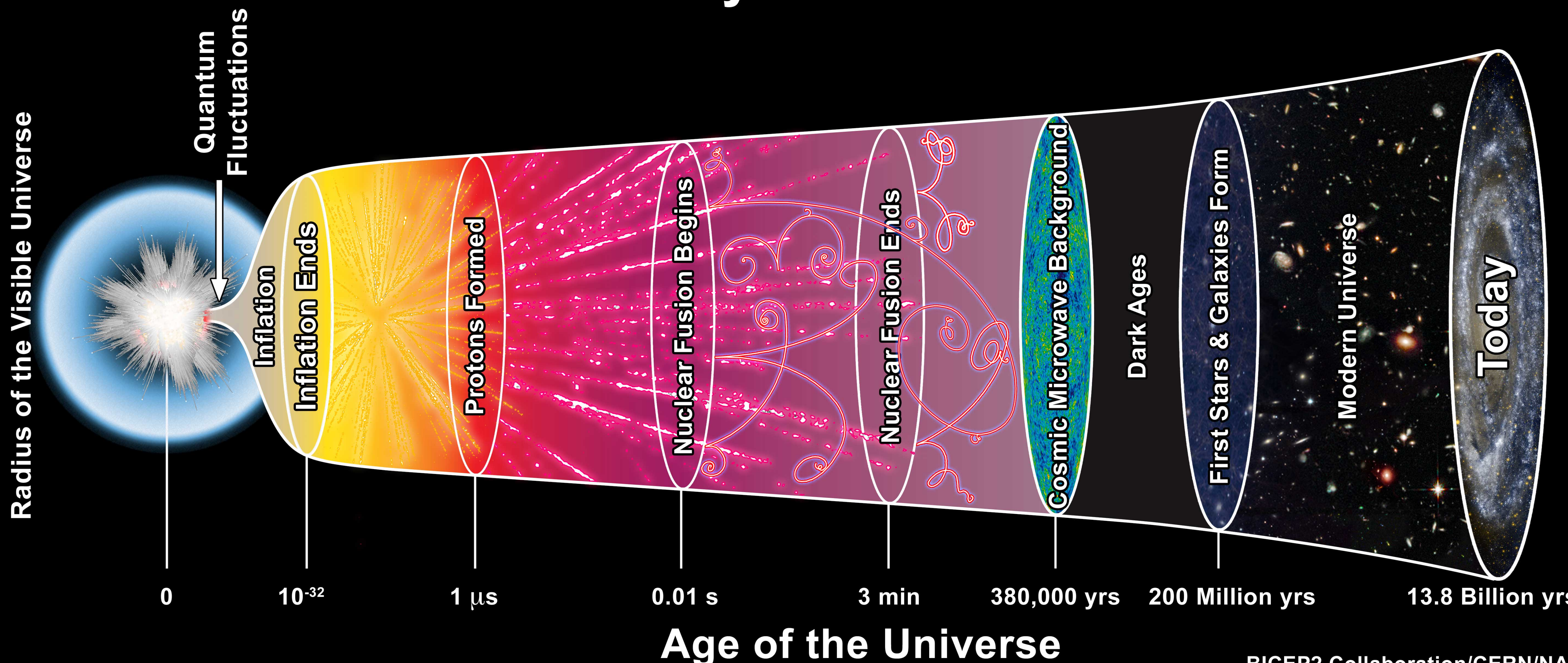
Universal Infrared Scaling of Stochastic Gravitational Wave Background

Shi Pi (皮石)

Kavli IPMU, University of Tokyo

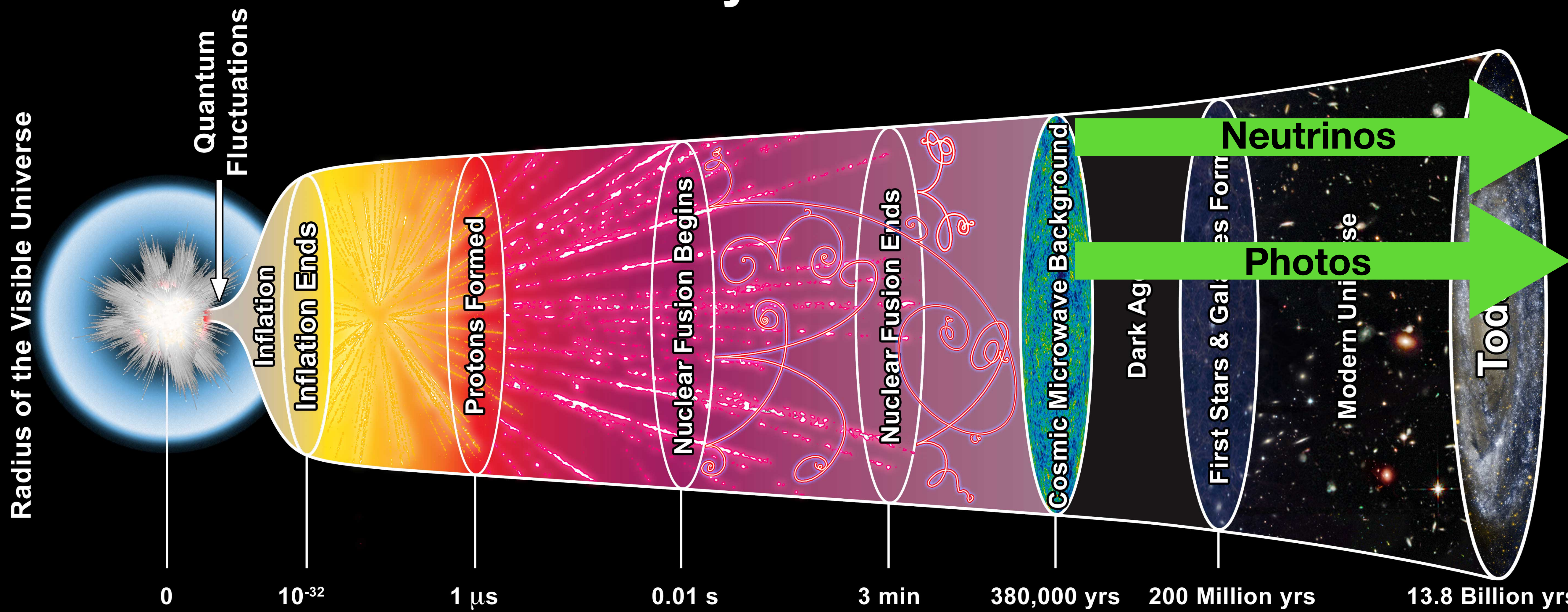
Based on
Rong-Gen Cai, SP, and Misao Sasaki, arXiv:1909.13728.

History of the Universe

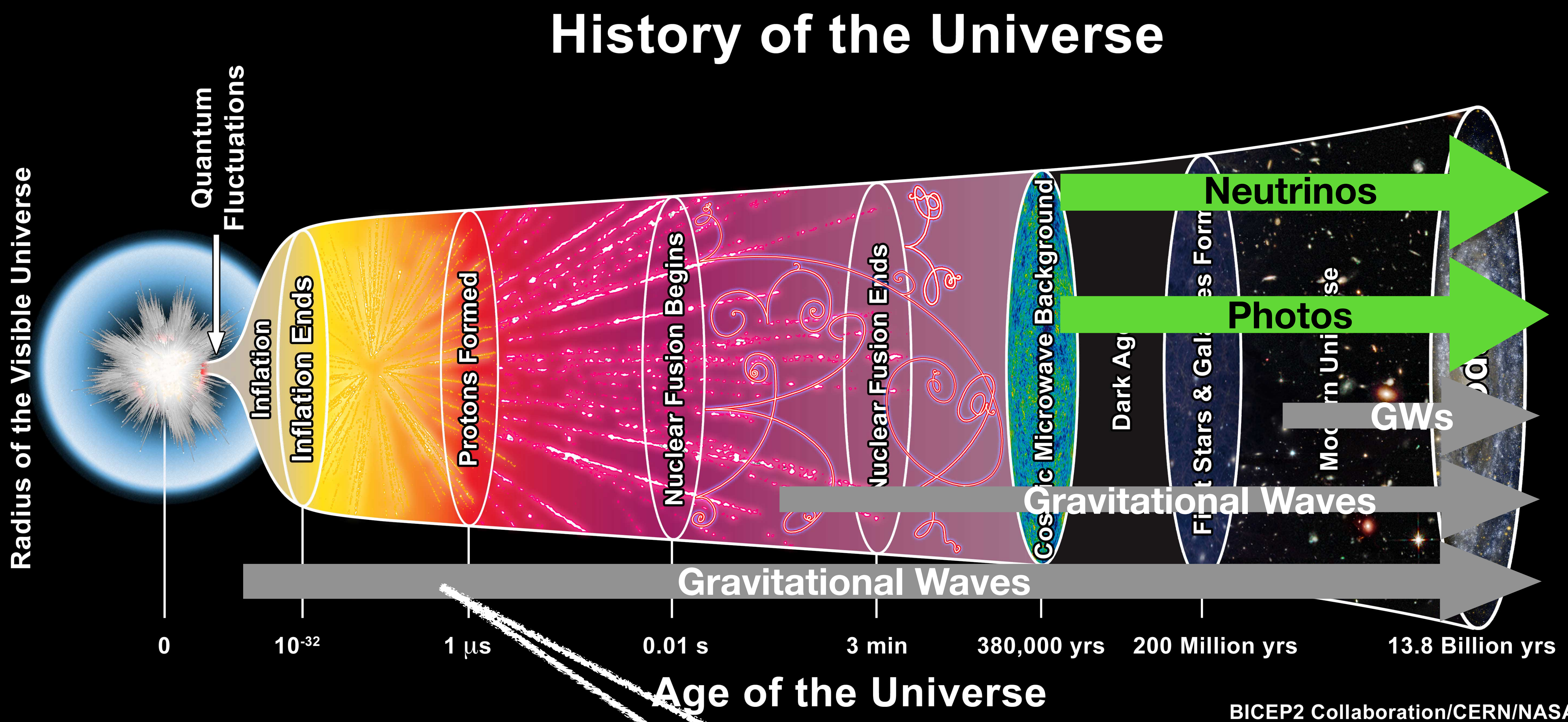


BICEP2 Collaboration/CERN/NASA

History of the Universe

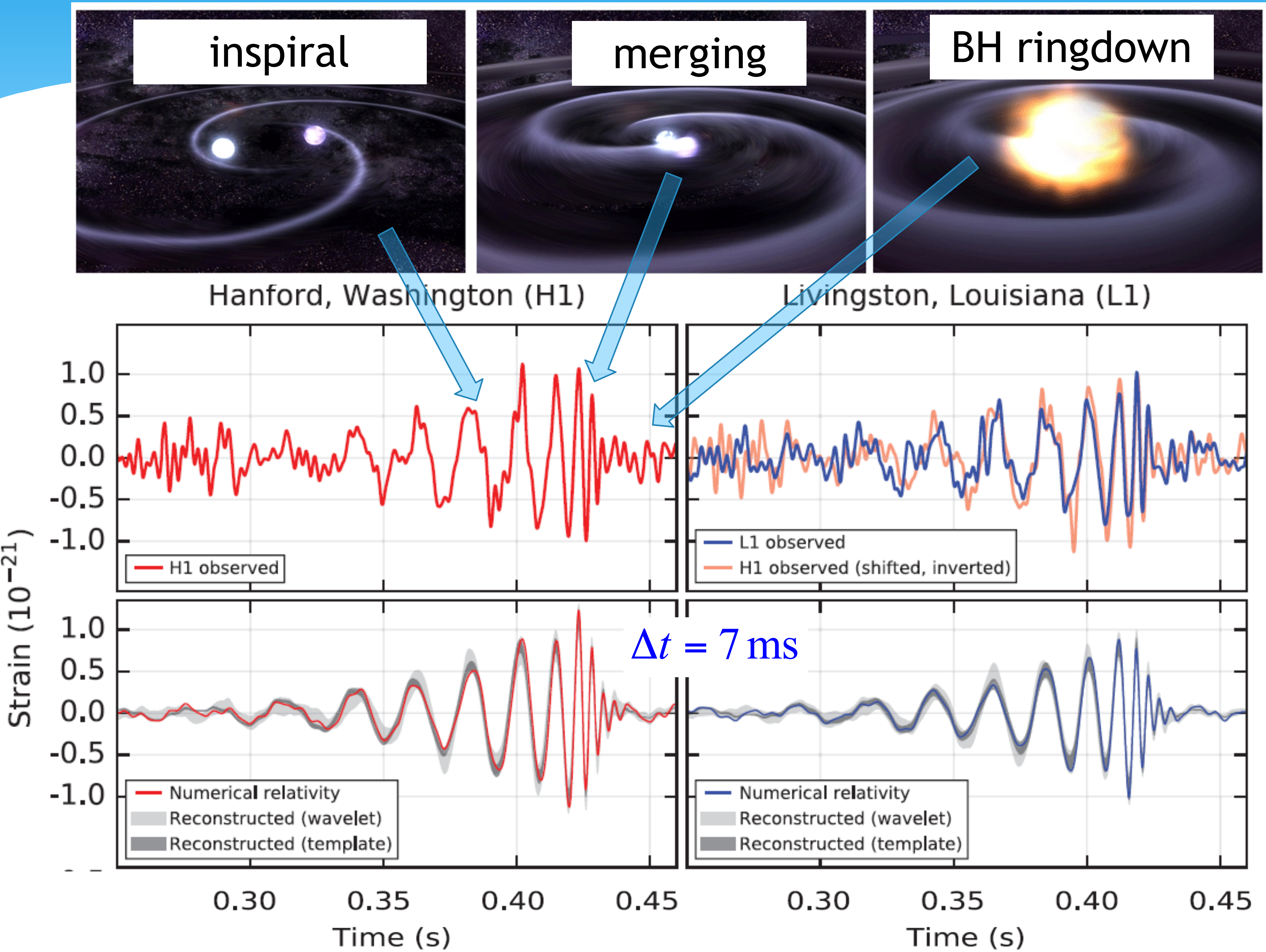


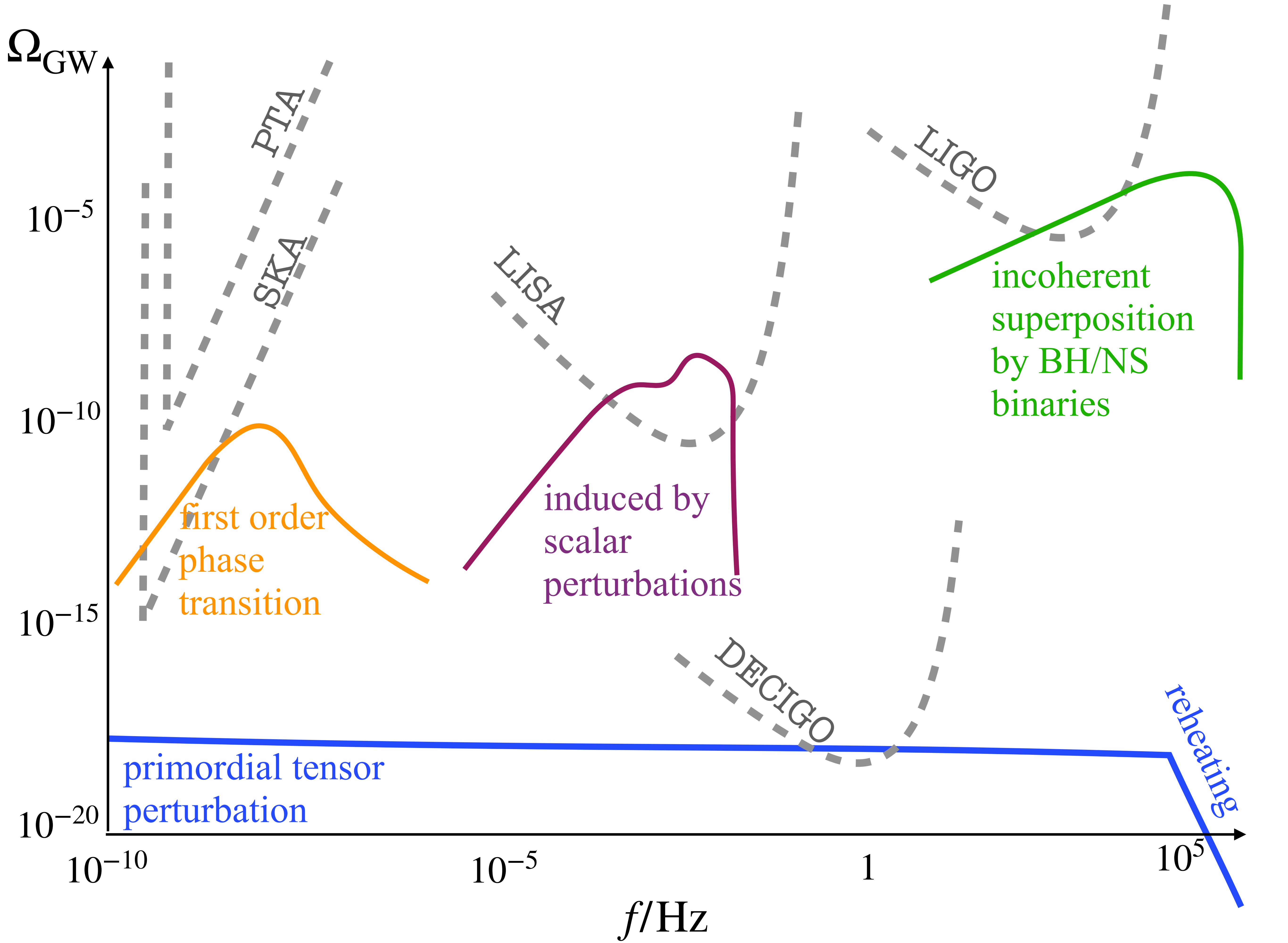
BICEP2 Collaboration/CERN/NASA



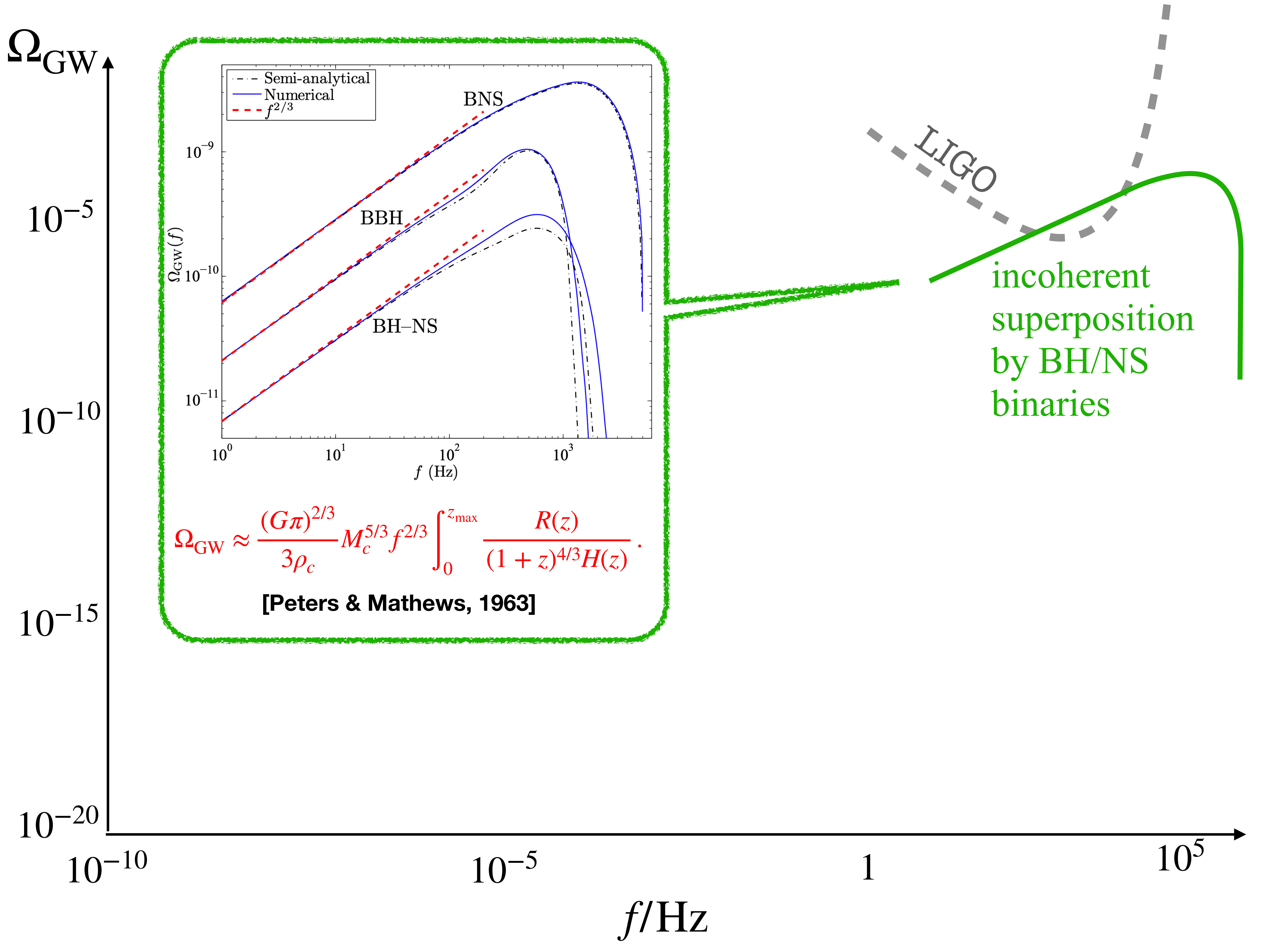
GWs propagates freely all the time!

observed GW signal

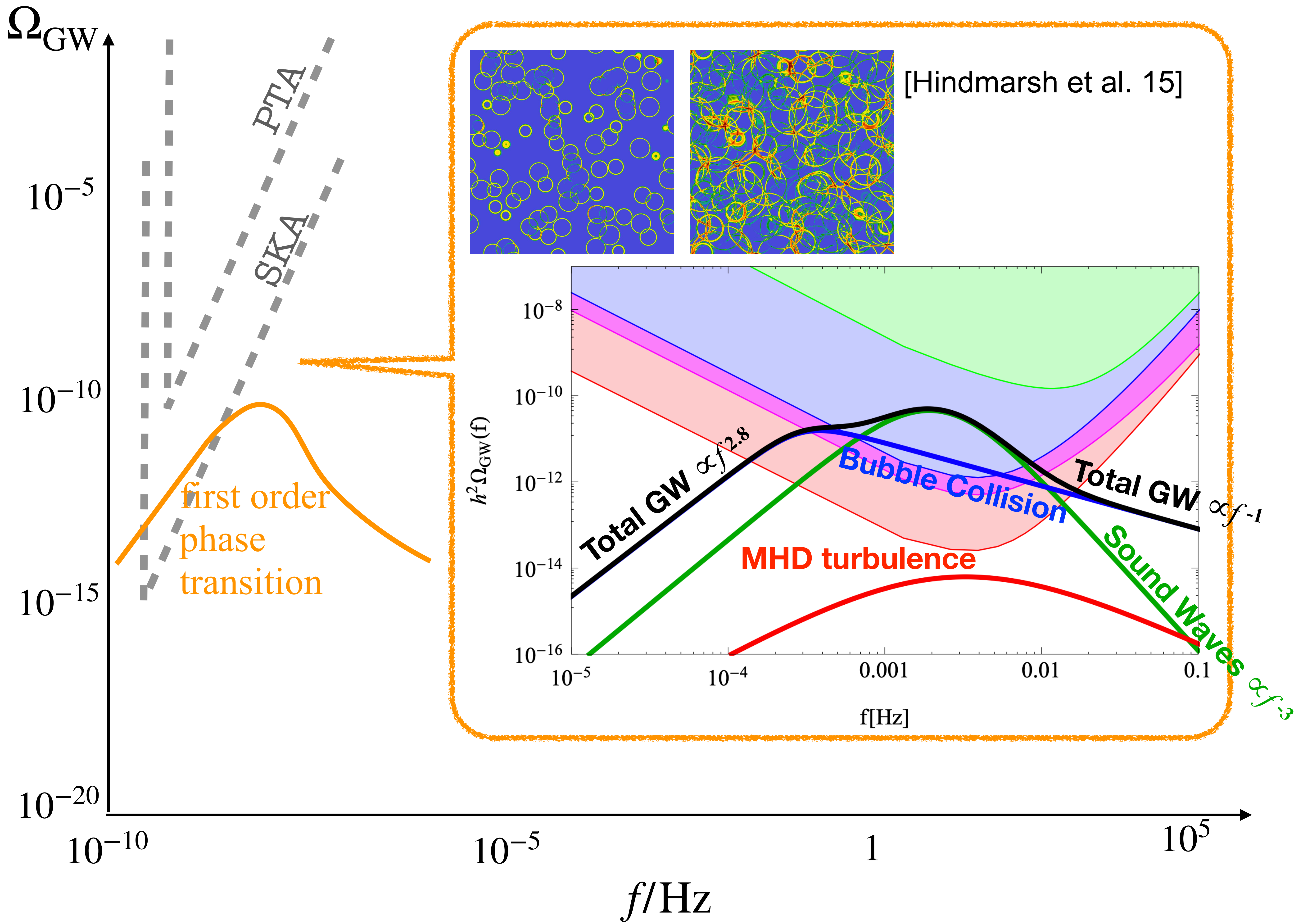




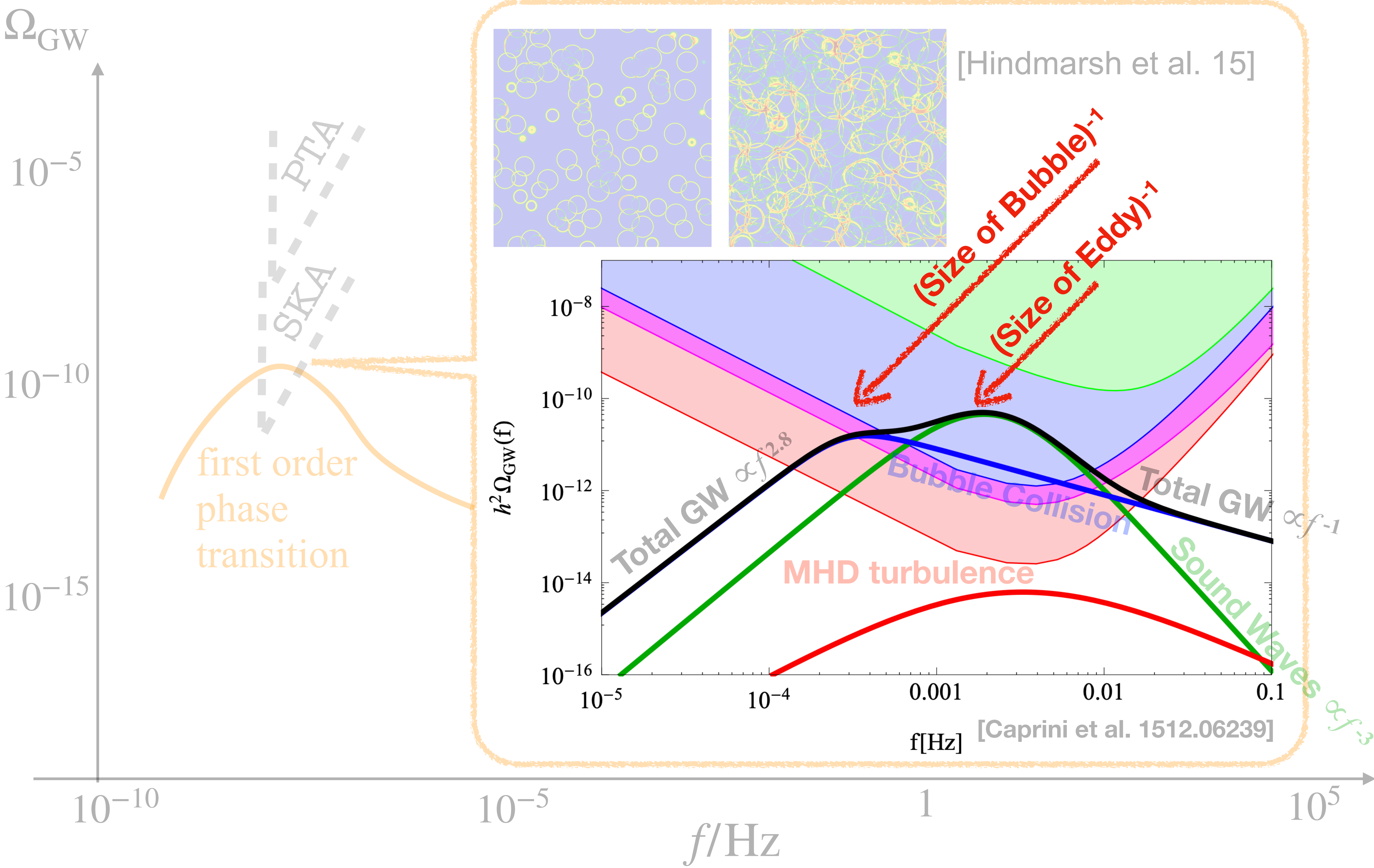
Ω_{GW}
10^{-5}
10^{-10}
10^{-15}
10^{-20}
PTA
SKA
LISA
LIGO
DECIGO
incoherent superposition by BH/NS binaries
induced by scalar perturbations
first order phase transition
primordial tensor perturbation
reheating
10^{-10}
10^{-5}
1
10^{5}
f/Hz



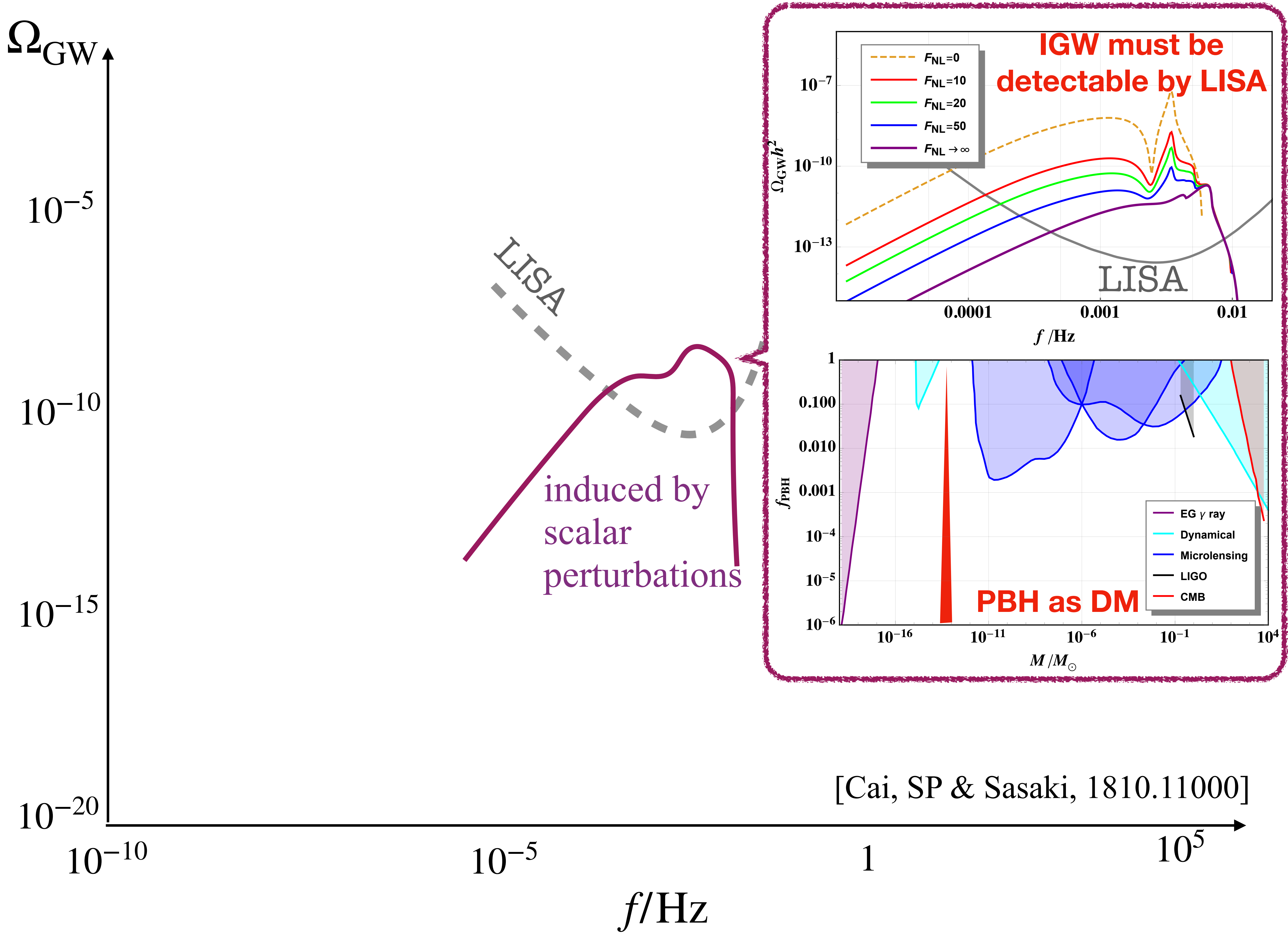
$\Omega_{\rm GW}$
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-10}
10^{-5}
1
10^{5}
f/Hz
LIGO
incoherent superposition by BH/NS binaries
Semi-analytical
Numerical
$f^{2/3}$
BNS
BBH
BH–NS
$\Omega_{\rm GW}(f)$
f (Hz)
$$\Omega_{\rm GW} \approx \frac{(G\pi)^{2/3}}{3\rho_c} M_c^{5/3} f^{2/3} \int_0^{z_{\max}} \frac{R(z)}{(1+z)^{4/3} H(z)} \,.$$
[Peters & Mathews, 1963]



Ω_{GW}
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-10}
10^{-5}
1
10^{5}
f/Hz
PTA
SKA
first order
phase
transition
[Hindmarsh et al. 15]
$h^2\Omega_{GW}(f)$
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-5}
10^{-4}
0.001
0.01
0.1
f[Hz]
Total GW $\propto f^{2.8}$
Bubble Collision
Total GW $\propto f^{-1}$
MHD turbulence
Sound Waves $\propto f^{-3}$

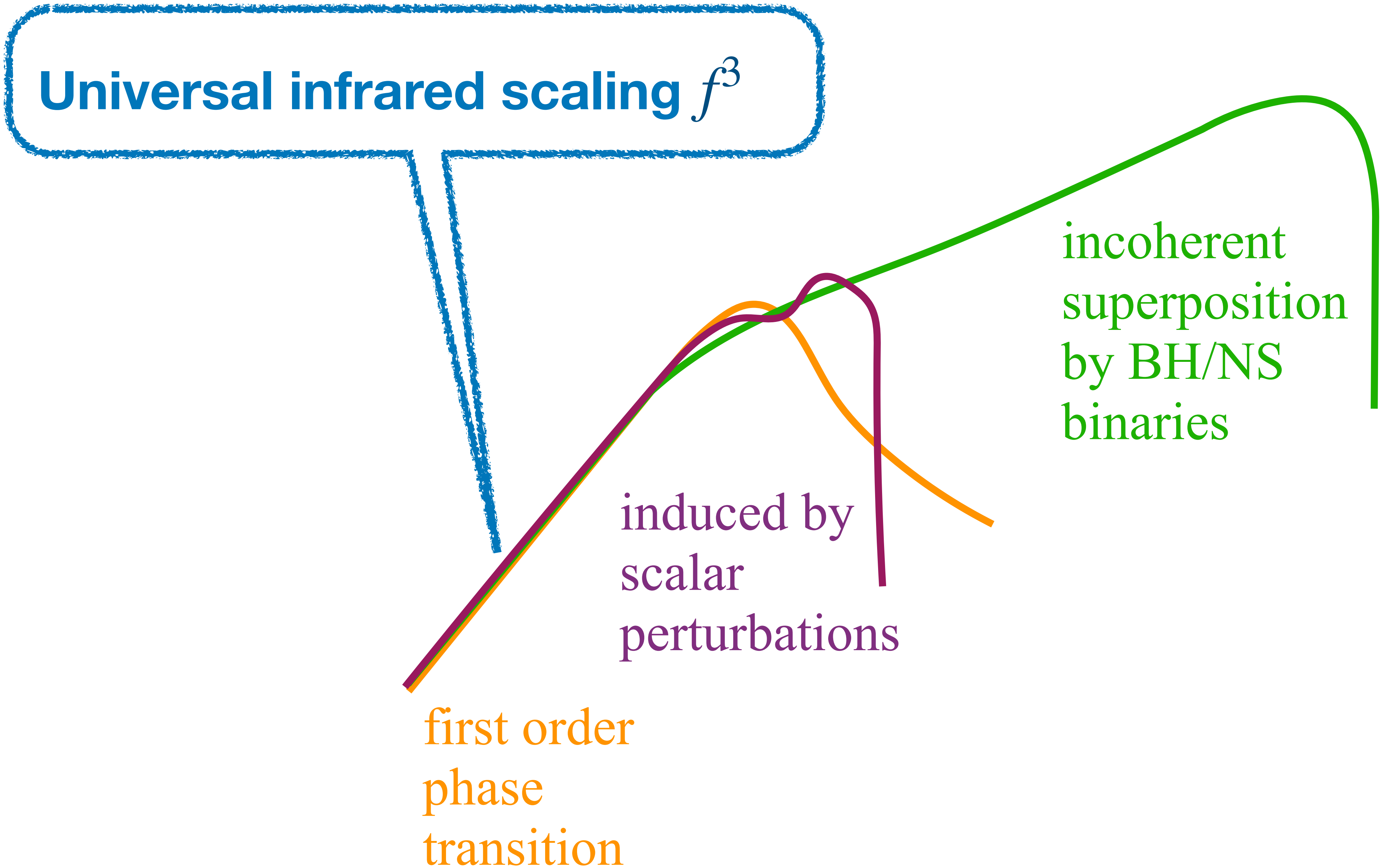


Ω_{GW}
10^{-5}
10^{-10}
10^{-15}
PTA
SKA
first order phase transition
10^{-10}
10^{-5}
1
10^{5}
f/Hz
[Hindmarsh et al. 15]
(Size of Bubble)$^{-1}$
(Size of Eddy)$^{-1}$
$h^2\Omega_{GW}(f)$
Total GW $\propto f^{2.8}$
Bubble Collision
Total GW $\propto f^{-1}$
MHD turbulence
Sound Waves $\propto f^{-3}$
f[Hz]
[Caprini et al. 1512.06239]



Ω_{GW}
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-10}
10^{-5}
1
10^{5}
f/Hz
LISA
induced by
scalar
perturbations
IGW must be
detectable by LISA
$F_{NL}=0$
$F_{NL}=10$
$F_{NL}=20$
$F_{NL}=50$
$F_{NL}\to\infty$
$\Omega_{GW}h^2$
10^{-7}
10^{-10}
10^{-13}
LISA
0.0001
0.001
0.01
f /Hz
f_{PBH}
1
0.100
0.010
0.001
10^{-4}
10^{-5}
10^{-6}
10^{-16}
10^{-11}
10^{-6}
10^{-1}
10^{4}
$M/M_\odot$
EG γ ray
Dynamical
Microlensing
LIGO
CMB
PBH as DM
[Cai, SP & Sasaki, 1810.11000]

Infrared Scaling



[Rong-Gen Cai, SP, and Misao Sasaki, arXiv:1909.13728]

Calculate Ω_{GW}

- Define the power spectra of SGWB: $\Omega_{\mathrm{GW}} = \frac{1}{\rho_{\mathrm{crit}}} \frac{d\rho_{\mathrm{GW}}}{d \ln k}$.

- Metric: $ds^2 = a^2(\eta)(-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j)$,

- Source term:
  $T_{ab}(\eta, \mathbf{x}) = \partial_a \phi(\eta, \mathbf{x}) \partial_b \phi(\eta, \mathbf{x}) + v_a(\eta, \mathbf{x}) v_b(\eta, \mathbf{x})$.

- ϕ is a scaler field, while v_a is a vector field, and
  $v_a = \partial_a v + w_a$

Calculate $\Omega_{\rm GW}$

- The GW spectrum is
- $$\Omega_{\mathsf{GW}}(\eta, k) = \frac{k^3}{45a^4H^2M_{\rm Pl}^4}\int_0^\eta d\eta_1 \int_0^\eta d\eta_2\, a^3(\eta_1)a^3(\eta_2)$$
$$\cos k(\eta_1-\eta_2)\Lambda_{ab}^{cd}(\hat{k})\langle T_{\mathbf{k}}^{ab}(\eta_1)T^*_{cd,\mathbf{p}}(\eta_2)\rangle' .$$
- $T_{\mathbf{k}}^{ab}(\eta) = \int \frac{d^3\ell}{(2\pi)^{3/2}} v_{\mathbf{l}}^a(\eta)v_{\mathbf{k-l}}(\eta) + \cdots$
- $\langle T_{\mathbf{k}}^{ab}(\eta_1)T_{\mathbf{p}}^{cd*}(\eta_2)\rangle = \int \frac{d^3\ell\, d^3q}{(2\pi)^3}\Big[\langle v_{\mathbf{l}}^a(\eta_1)v_{\mathbf{k-l}}^b(\eta_1)v_{\mathbf{q}}^{c*}(\eta_2)v_{\mathbf{p-q}}^{d*}(\eta_2)\rangle + \cdots$

Calculate $\Omega_{\rm GW}$

$$
\begin{aligned}
&\left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^b_{\boldsymbol{k}-\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2) v^{d*}_{\boldsymbol{p}-\boldsymbol{q}}(\eta_2) \right\rangle \\
&= \left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2) \right\rangle \left\langle v^b_{\boldsymbol{k}-\boldsymbol{\ell}}(\eta_1) v^{d*}_{\boldsymbol{q}-\boldsymbol{p}}(\eta_2) \right\rangle \\
&+ \left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^{d*}_{\boldsymbol{p}-\boldsymbol{q}}(\eta_2) \right\rangle \left\langle v^b_{\boldsymbol{k}-\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2) \right\rangle \\
&+ \left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^b_{\boldsymbol{k}-\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2) v^{d*}_{\boldsymbol{p}-\boldsymbol{q}}(\eta_2) \right\rangle_c,
\end{aligned}
$$

Calculate Ω_{GW}

$$
\begin{aligned}
&\left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^b_{\boldsymbol{k}-\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2) v^{d*}_{\boldsymbol{p}-\boldsymbol{q}}(\eta_2)\right\rangle \\
&= \left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2)\right\rangle \left\langle v^b_{\boldsymbol{k}-\boldsymbol{\ell}}(\eta_1) v^{d*}_{\boldsymbol{q}-\boldsymbol{p}}(\eta_2)\right\rangle \\
&+ \left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^{d*}_{\boldsymbol{p}-\boldsymbol{q}}(\eta_2)\right\rangle \left\langle v^b_{\boldsymbol{k}-\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2)\right\rangle \\
&\cancel{+ \left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^b_{\boldsymbol{k}-\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2) v^{d*}_{\boldsymbol{p}-\boldsymbol{q}}(\eta_2)\right\rangle_c},
\end{aligned}
$$

By symmetry

Calculate $\Omega_{\rm GW}$

$$
\begin{aligned}
&\left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^b_{\cancel{\boldsymbol{k}}-\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2) v^{d*}_{\cancel{\boldsymbol{p}}-\boldsymbol{q}}(\eta_2)\right\rangle \\
&= \left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2)\right\rangle \left\langle v^b_{\cancel{\boldsymbol{k}}-\boldsymbol{\ell}}(\eta_1) v^{d*}_{\boldsymbol{q}-\cancel{\boldsymbol{p}}}(\eta_2)\right\rangle \\
&+ \left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^{d*}_{\cancel{\boldsymbol{p}}-\boldsymbol{q}}(\eta_2)\right\rangle \left\langle v^b_{\cancel{\boldsymbol{k}}-\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2)\right\rangle \\
&\cancel{+ \left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^b_{\boldsymbol{k}-\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2) v^{d*}_{\boldsymbol{p}-\boldsymbol{q}}(\eta_2)\right\rangle_c},
\end{aligned}
$$

When $k \ll l$

Calculate $\Omega_{\rm GW}$

$$
\begin{aligned}
&\left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^b_{\cancel{\boldsymbol{k}}-\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2) v^{d*}_{\cancel{\boldsymbol{p}}-\boldsymbol{q}}(\eta_2)\right\rangle \\
&= \left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2)\right\rangle \left\langle v^b_{\cancel{\boldsymbol{k}}-\boldsymbol{\ell}}(\eta_1) v^{d*}_{\boldsymbol{q}-\cancel{\boldsymbol{p}}}(\eta_2)\right\rangle \\
&+ \left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^{d*}_{\cancel{\boldsymbol{p}}-\boldsymbol{q}}(\eta_2)\right\rangle \left\langle v^b_{\cancel{\boldsymbol{k}}-\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2)\right\rangle \\
&\cancel{+ \left\langle v^a_{\boldsymbol{\ell}}(\eta_1) v^b_{\boldsymbol{k}-\boldsymbol{\ell}}(\eta_1) v^{c*}_{\boldsymbol{q}}(\eta_2) v^{d*}_{\boldsymbol{p}-\boldsymbol{q}}(\eta_2)\right\rangle_c},
\end{aligned}
$$

Define as un-equal time power spectrum $\mathscr{P}_{v,w,\phi}(\ell, \eta_1, \eta_2)$

Infrared Scaling

- The GW spectrum in the infrared region when k is smaller than all the scales of the source term ($k \ll l$) is
- $$\Omega_{\text{GW}}(\eta, k) = \frac{k^3}{45 a^4 H^2 M_{\text{Pl}}^4} \int_0^\eta d\eta_1 \int_0^\eta d\eta_2\, a^3(\eta_1) a^3(\eta_2)$$
  $$\cos k(\eta_1 - \eta_2) \int d\ell \left[\left(2\mathscr{P}_v + 3\mathscr{P}_w\right)^2 + 5\mathscr{P}_w^2 + 4\mathscr{P}_\phi^2 \right].$$
- $\mathscr{P}_{v,w,\phi}(\ell, \eta_1, \eta_2)$ are unequal time correlators of v, w, and ϕ.
- $\langle v_{\mathbf{l}}(\eta_1) v_{\mathbf{l}}(\eta_2) \rangle \sim \mathscr{P}_v(\ell, \eta_1, \eta_2)$

Infrared Scaling

- The GW spectrum in the infrared region when k is smaller than all the scales of the source term and the inverse time duration of the source term ($k \ll 1/\Delta\eta$) is

- $$\Omega_{\mathrm{GW}}(\eta, k) = \frac{k^3}{45 a^4 H^2 M_{\mathrm{Pl}}^4} \int_0^{\eta} d\eta_1 \int_0^{\eta} d\eta_2\, a^3(\eta_1) a^3(\eta_2) \cancel{\cos k(\eta_1 - \eta_2)} \int d\ell \left[\left(2\mathscr{P}_v + 3\mathscr{P}_w\right)^2 + 5\mathscr{P}_w^2 + 4\mathscr{P}_\phi^2 \right].$$

- $\mathscr{P}_{v,w,\phi}(\ell, \eta_1, \eta_2)$ are unequal time correlators of v, w, and ϕ.

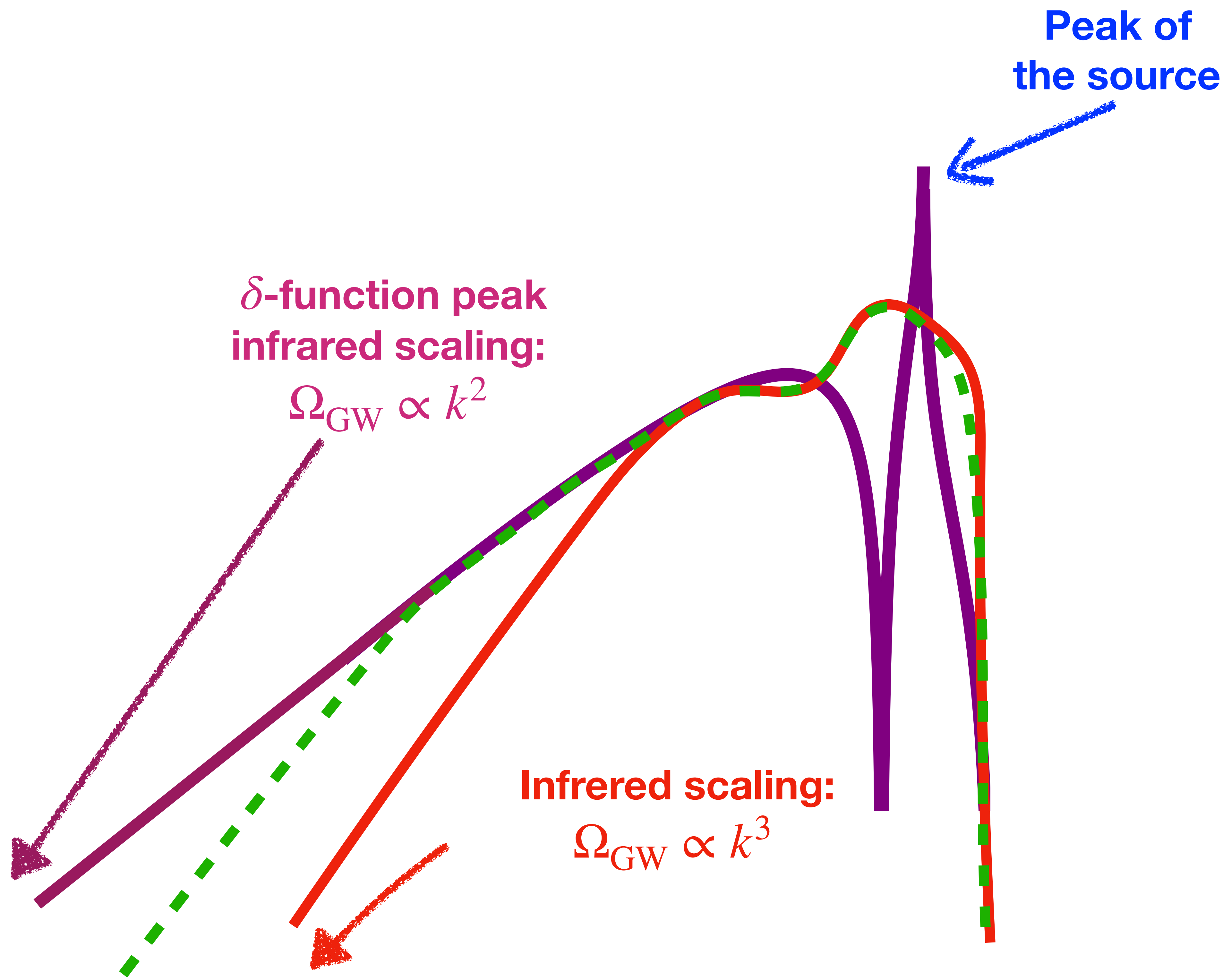
- $\langle v_{\mathbf{l}}(\eta_1) v_{\mathbf{l}}(\eta_2) \rangle \sim \mathscr{P}_v(\ell, \eta_1, \eta_2)$

Example: Induced GWs

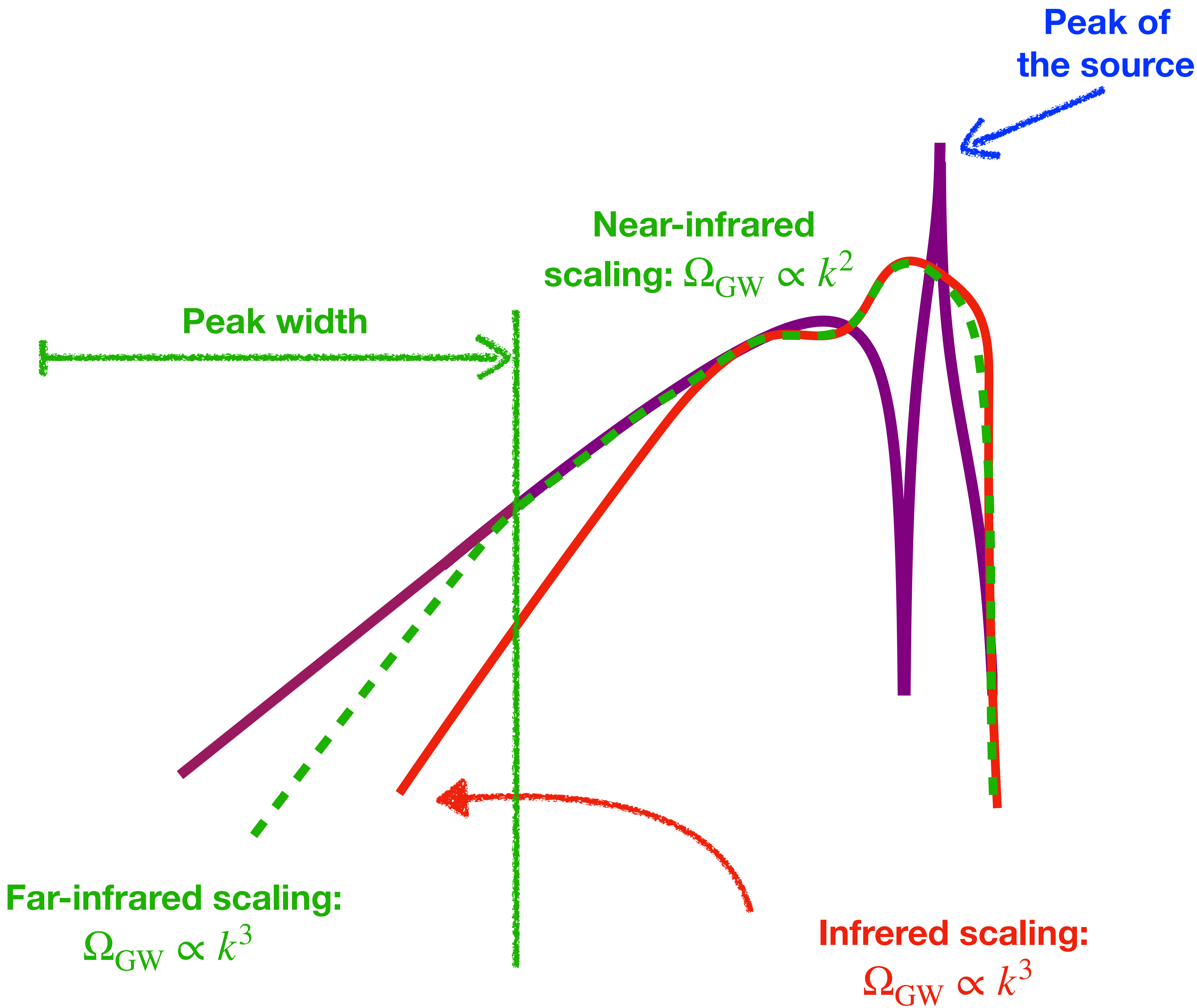
- Broad peak: In the infrared region the integral is redular, so we have $\Omega_{\rm GW} \propto k^3$.

- δ-function peak: The integral is linearly divergent in the infrared region: $\int d\ell(\cdots) \sim 1/k$. Therefore we have $\Omega_{\rm GW} \propto k^2$.

- Narrow peak with width σ: In the far infrared region ($k \ll \sigma \ll k_{\rm peak}$), $\Omega_{\rm GW} \propto k^3$. In the near infrared region ($\sigma \ll k \ll k_{\rm peak}$), $\Omega_{\rm GW} \propto k^2$.



Peak of
the source
δ-function peak
infrared scaling:
$\Omega_{\rm GW} \propto k^2$
Infrered scaling:
$\Omega_{\rm GW} \propto k^3$



Peak of
the source
Near-infrared
scaling: $\Omega_{\rm GW} \propto k^2$
Peak width
Far-infrared scaling:
$\Omega_{\rm GW} \propto k^3$
Infrered scaling:
$\Omega_{\rm GW} \propto k^3$

Conclusion

- If the following 3 conditions are satisfied, we have $\Omega_{\mathrm{GW}} \propto k^3$.

- (1) $0 < \int d\ell \left[\left(2\mathscr{P}_v + 3\mathscr{P}_w\right)^2 + 5\mathscr{P}_w^2 + 4\mathscr{P}_\phi^2 \right] < \infty$.

- (2) k is smaller than all the scales associate with the source term, for instance k_{peak}, Δk, $1/\Delta\eta$, etc.

- (3) Modes of interest reenters the horizon during radiation dominated universe.

Possible Violations

- GWs from BBH/BNS/… Resonance $k = 2\omega$.
- GWs induced by scale-invariant scalar perturbations. Integral is divergent ($\propto k^{-3}$) (Baumann et al 2007)
- GWs induced by scalar perturbations with δ-function peak. Integral is divergent ($\propto k^{-1}$). No infrared region. (Ananda et al 2006)
- Induced GW in a slow transition of matter- to radiation-dominant universe. Resonance. (Inomata et al 2019)
- GWs from sound waves which last for a long period of time. Resonance. (Hindmarsh 2016, Hindmarsh et al 2019)