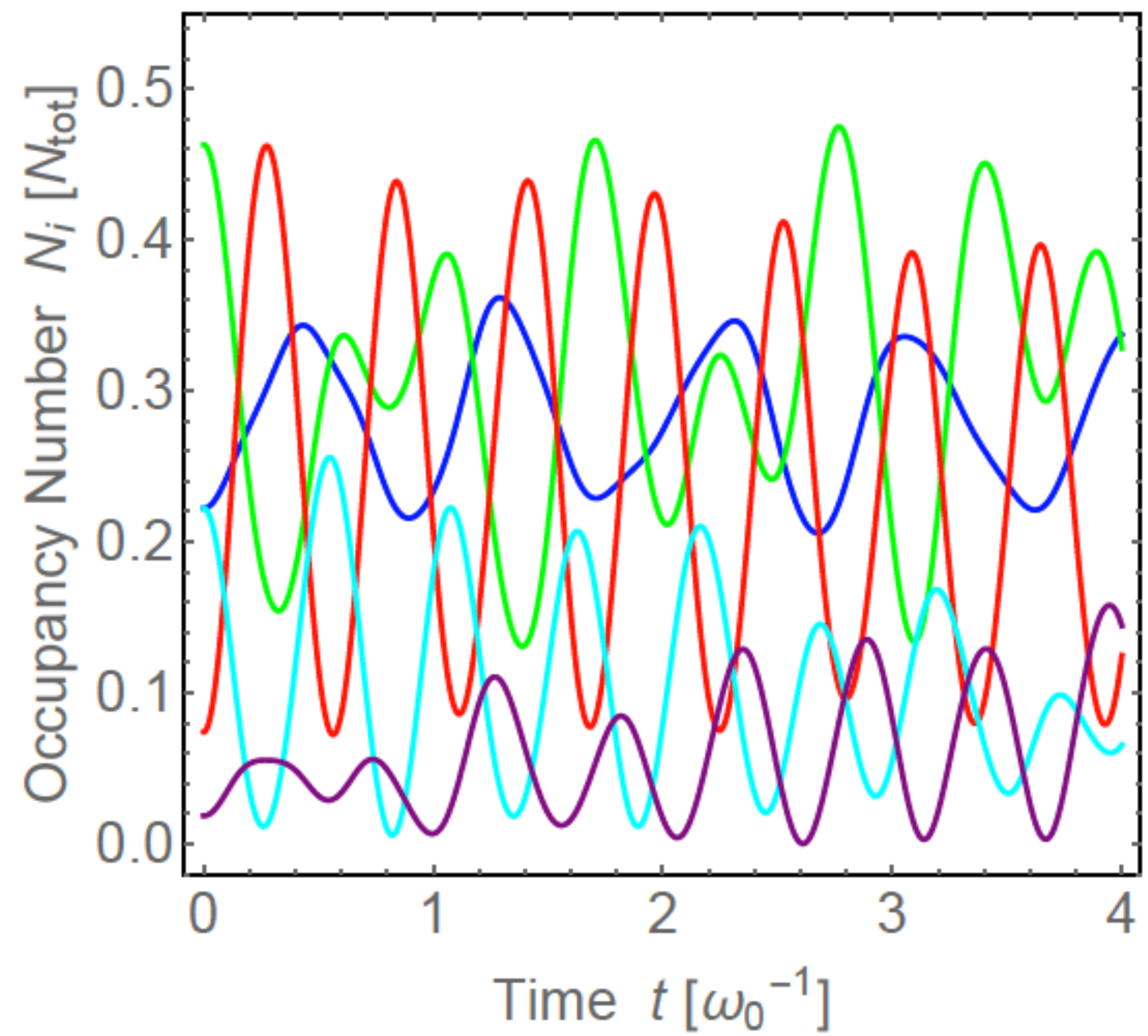
Initial quantum state $\quad |\{N_i\}\rangle = |12, 25, 4, 12, 1\rangle$

Initial classical state $\quad a_i = \sqrt{N_i}$

Sikivie, Todarello, 1607.00949

Quantum vs Classical??





Sikivie, Todarello, 1607.00949

Correct Classical Treatment

Initial classical state $a_i = \sqrt{N_i}\, e^{I\theta_i}, \quad \theta_i \in [0, 2\pi)$

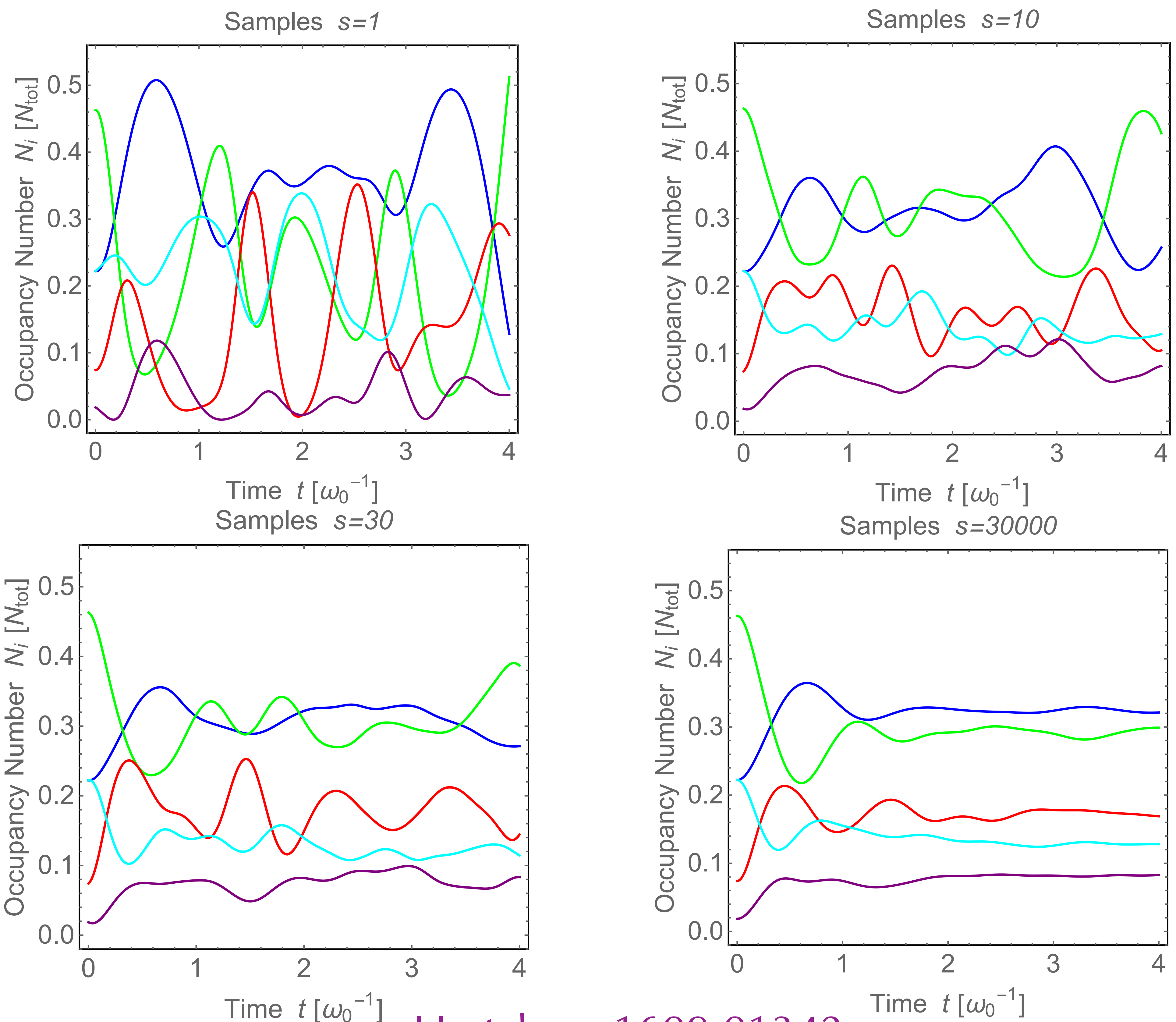
Ensemble average over random initial phases

Meaningful comparison

Connects to uncertainty in branch of wavefunction

Hertzberg 1609.01342

Correct Classical Treatment



Hertzberg 1609.01342

Implication for Correlation Functions

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At high occupancy

$$\langle \{N_i\} | \hat{\psi}^\dagger(\mathbf{x}, t)\, \hat{\psi}(\mathbf{y}, t) | \{N_i\} \rangle \approx \langle \psi^*(\mathbf{x}, t)\, \psi(\mathbf{y}, t) \rangle_{ens}$$

Ergodic theorem

$$\langle \psi^*(\mathbf{x}, t)\, \psi(\mathbf{y}, t) \rangle_{ens} = \frac{1}{V} \int_V d^3z\, \psi_\mu^*(\mathbf{x} + \mathbf{z}, t)\, \psi_\mu(\mathbf{y} + \mathbf{z}, t)$$

Hertzberg 1609.01342

Implication for Axion Simulations

Correlation functions of quantum and classical micro-states agree at high occupancy, despite the macroscopic spreading of wave-functions in these chaotic systems

Note: this is not some trivial consequence of Ehrenfest theorem;
More akin to billiard balls which exhibit chaos; Albrecht, Phillips 2012

Hertzberg 1609.01342

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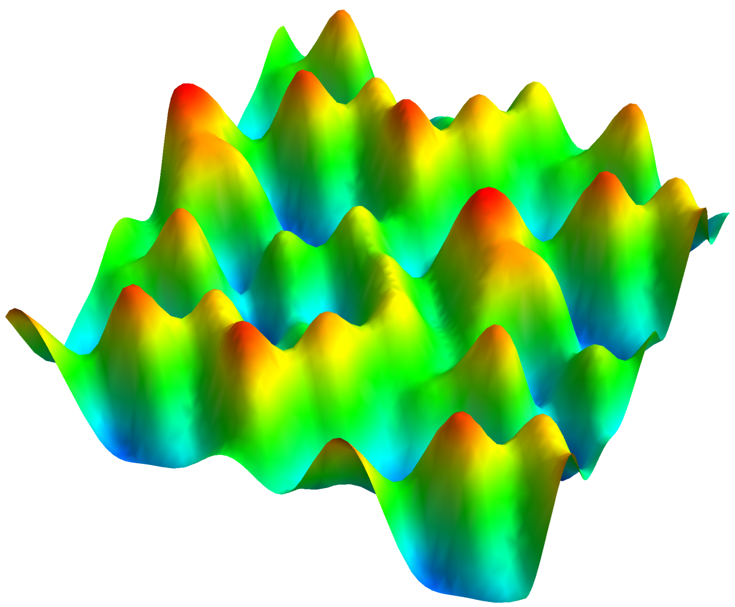

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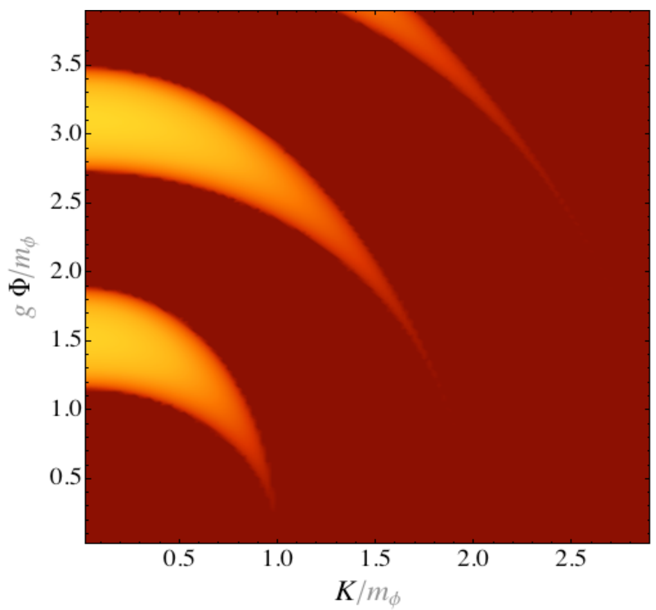

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Beginning with low occupancy modes

Preheating - Mode Functions

$$S = S_{free} + \int d^4x\sqrt{-g}\left[\frac{\gamma}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 + \frac{\sigma}{2}\phi\chi^2 - \frac{g_{a\gamma}}{4}\phi F_{\mu\nu}\tilde{F}^{\mu\nu} + \ldots\right]$$

Heisenberg

$$\hat{\chi}(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3}\left[e^{i\mathbf{k}.\mathbf{x}}v_k(t)\hat{a}_k + h.c\right]$$

Initial conditions

$$v_k(t) \to \frac{e^{-i\omega_k t}}{\sqrt{2\omega_k}}$$

(Bunch-Davies)

Evolution

$$\ddot{v}_k + 3H\dot{v}_k + \frac{k^2}{a^2}v_k + \sigma\phi(t)v_k = 0$$

Adiabatic

$$Q = \delta\phi + \frac{\dot{\phi}}{H}\psi$$

(Mukhanov-Sasaki)

Kofman, Linde, & Starobinsky PRL (1994); PRD (1997)

Preheating - Mode Functions

$$S = S_{free} + \int d^4x\sqrt{-g}\left[\frac{\gamma}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 + \frac{\sigma_{ij}}{2}\phi\chi_i\chi_j - \frac{g_{a\gamma}}{4}\phi F_{\mu\nu}\tilde{F}^{\mu\nu} + \dots\right]$$

Heisenberg $\hat{\chi}_j(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3}\left[e^{i\mathbf{k}.\mathbf{x}} v_{k,ji}(t)\hat{a}_{k,i} + h.c\right]$

Initial conditions $v_{k,ji}(t) \to \frac{e^{-i\omega_k t}}{\sqrt{2\omega_k}}\delta_{ji}$ (Bunch-Davies)

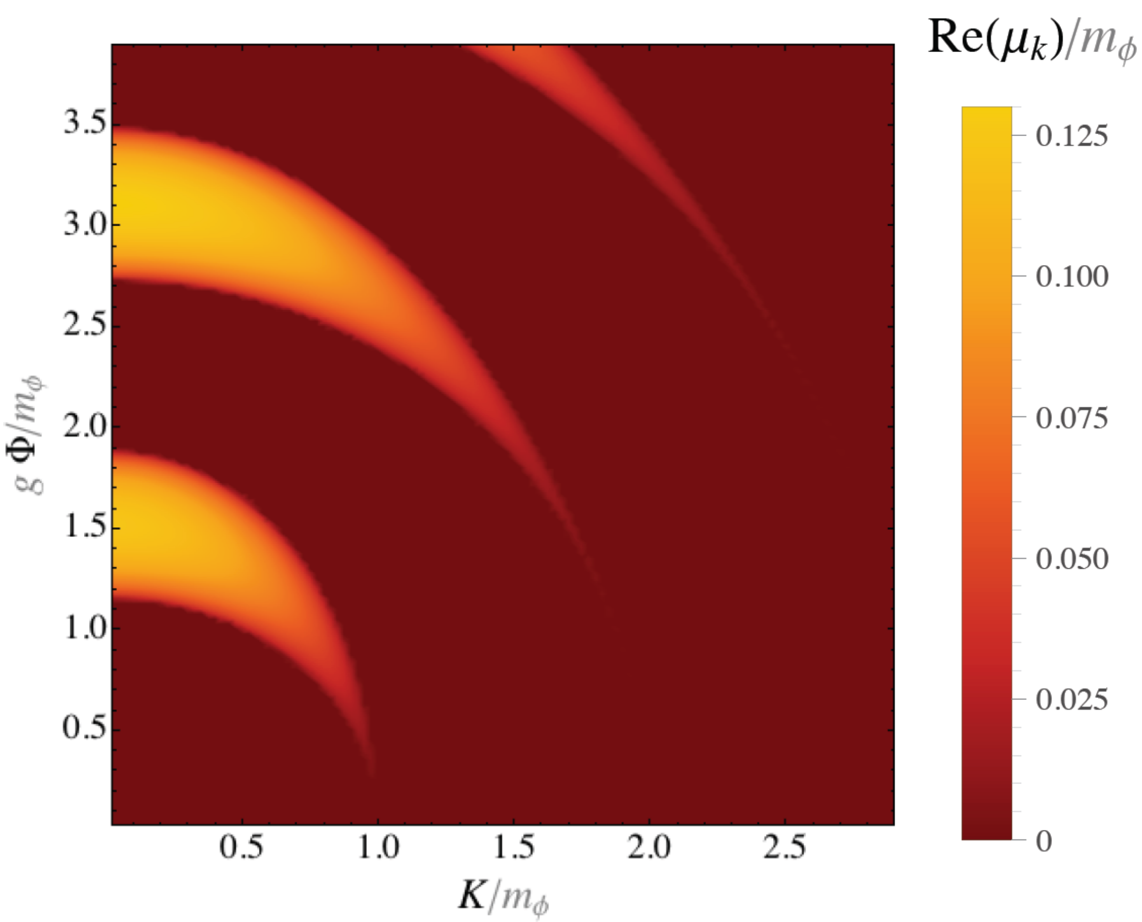
Evolution $\ddot{v}_{k,ji} + 3H\dot{v}_{k,ji} + \frac{k^2}{a^2}v_{k,ji} + \sigma_{jb}\phi(t)v_{k,bi} = 0$

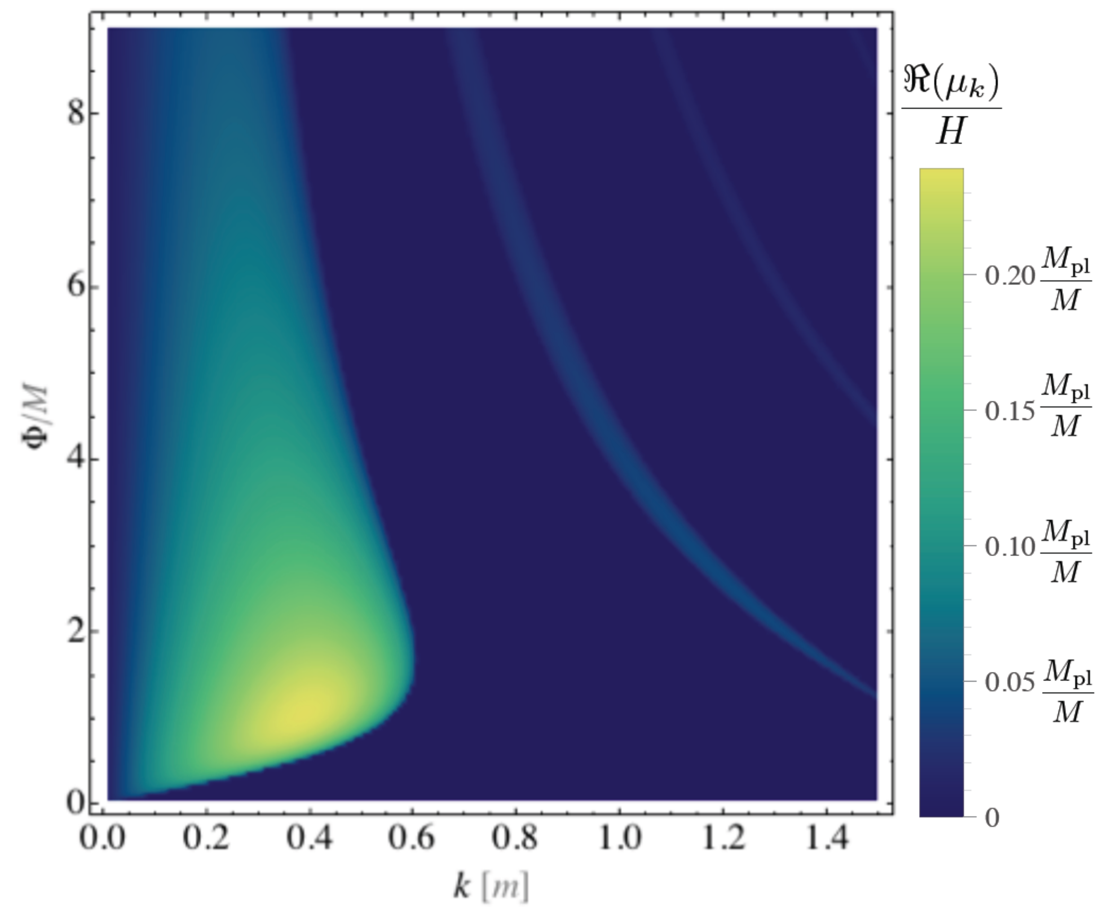
Kofman, Linde, & Starobinsky PRL (1994); PRD (1997); Amin, Hertzberg, Kaiser, Karouby 1410.3808

Floquet Theory

$$\ddot{V}_k + [C_k + F(t)]\, V_k = 0 \qquad \text{with } F(t) \text{ periodic:}$$

$$\longrightarrow \quad V_k(t) = P_{1,k}(t) e^{\mu_k t} + P_{2,k}(t) e^{-\mu_k t}$$





$$V(\phi,\chi) = \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{2}g^2\phi^2\chi^2$$

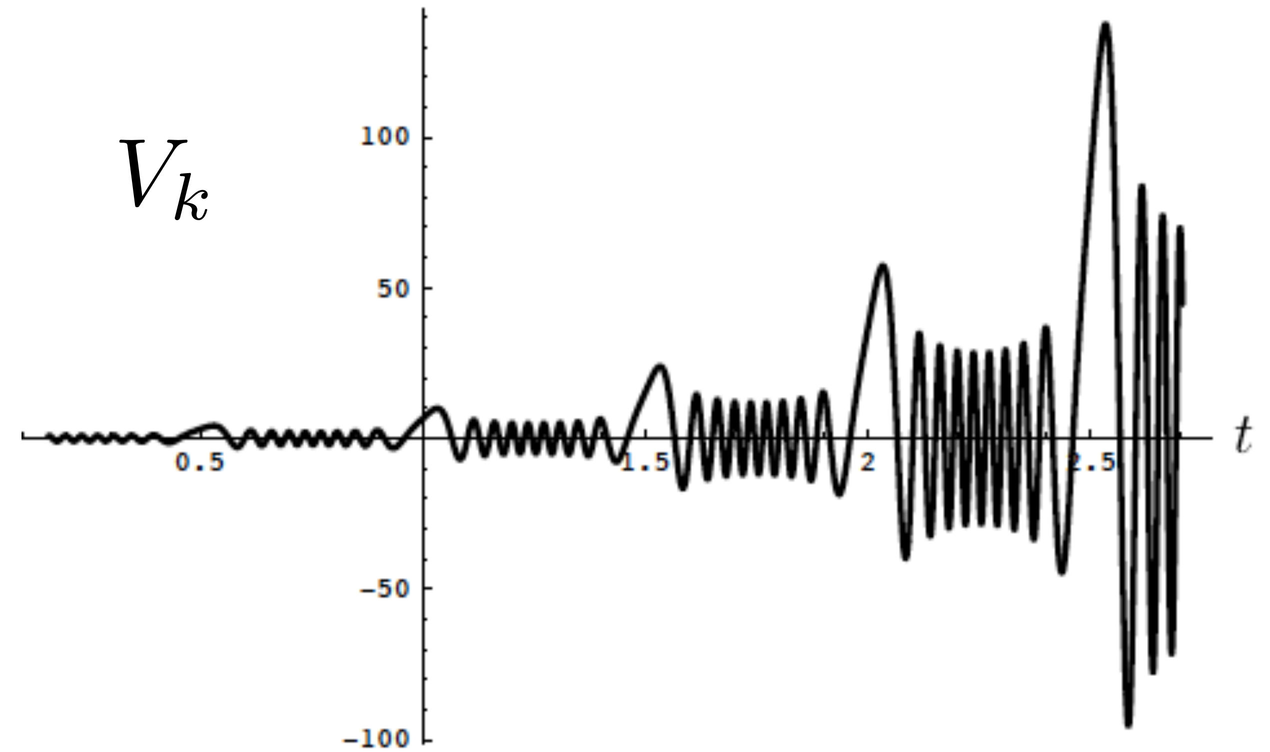
$$V(\phi) = m^2 M^2 \left[\sqrt{1+(\phi/M)^2} - 1\right]$$

General algorithm: Amin, Hertzberg, Kaiser, Karouby, 1410.3808

Exponential Growth in Fields

$$V_k(t) \to e^{\mu_k t} \text{ with } \mathbb{R}[\mu_k] > 0$$

Sensitive dependence on wavenumber *k*



Kofman, Linde, & Starobinsky, arXiv:hep-ph/9704452;

Nonlinear Effects

Exponential growth means that linear evolution eventually breaks down, requiring full non-linear dynamics

Preheating - Ensemble Average

$$S = S_{free} + \int d^4x\sqrt{-g}\left[\frac{\gamma}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 + \frac{\sigma_{ij}}{2}\phi\chi_i\chi_j - \frac{g_{a\gamma}}{4}\phi F_{\mu\nu}\tilde{F}^{\mu\nu} + \ldots\right]$$

Heisenberg $$\hat{\chi}_j(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3}\left[e^{i\mathbf{k}.\mathbf{x}}v_{k,ji}(t)\hat{a}_{k,i} + h.c\right]$$

Initial conditions $$v_{k,ji}(t) \rightarrow \frac{e^{-i\omega_k t}}{\sqrt{2\omega_k}}\delta_{ji}$$ (Bunch-Davies)

Evolution $$\ddot{v}_{k,ji} + 3H\dot{v}_{k,ji} + \frac{k^2}{a^2}v_{k,ji} + \sigma_{jb}\phi(t)v_{k,bi} = 0$$

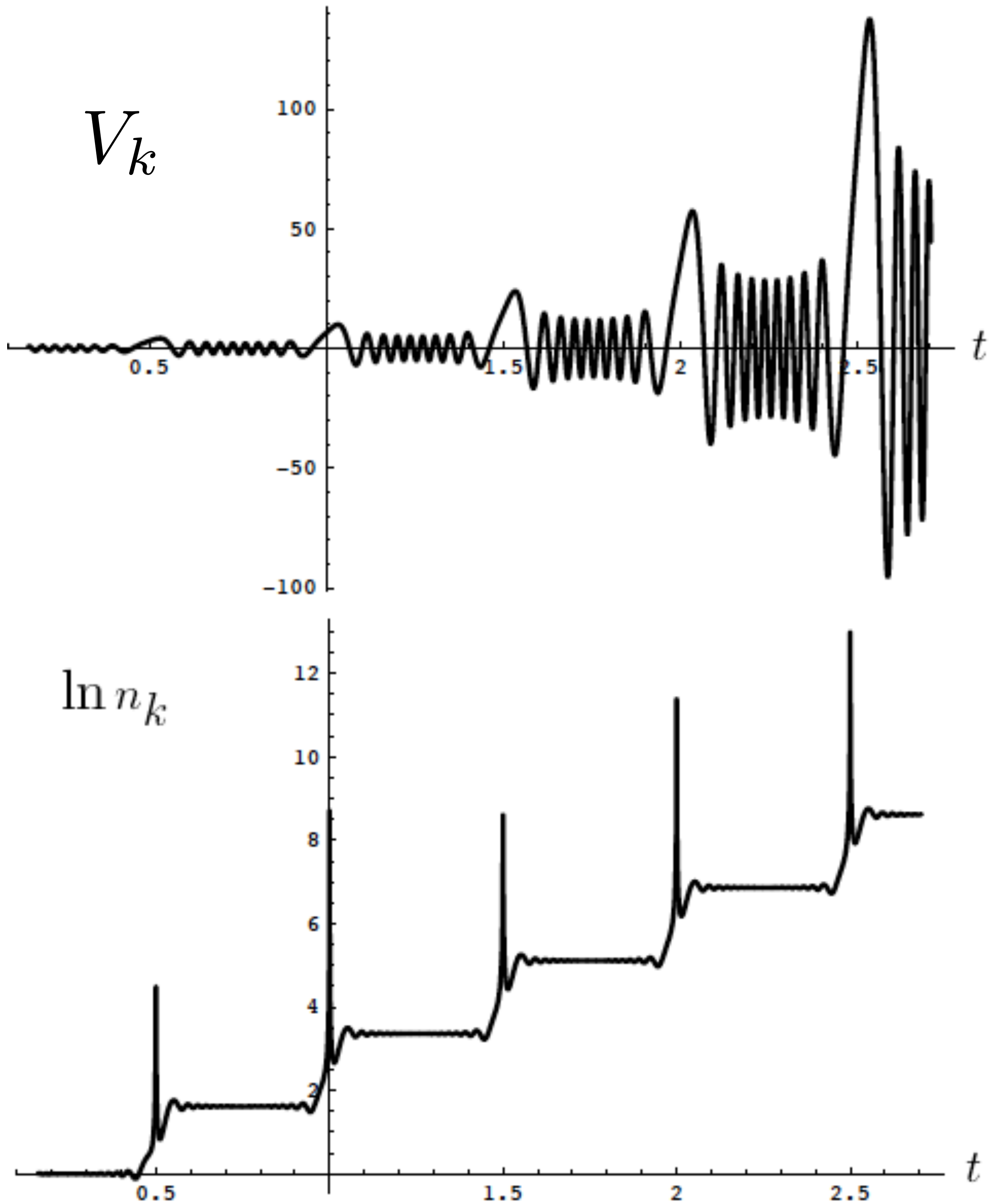
Ensemble $$\Psi[\chi_j] \propto \exp\left[-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\omega_k|\chi_{k,j}|^2\right] \quad \Psi[\pi_j] \propto \exp\left[-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k}|\pi_{k,j}|^2\right]$$

Correlations functions are exact $$\langle\chi_i(\mathbf{x},t)\chi_j(\mathbf{y},t)\rangle_{ens}$$

Exponential Growth in Fields

$$V_k(t) \rightarrow e^{\mu_k t} \text{ with } \mathbb{R}[\mu_k] > 0$$

Sensitive dependence on wavenumber *k*



Exponential growth in occupancy

Kofman, Linde, & Starobinsky, arXiv:hep-ph/9704452;

Nonlinear Effects

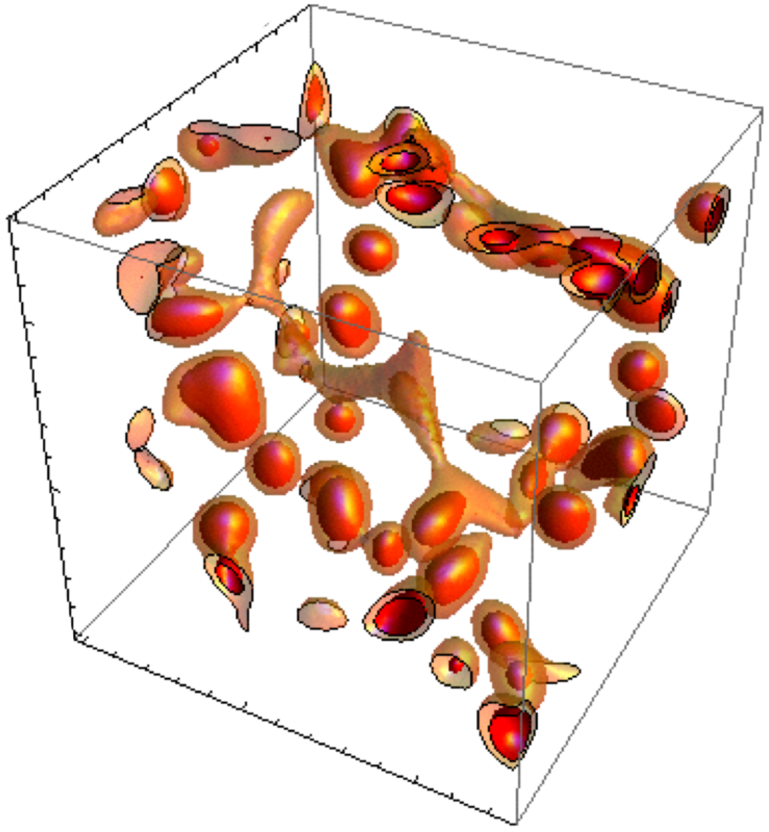
Exponential growth means that linear evolution eventually breaks down, requiring full non-linear dynamics

- Lattice Easy (Felder and Tkachev, hep-ph/0011159)
- Defrost (Frolov, 0809.4904)
- CudaEasy (Sainio, 0911.5692)
- PSpectre (Easther, Finkel, and Roth, 1005.1921)
- HLattice (Huang, 1102.0227)
- PyCool (Sainio, 1201.5029)
- GABE (Child, Giblin, Ribeiro, and Seery, 1305.0561)

Classical lattice simulations

Localized structures, e.g., clumps/oscillons

Amin, Easther, Finkel, Flauger, Hertzberg, 1106.3335



Late-Time Decays

Consider Inflaton to Photon Coupling

EM Lagrangian

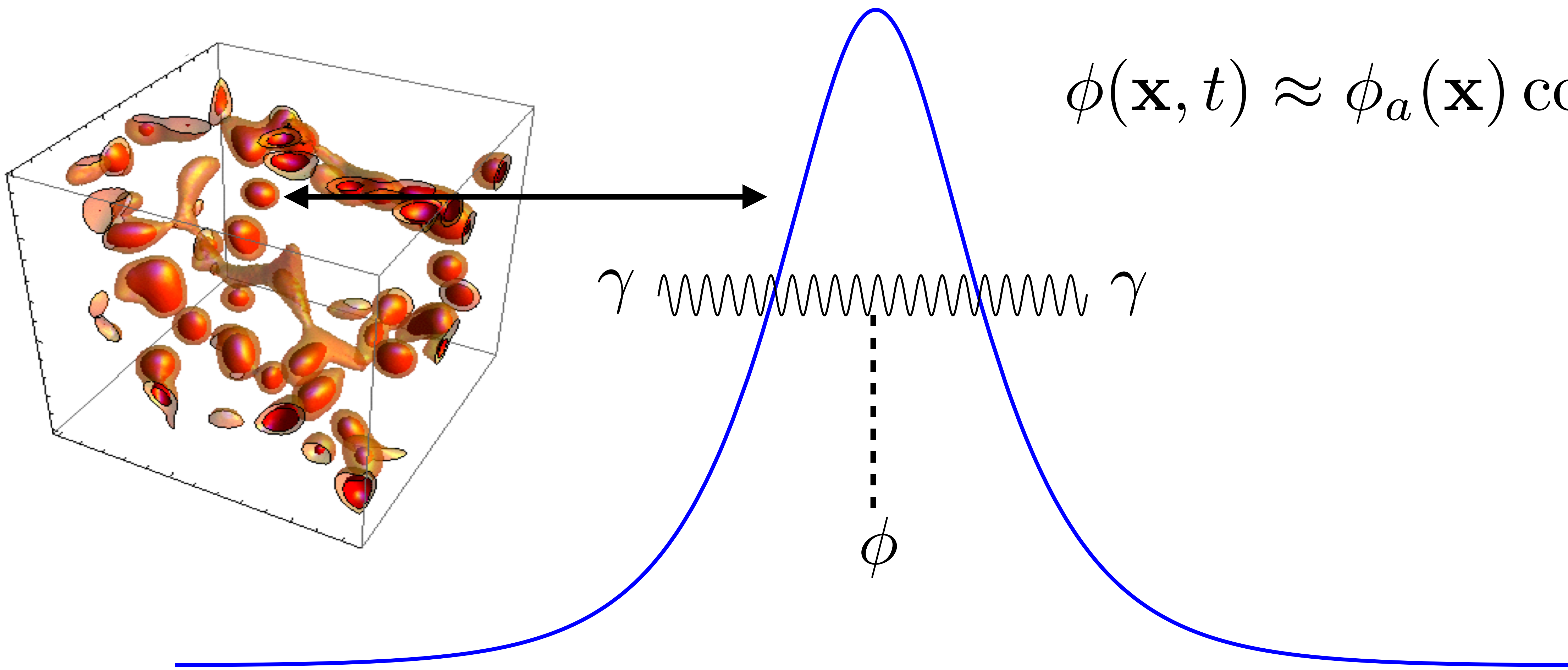
$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{g_{a\gamma}}{4}\phi F_{\mu\nu}\tilde{F}^{\mu\nu}$$

(Sikivie 1983; Adshead, Giblin, Scully, Sfakianakis 2015, 2016; Masaki, Aoki, Soda 2017)

Equation of motion

$$\ddot{\mathbf{A}} - \nabla^2\mathbf{A} + g_{a\gamma}\,\partial_t\phi\nabla\times\mathbf{A} = 0$$

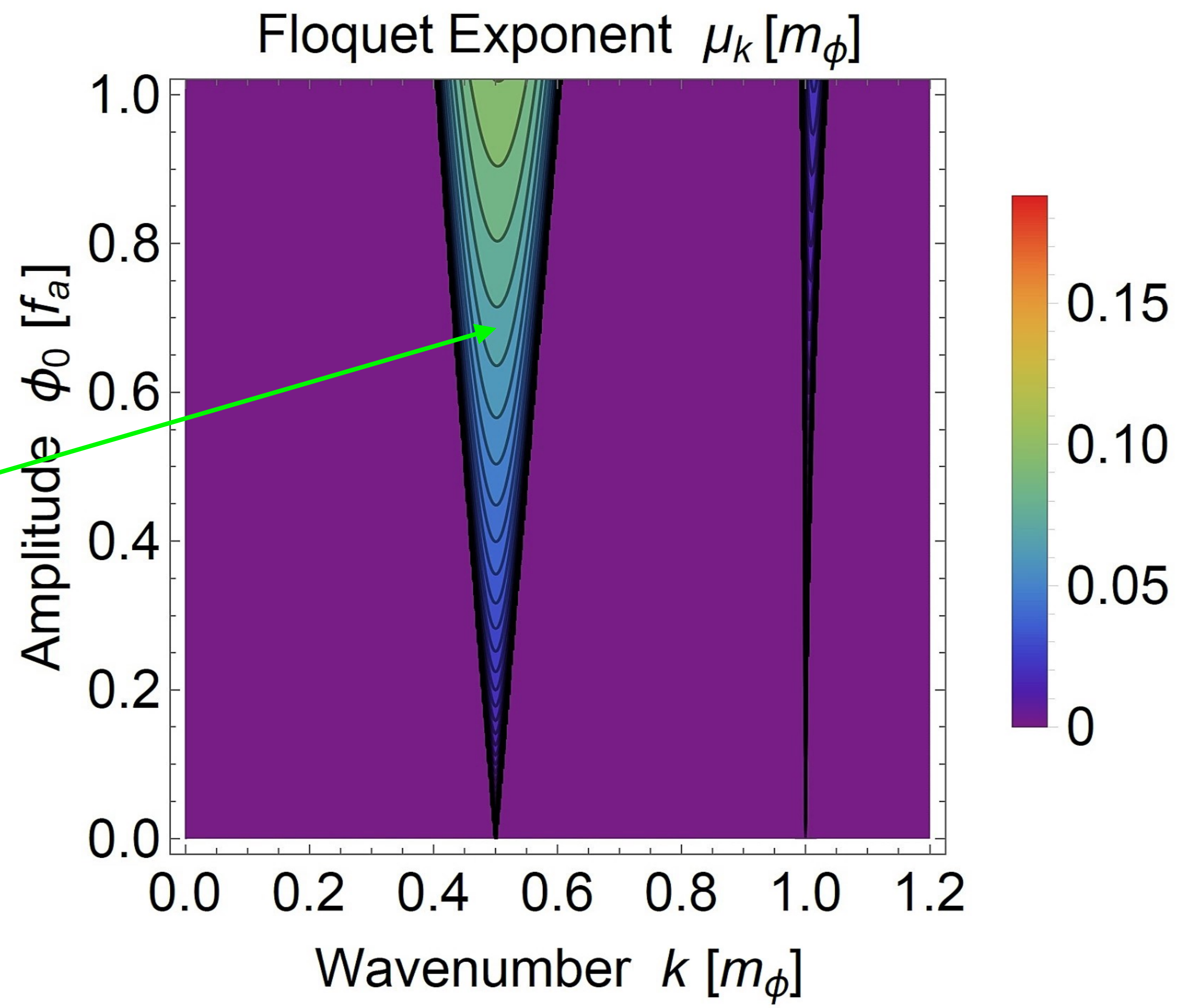
$$\phi(\mathbf{x},t) \approx \phi_a(\mathbf{x})\cos(m_\phi t)$$



Homogeneous Inflaton Field

Mathieu Equation

$$\ddot{\mathbf{A}}^T_{\mathbf{k}} + k^2 \mathbf{A}^T_{\mathbf{k}} + g_{a\gamma}\, k\, \partial_t \phi(t) \mathbf{A}^T_{\mathbf{k}} = 0$$



Parametric resonance
always present

$$k \approx \frac{m_a}{2}$$

$$\mu^*_H \approx \frac{1}{4} g_{a\gamma}\, m_\phi\, \phi_a$$

e.g., Yoshimura 1996

Inhomogeneous (Spherical) Clump

Decomposition into vector spherical harmonics

$$\mathbf{A}(\mathbf{r},t) = \sum_{lm} \int \frac{d^3k}{(2\pi)^3} \left[a_{lm}(k,t)\mathbf{N}_{lm}(k,\mathbf{r}) + b_{lm}(k,t)\mathbf{M}_{lm}(k,\mathbf{r})\right]$$

where

$$\mathbf{M}_{lm}(k,\mathbf{r}) = i\frac{j_l(kr)}{\sqrt{l(l+1)}}\nabla \times [Y_{lm}(\theta,\varphi)\mathbf{r}]$$

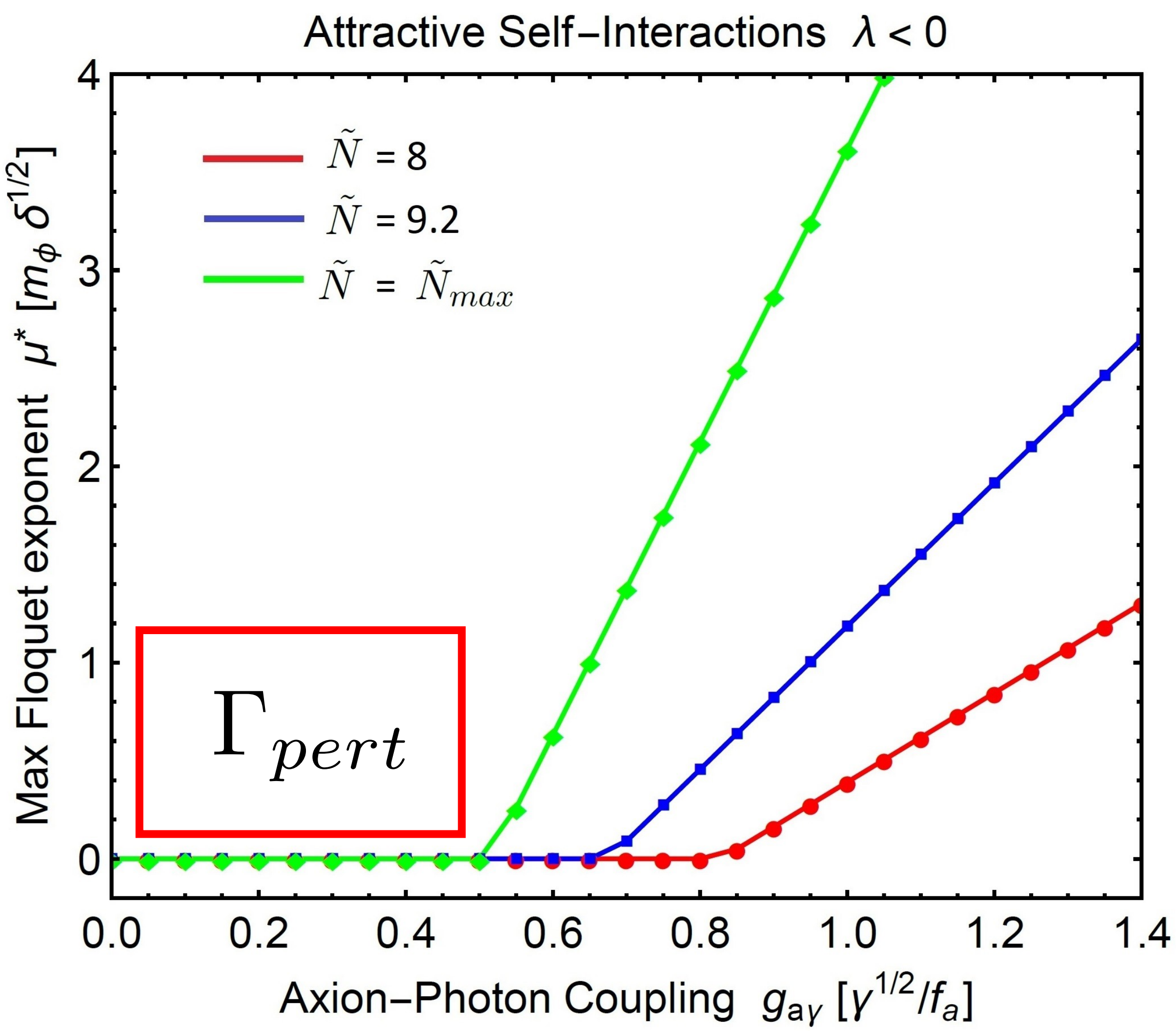
$$\mathbf{N}_{lm}(k,\mathbf{r}) = \frac{i}{k}\nabla \times \mathbf{M}_{lm}$$

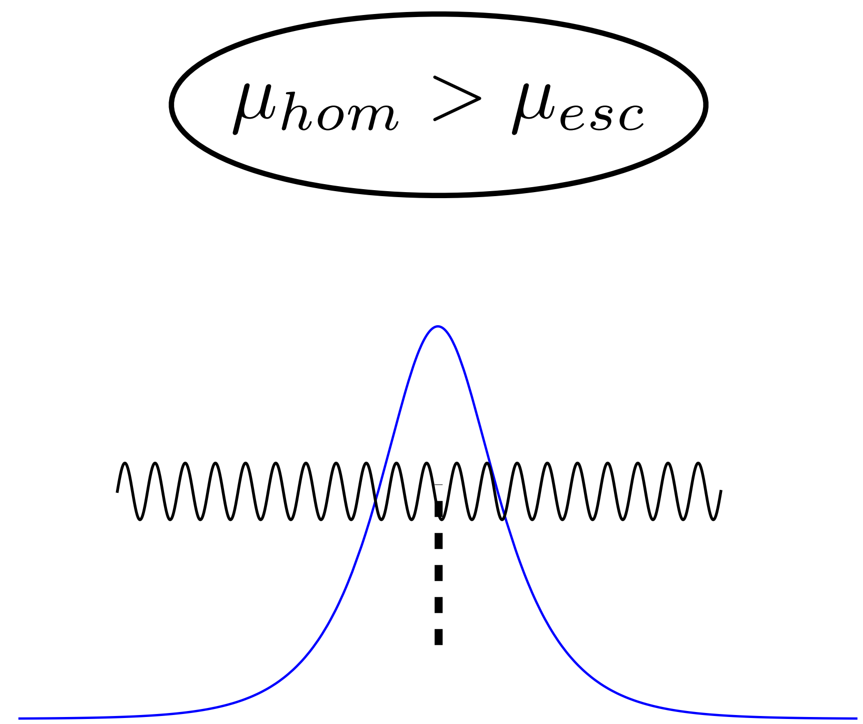
Hertzberg, Schiappacasse 1805.00430

Inhomogeneous (Spherical) Clump

Instability channel $\quad l = 1, m = 0, \quad b_{10} = -i\, a_{10}$

$$\ddot{a}_{10}(k,t) + k^2 a_{10}(k,t) + g_{a\gamma} k \int \frac{dk'}{(2\pi)} \partial_t \tilde{\phi}(k-k') a_{10}(k',t) = 0$$





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Hertzberg 1003.3459

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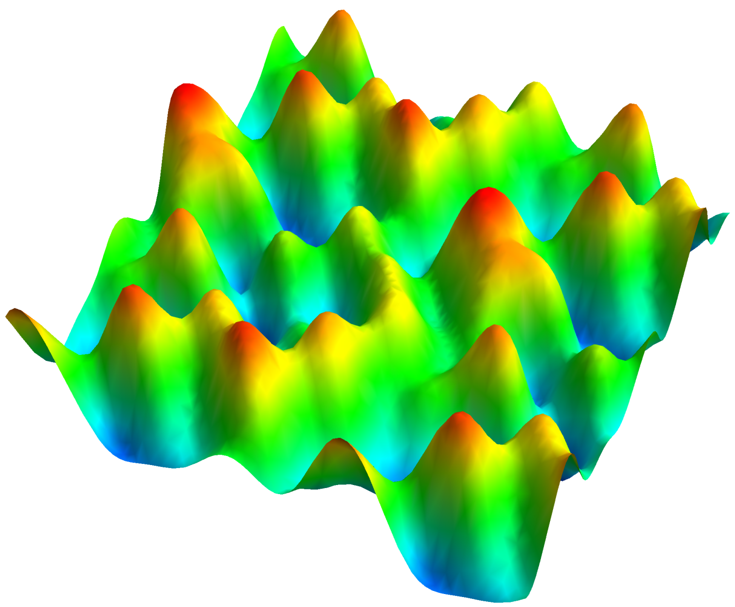

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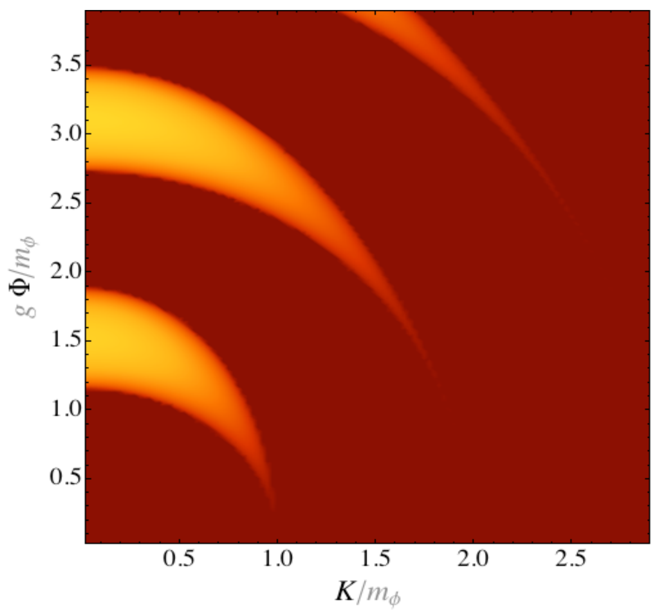

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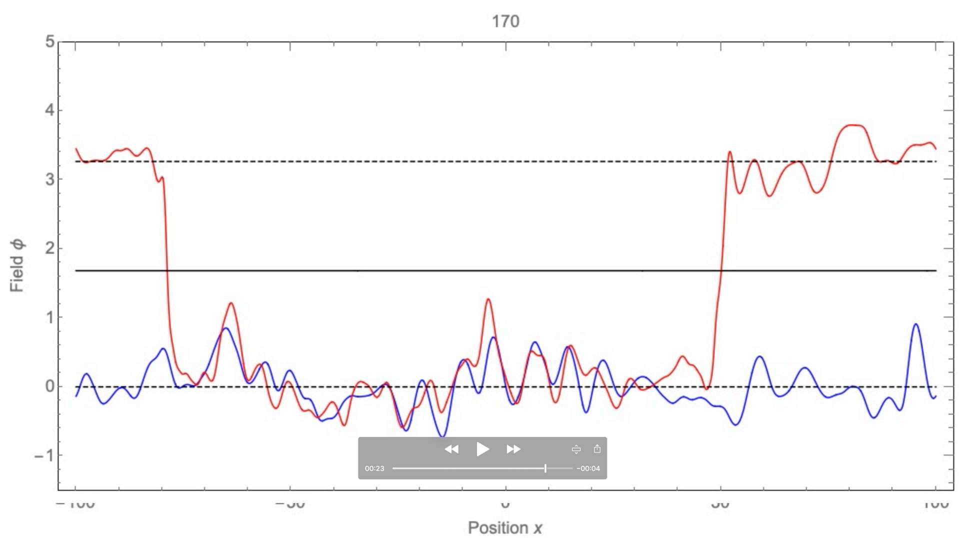
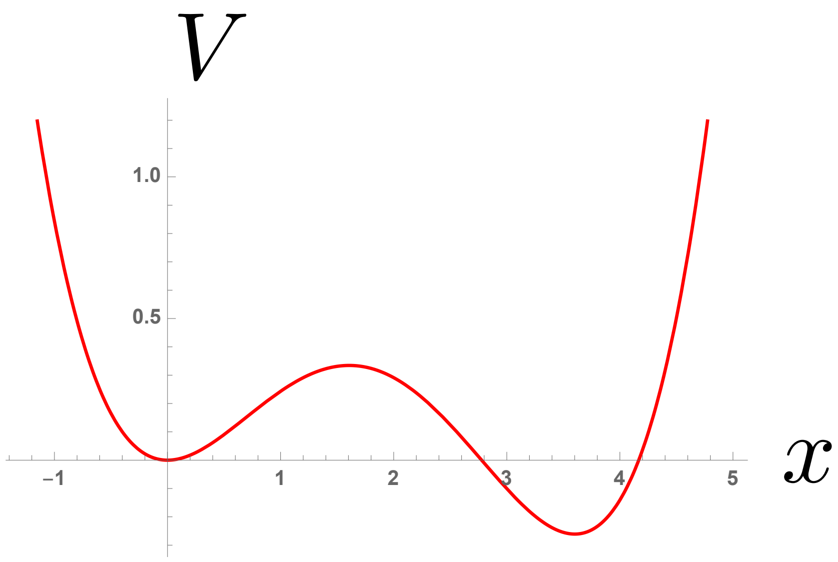


- Hertzberg, Yamada 1904.08565;

Usually thought to be forbidden classically

“Classical Tunneling” in Quantum Mechanics?

Single Particle Behaviour

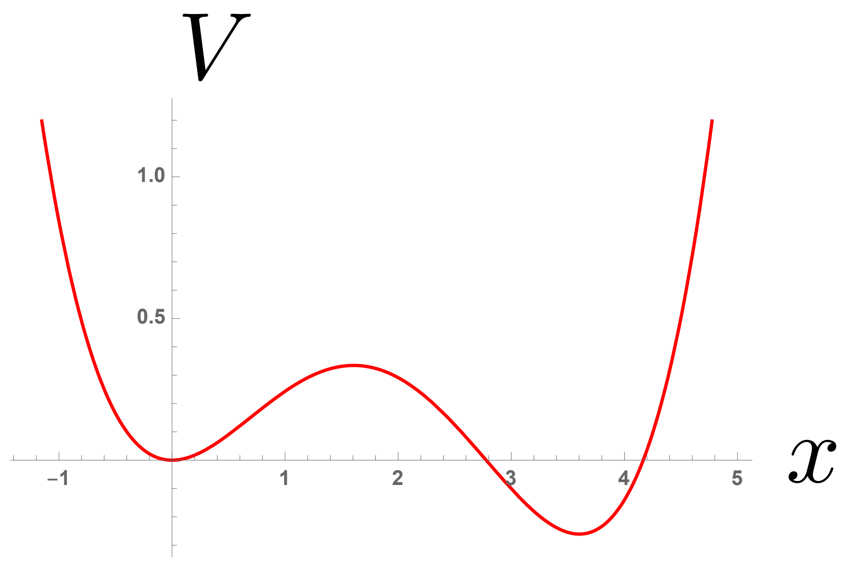


Initial conditions drawn from quadratic theory $V(x) = \frac{1}{2} m\omega^2 x^2 + \ldots$

$$\Psi[x] \propto \exp\left[-\frac{1}{2} m\,\omega\,x^2\right]$$

$$\Psi[p] \propto \exp\left[-\frac{1}{2}\frac{1}{m\,\omega} p^2\right]$$

Single Particle Behaviour



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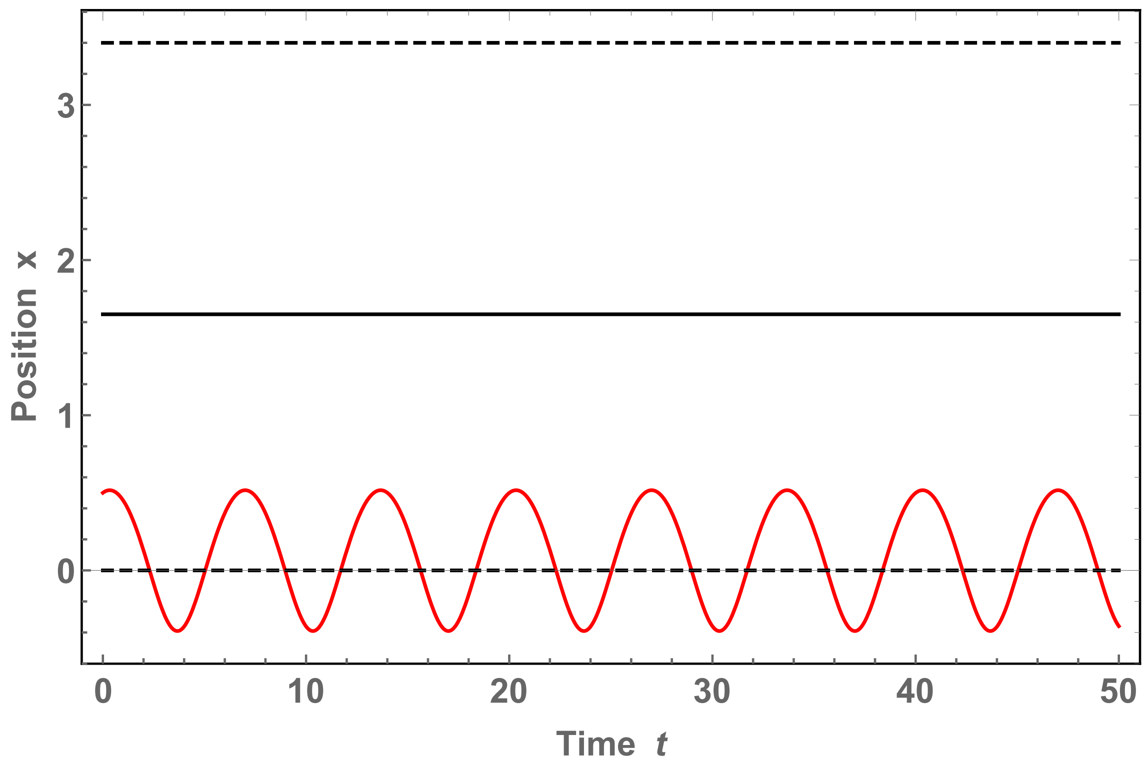
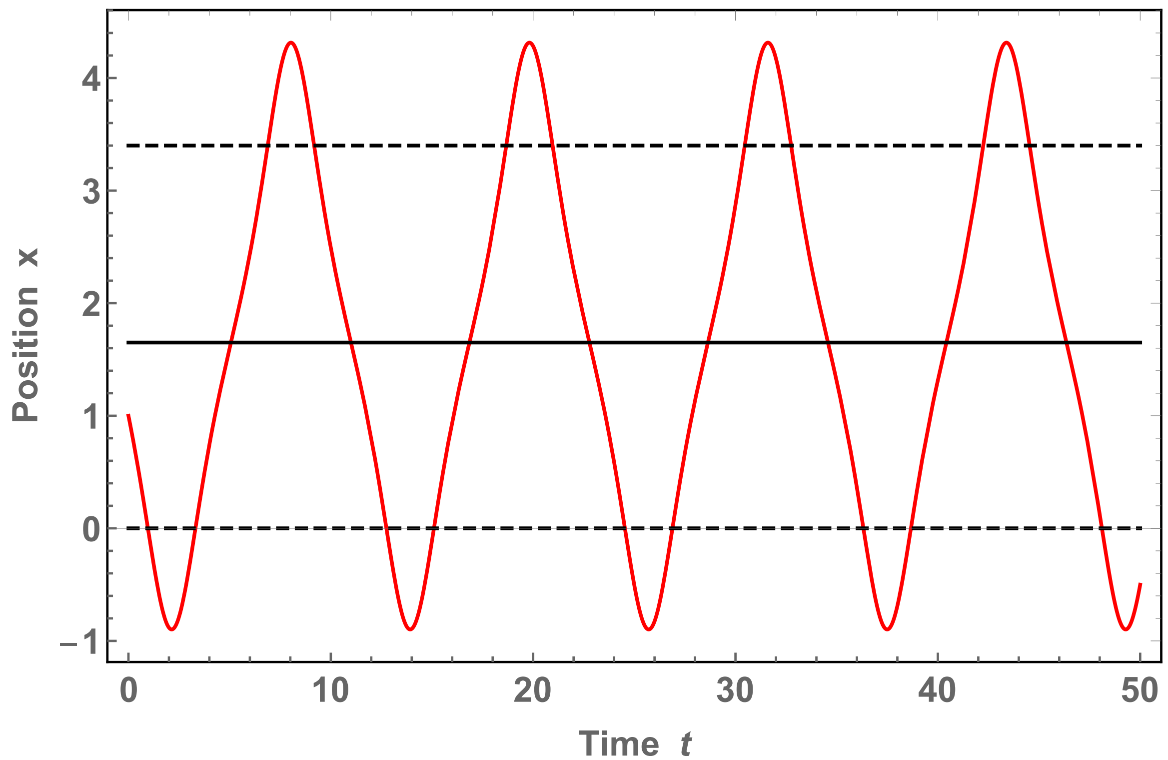
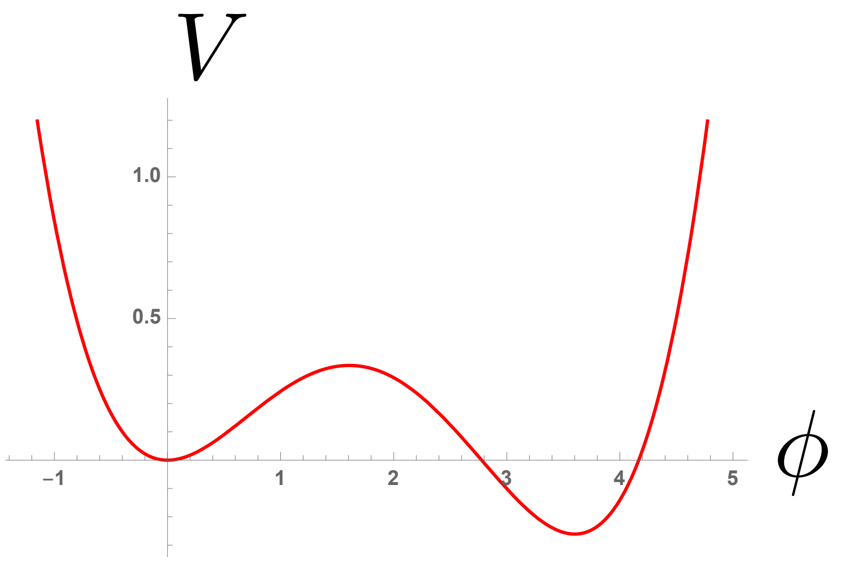




“Classical Tunneling” in Quantum Field Theory?

Scalar Field Theory Behaviour



Initial conditions drawn from free theory

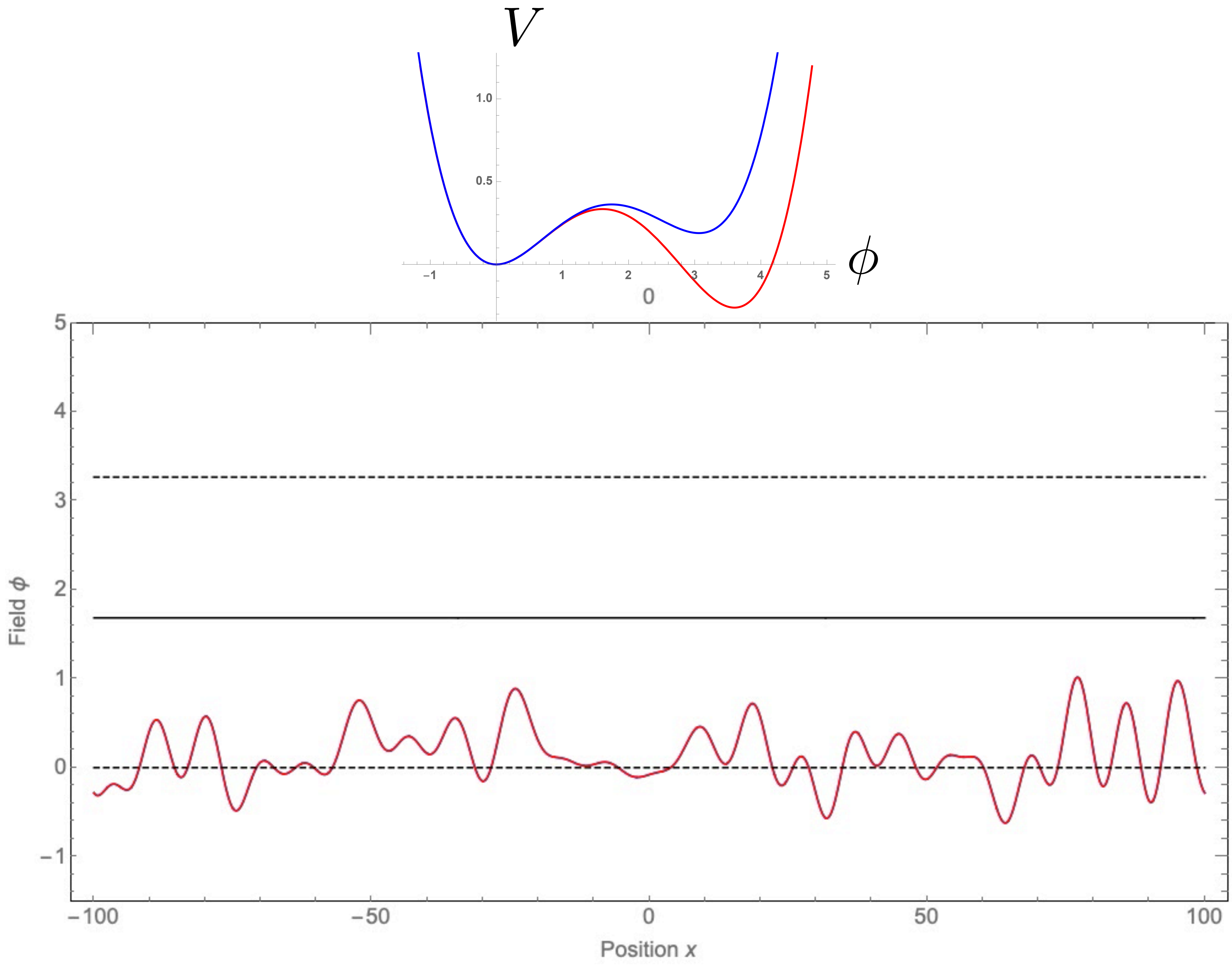
$$V(\phi) = \frac{1}{2}m^2\phi^2 + \dots$$

$$\Psi[\phi] \propto \exp\left[-\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\omega_k|\phi_k|^2\right]$$

$$\Psi[\pi] \propto \exp\left[-\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{\omega_k}|\pi_k|^2\right]$$

$$\omega_k = \sqrt{m^2 + k^2}$$

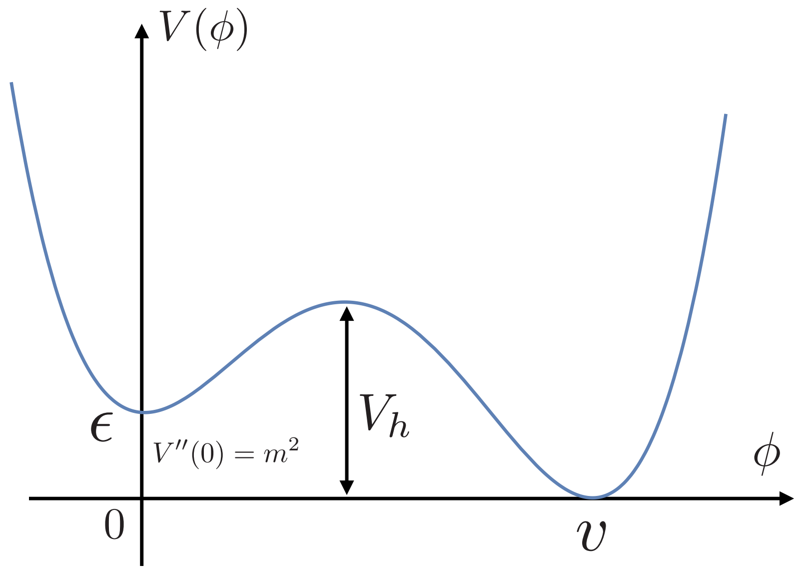
Scalar Field Theory Behaviour



Scalar Field Theory Behaviour

- Approximate numerical agreement in decay rates found in:
Braden, Johnson, Peiris, Pontzen, Weinfurtner 2018

Some (Partial) Understanding



- In LINEAR regime, ensemble average is correct (despite small occupancy)

- In NONLINEAR regime; ensemble average can (sometimes) describe tunneling (since bubble's have high occupancy)

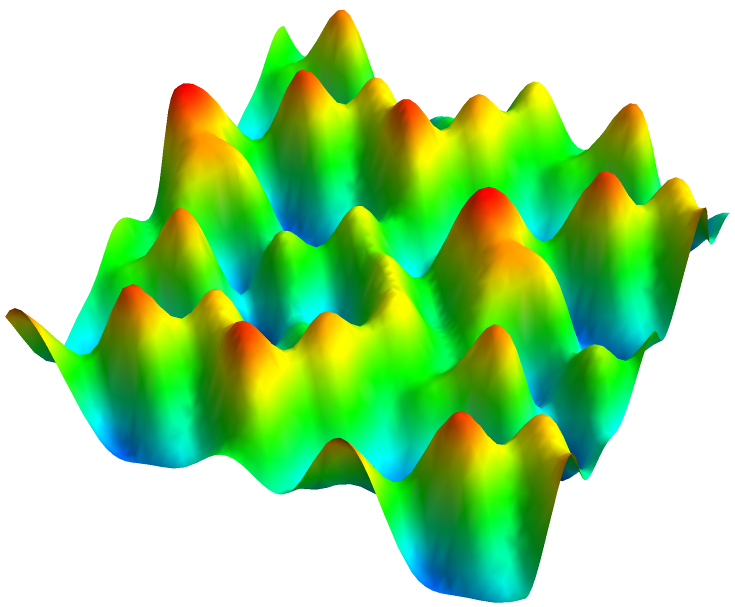
$$\mathcal{N} \sim \left(\frac{V_h}{\epsilon}\right)^{d-1} \left(\frac{v^{2(d+1)/(d-1)}}{V_h}\right)^{d/2-1} \left(\frac{v}{m^{(d-1)/2}}\right)^{2/(d-1)} \gg 1$$

- Tunneling rate is parametrically correct from bubble statistics

Hertzberg, Yamada 1904.08565

Thank you

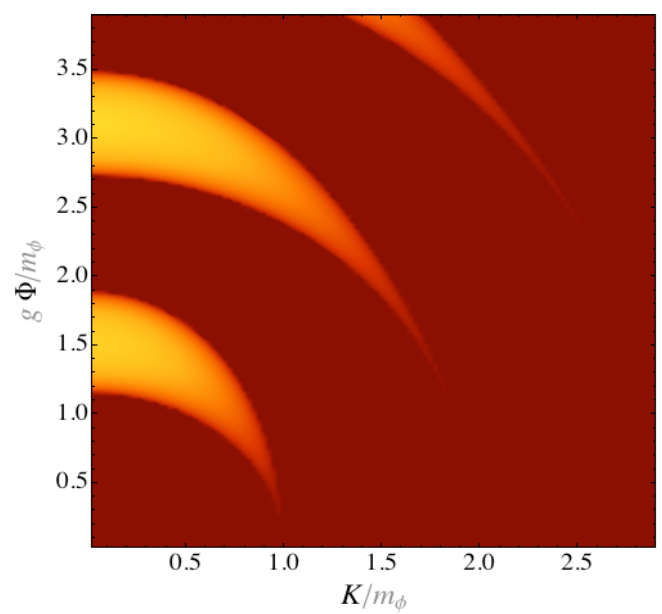
Part 1: Axion Dark Matter



- Hertzberg, Tegmark, Wilczek 0807.1726;
- Guth, Hertzberg, Prescod-Weinstein 1412.5930;
- Hertzberg 1609.01342

Macroscopic spreading of wave-function

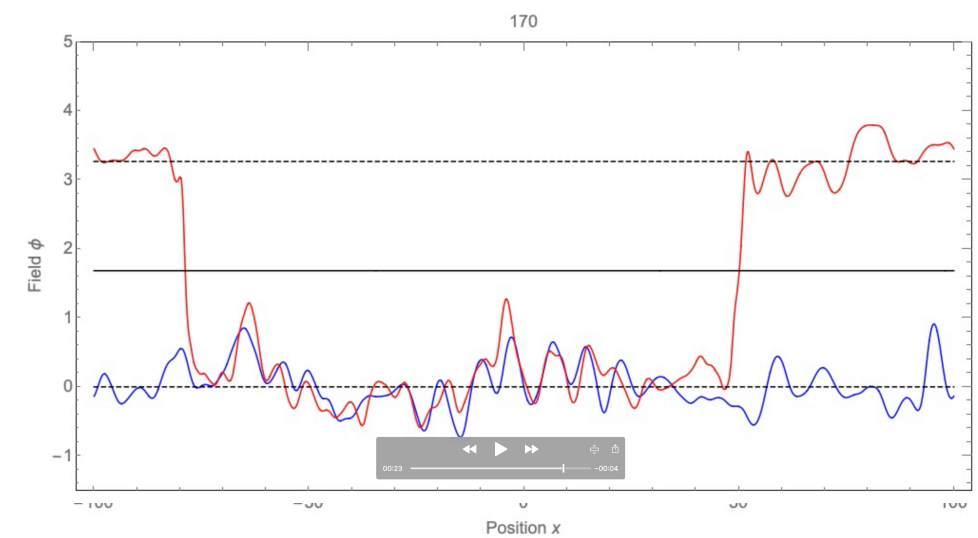
Part 2: Post-Inflation



- Hertzberg 1003.3459;
- Amin, Easther, Finkel, Flauger, Hertzberg 1106.335;
- Amin, Hertzberg, Kaiser, Karouby 1410.3808;
- Hertzberg, Schiappacasse 1805.00430

Beginning with low occupancy modes

Part 3: Tunneling



- Hertzberg, Yamada 1904.08565;

Usually thought to be forbidden classically