Focus Week on Primordial Black Holes, December 2019 @IPMU

Statistical bias for black hole mass functions from the inflationary power spectrum

Yi-Peng Wu

with Jun'ichi Yokoyama, in preparation

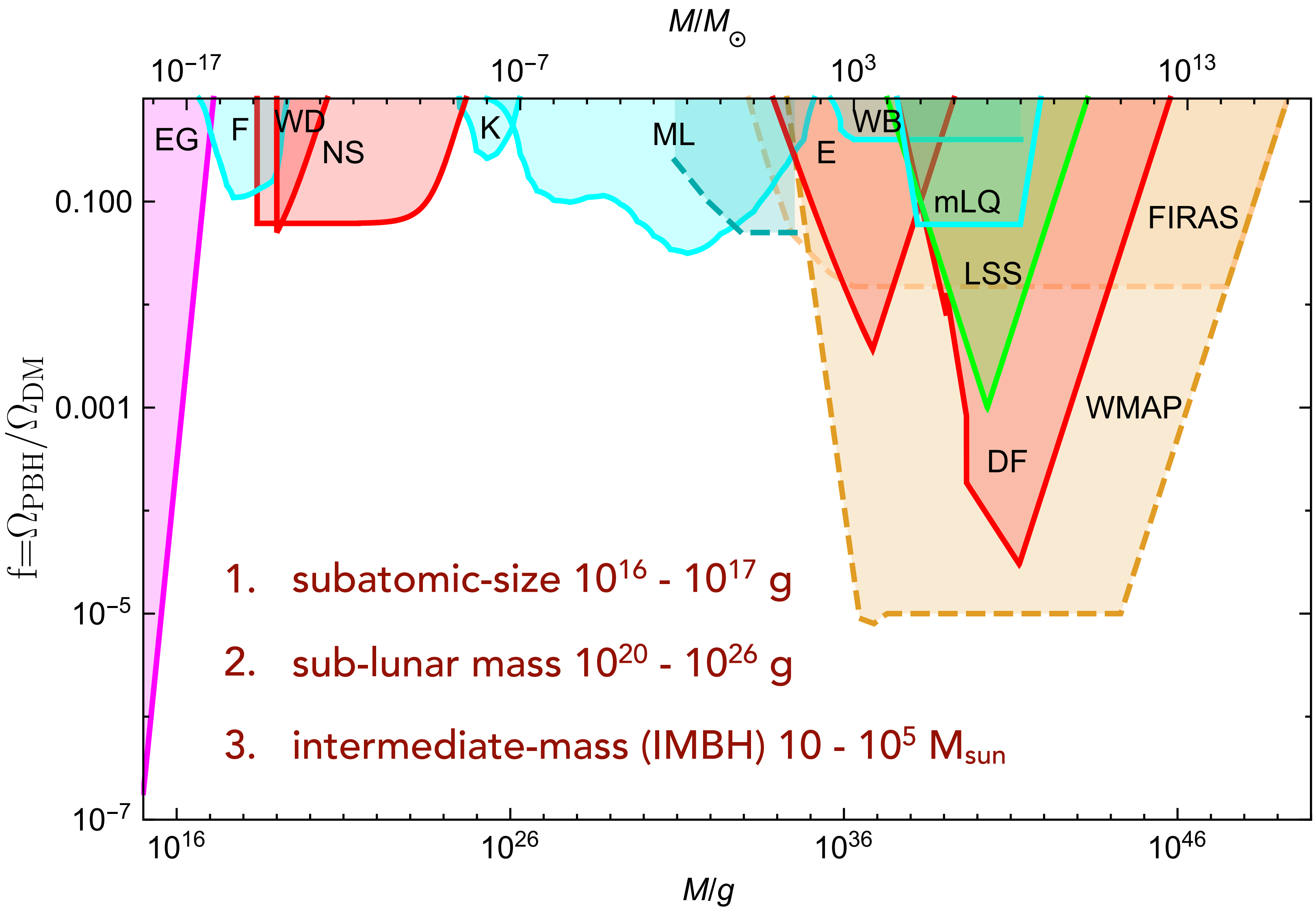
partially based on Young, Byrnes & Sasaki 2014

Research Center for the Early Universe (RESCEU)
The University of Tokyo

Laboratoire de Physique Theorique et Hautes Energies (LPTHE)
Sorbonne University

PBHs as DM in 2016

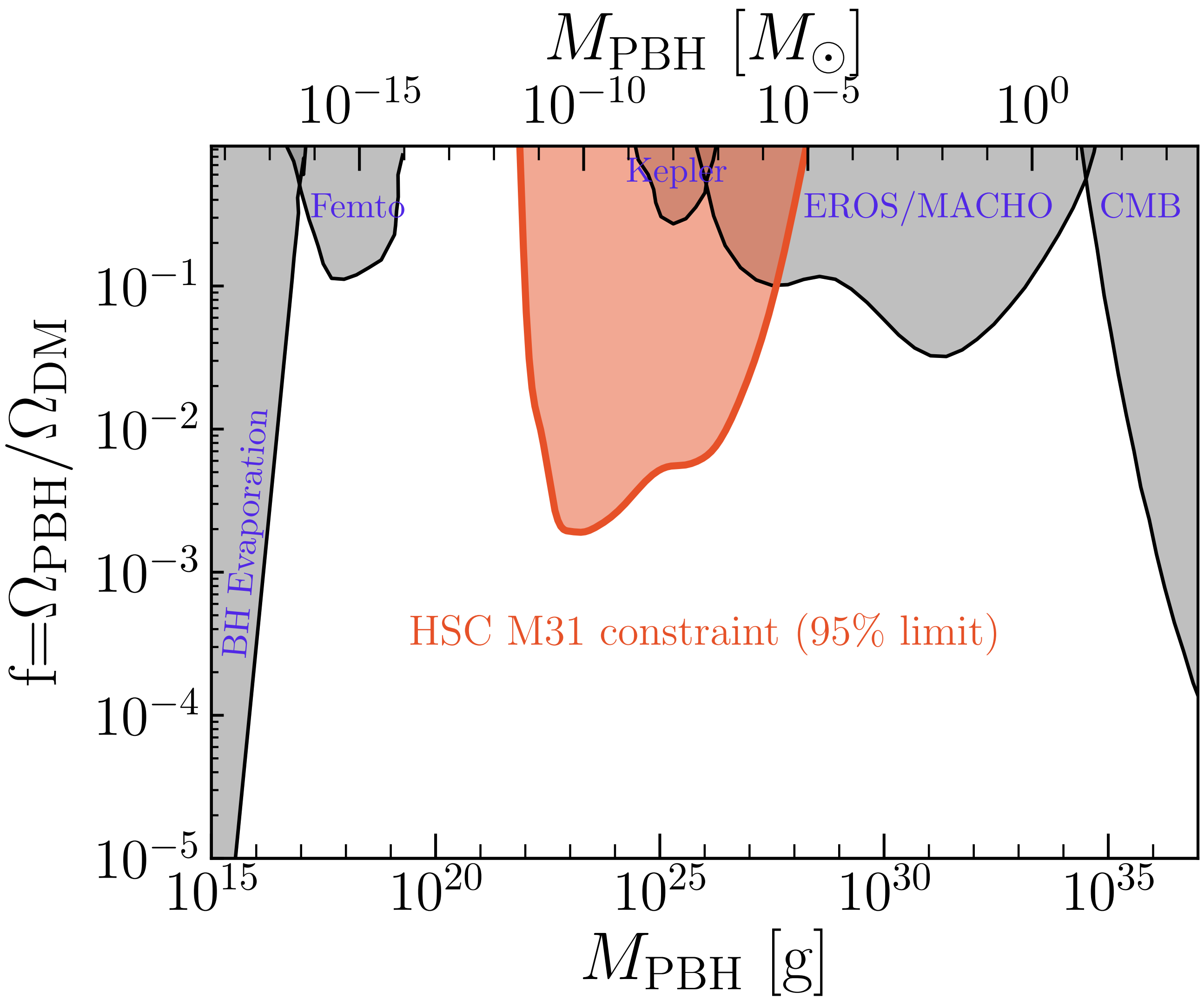
Carr, Kuhnel & Sandstad PRD 2016



1. subatomic-size 10^{16} - 10^{17} g
2. sub-lunar mass 10^{20} - 10^{26} g
3. intermediate-mass (IMBH) 10 - 10^{5} M_{sun}

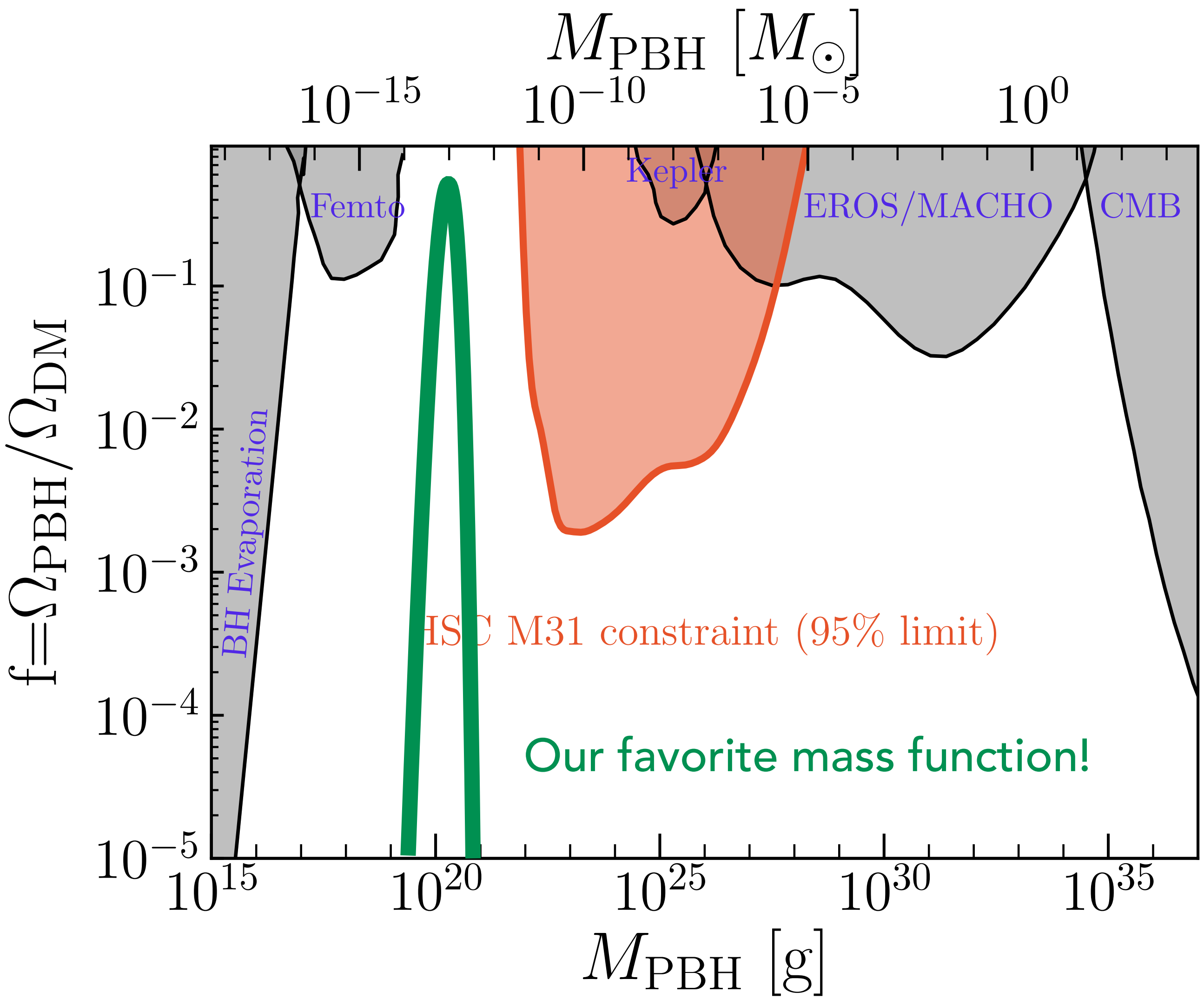
PBHs as DM in 2019

Niikura et al. Nature Astronomy 2019



PBHs as DM in 2019

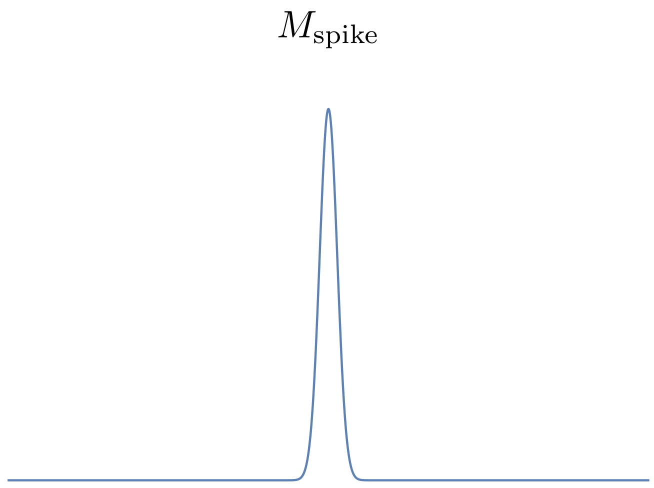
Niikura et al. Nature Astronomy 2019



PBH mass functions

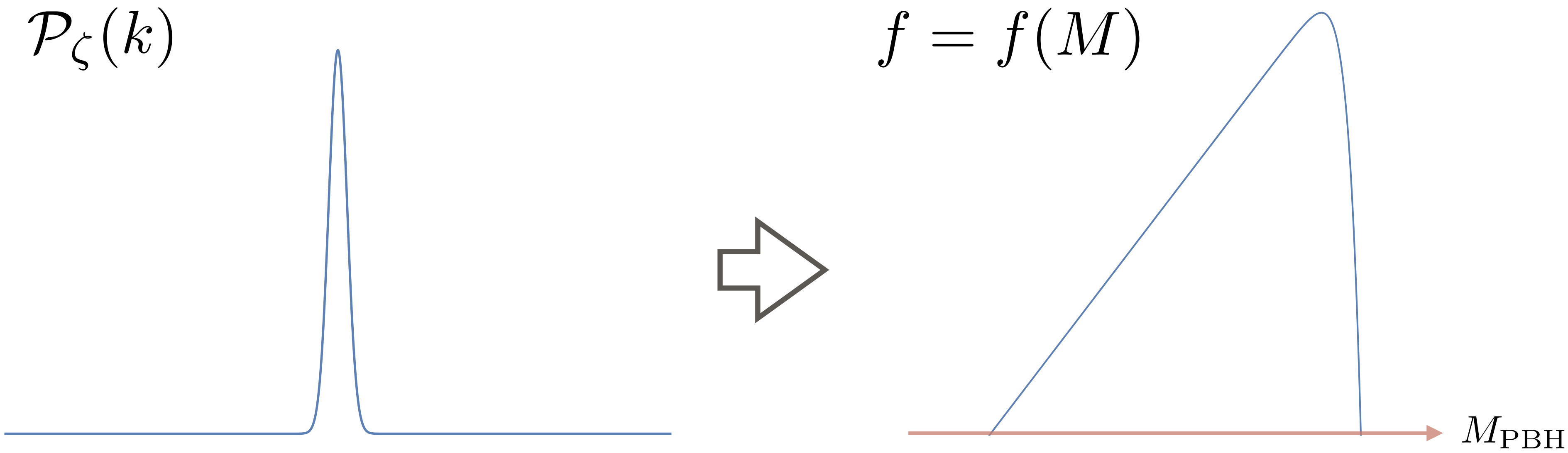
inflationary power spectrum $\mathcal{P}_\zeta(k) \Rightarrow \mathcal{P}_\Delta(k,t) \Rightarrow \sigma_\Delta(M_H(t))$

selecting processes for valid density peaks $\Rightarrow \beta(M, M_H) \Rightarrow f(M)$



$$f \approx \beta(M = M_H = M_{\rm spike})$$

PBH mass functions (in reality)



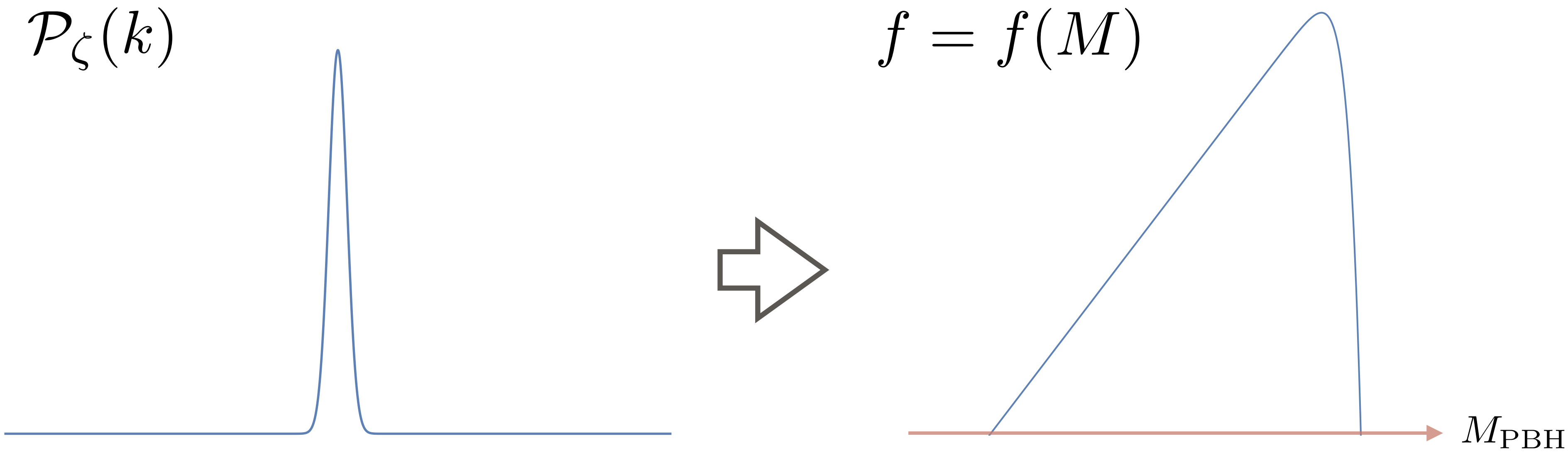
The effect of critical collapse:

$$M = K M_H \left(\Delta - \Delta_c\right)^\gamma$$

Niemeyer & Jedamzik 1998

Yokoyama 1998

PBH mass functions (in reality)



more uncertainties: threshold of Δ, f_{NL}, non-linearity from ζ to Δ, window functions…

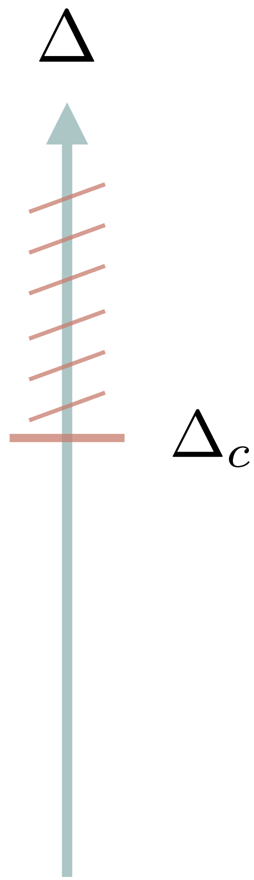
Press-Schechter method

$$P(\Delta) = \frac{1}{\sqrt{2\pi}\sigma_\Delta} \exp\left[-\frac{\Delta^2}{2\sigma_\Delta^2}\right]$$

univariate-Gaussian (one-dimensional)

$$\Omega_{\rm PS}(M_H) = \frac{1}{M_H} \int_{\Delta_c}^{\Delta_{\rm max}} P(\Delta) M(\Delta) d\Delta.$$

$$\beta(M, M_H) \equiv d\Omega(M_H)/d\ln M$$



Peak statistics

Bardeen et al. 1986

$$n_{\rm pk}(\mathbf{r},\nu)d\nu = \sum_p \delta^{(3)}(\mathbf{r}-\mathbf{r}_p)$$

Constraints on spatial configuration: $\delta^{(3)}(\mathbf{r}-\mathbf{r}_p) = \det|\mathbf{z}(\mathbf{r}_p)|\,\delta^{(3)}[\mathbf{u}(\mathbf{r})]$

$u_i = (\mathbf{u})_i = \partial_i \Delta$ extremum: $u_i = 0$

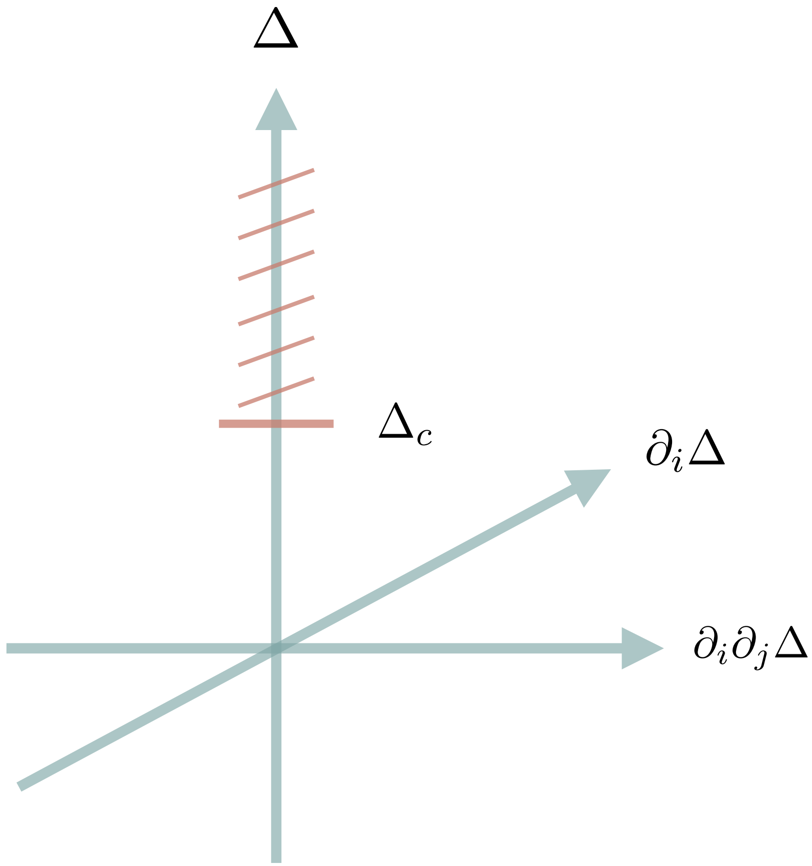
$z_{ij} = (\mathbf{z})_{ij} = \partial_i\partial_j \Delta$ maximum: principal axes of $\mathbf{z} \leq 0$

Peak statistics

Bardeen et al. 1986

$$P(\Delta, u_i, z_{ij}) = \frac{\exp(-B)}{\sqrt{(2\pi)^{10}\det|\mathcal{M}|}}$$

multivariate-Gaussian (10-dimensional)



$$B \equiv \sum \tfrac{1}{2}\Delta y_i (\mathcal{M}^{-1})_{ij} \Delta y_j$$

$$\Delta y_i = y_i - \langle y_i \rangle \text{ (for } i = 1, \cdots, 10)$$

Peak statistics

Bardeen et al. 1986

$$\langle k^2 \rangle = \sigma_1^2 / \sigma_\Delta^2$$

$$\gamma = \sigma_1^2 / (\sigma_2 \sigma_\Delta)$$

$$\mathcal{M}_{ij} = \begin{pmatrix} 1 & & & & \gamma & & & & & \\ & \frac{\langle k^2 \rangle}{3} & & & & & & & & \\ & & \frac{\langle k^2 \rangle}{3} & & & & & & & \\ & & & \frac{\langle k^2 \rangle}{3} & & & & & & \\ \gamma & & & & 1 & & & & & \\ & & & & & \frac{1}{15} & & & & \\ & & & & & & \frac{1}{5} & & & \\ & & & & & & & 0 & & \\ & & & & & & & & 0 & \\ & & & & & & & & & 0 \end{pmatrix}$$

non-vanished cross-correlation!

special moment: $\sigma_i^2(R) = \int_0^\infty k^{2i} W^2(kR) \mathcal{P}_\Delta(k, R) d \ln k.$

Peak statistics

Bardeen et al. 1986

homogeneity & isotropy of density perturbation:

$$P(\Delta, u_i, z_{ij}) \to P(\Delta)$$

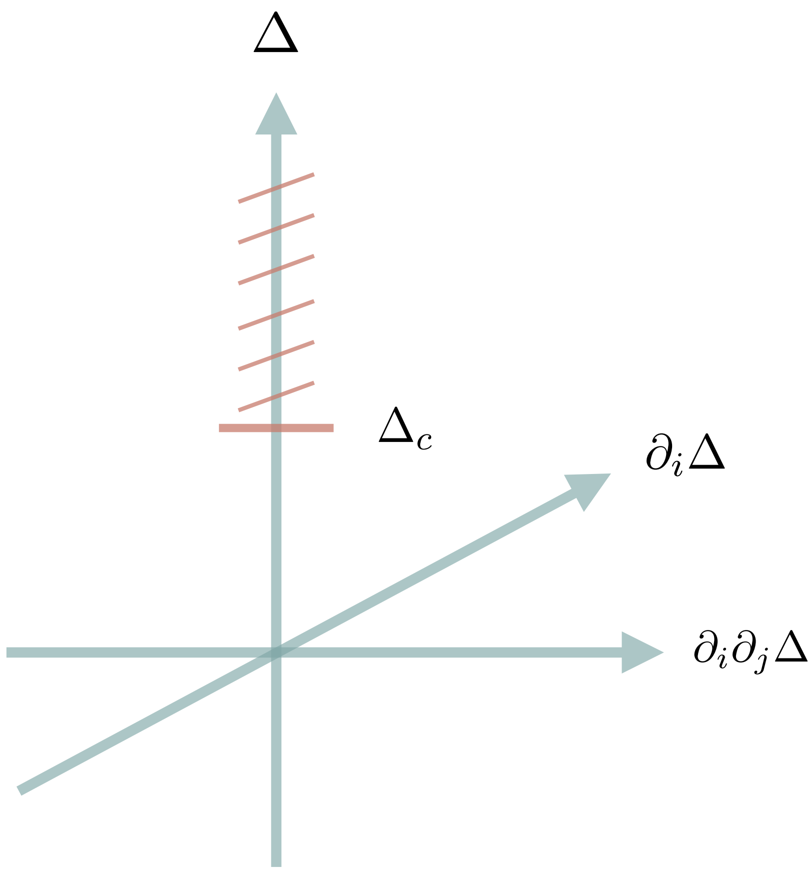
c.f. Germani & Musco 2019

$\nu = \Delta/\sigma_\Delta$

$$\Omega_{\rm pk}(M_H) = \frac{1}{\rho(M_H)} \int_{\nu_c}^{\nu_{\rm max}} \rho_{\rm PBH}(\nu) d\nu,$$
$$= \frac{V(R)}{M_H} \int_{\nu_c}^{\nu_{\rm max}} N_{\rm pk}(\nu) M(\nu) d\nu,$$

one-dimensional effective representation

$$N_{\rm pk}(\nu)d\nu \approx \frac{1}{(2\pi)^2 R_*^3} \left(\nu^3 - 3\nu\right) e^{-\nu^2/2} d\nu,$$



Peak statistics v.s. Press-Schechter

Wu & Yokoyama, in preparation

$$\beta_{\rm pk}(M, M_H) = \frac{K_m}{\sqrt{2\pi}\gamma_m \sigma_\Delta} \left(\frac{\sigma_1^2 R^2}{3\sigma_\Delta^2}\right)^{3/2} \left(\frac{M}{K_m M_H}\right)^{1+\frac{1}{\gamma_m}} \left(\nu^3 - 3\nu\right) e^{-\nu^2/2},$$

$$\beta_{\rm PS}(M, M_H) = \frac{K_m}{\sqrt{2\pi}\gamma_m \sigma_\Delta} \left(\frac{M}{K_m M_H}\right)^{1+1/\gamma_m} \exp\left[-\frac{\Delta^2(M)}{2\sigma_\Delta^2}\right],$$

For total mass distribution after matter-radiation equality:

$$f_{\rm PS}(M) = \frac{1}{\Omega_{\rm DM}} \int_{\ln M_{\rm min}}^{\ln M_{Heq}} \left(\frac{M_{Heq}}{M_H}\right)^{1/2} \beta_{\rm PS} d\ln M_H .$$

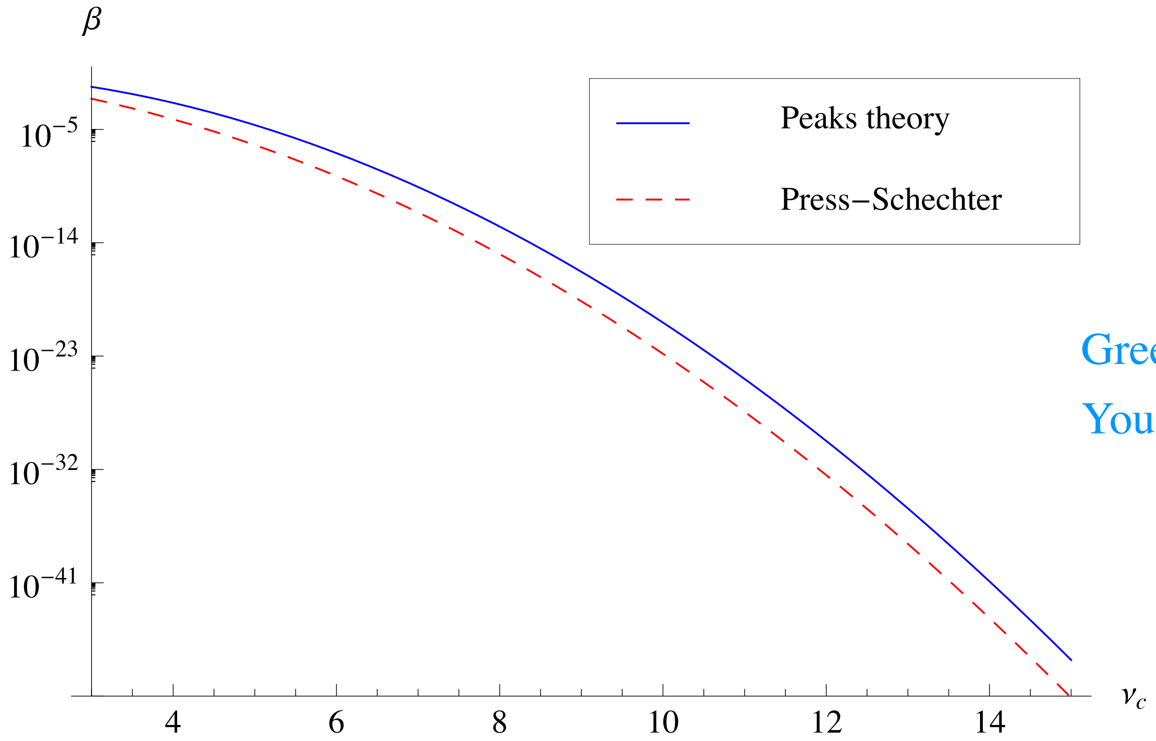
$\mathrm{PS} \leftrightarrow \mathrm{pk}$

Byrnes, Hindmarsh, Young & Hawkings 2018

Wang, Terada & Kohri 2019

power-law templates:

$$\mathcal{P}_\zeta(k) = A_\zeta(k/k_0)^{n-1}$$



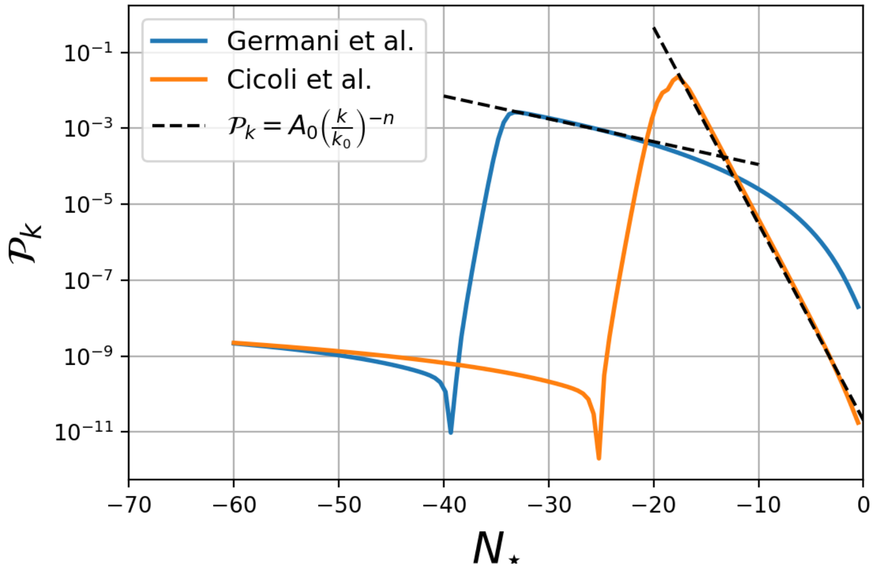
Green et. al. 2004

Young, Byrnes & Sasaki 2014

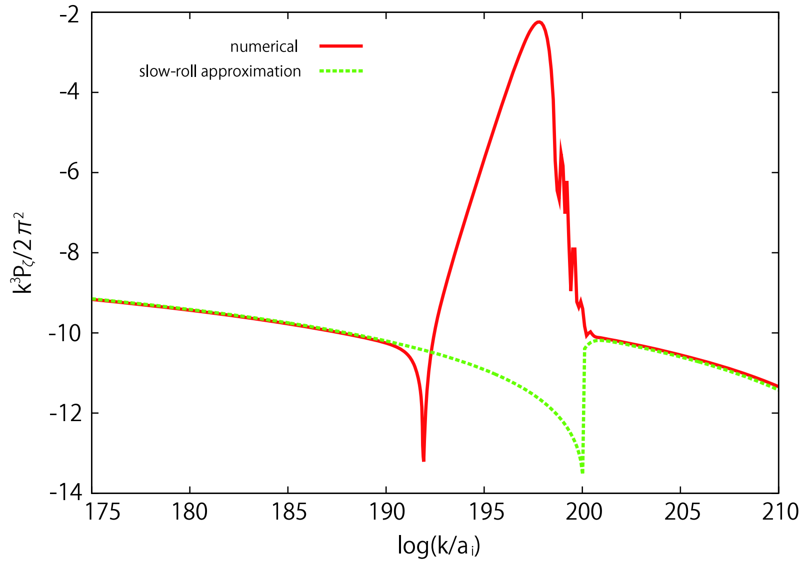
- Only blue-tilted (broad) spectra are considered.
- Monochromatic mass (no critical collapse).
- The discrepancy (a factor of order 10) is within the uncertainty.

broken power-law template:
(for single-field inflation)

$$\mathcal{P}_\zeta(k) = \begin{cases} A_\zeta\,(k/k_0)^{n-1} \;, & k \geq k_0, \\ 0 \;, & k < k_0, \end{cases}$$



Atal & Germani 2019



Saito, Yokoyama & Nagata 2008

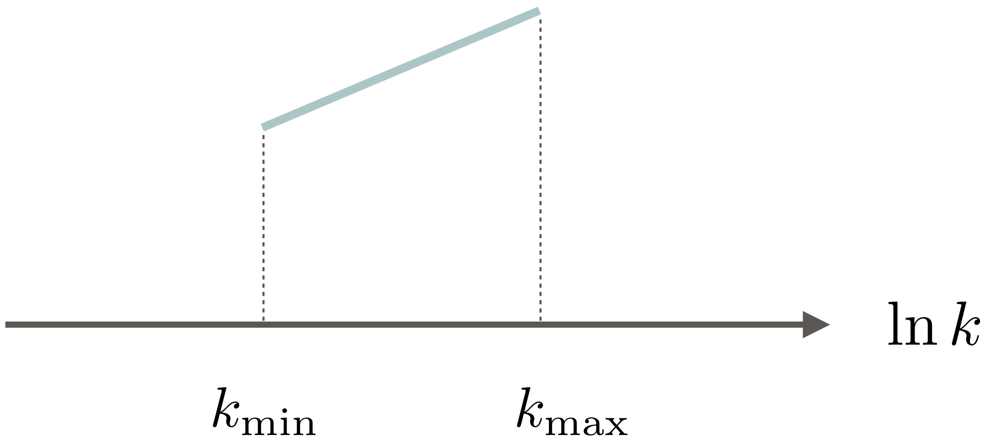
trapezoidal templates:

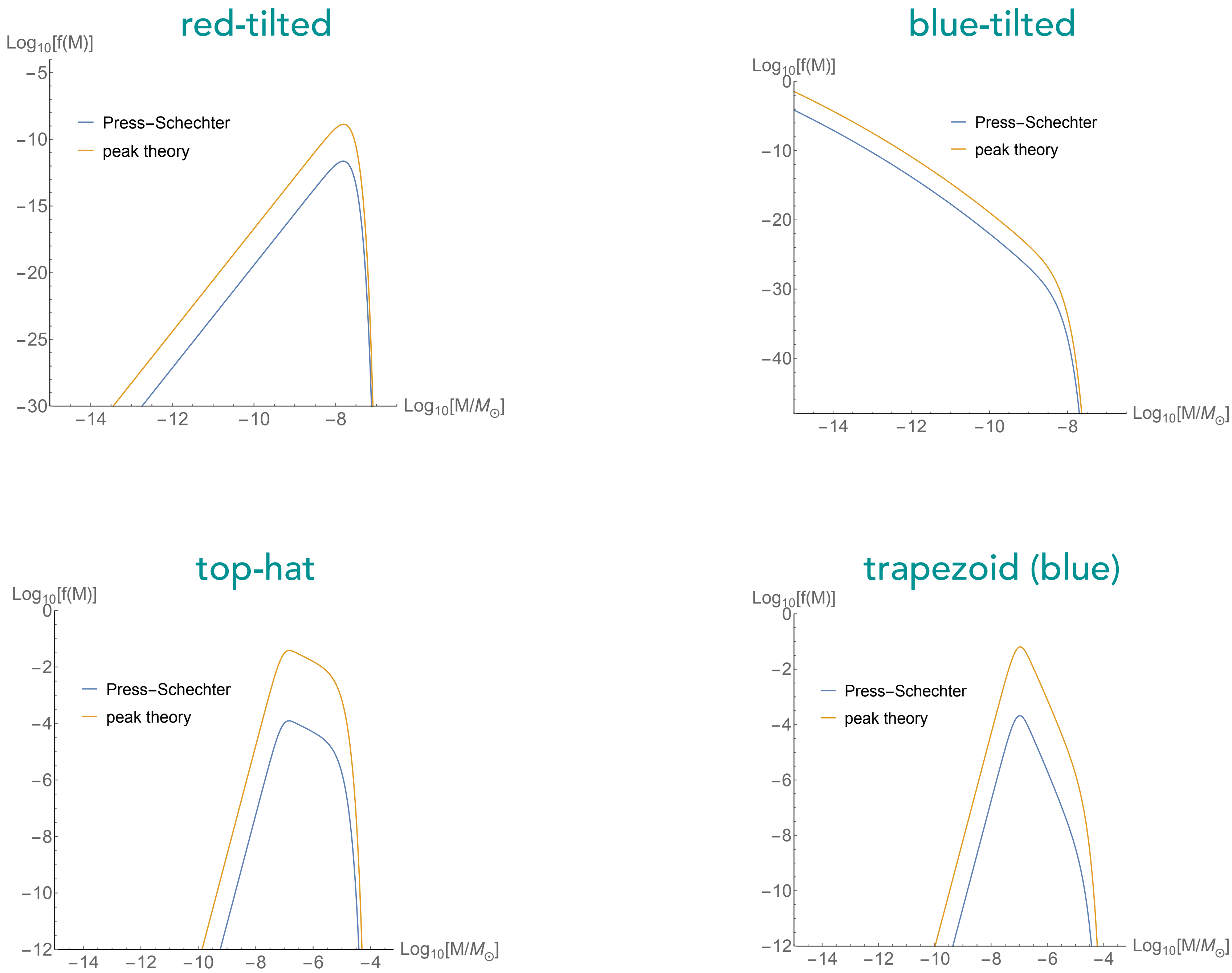
Wu & Yokoyama, in preparation

$$\mathcal{P}_\zeta(k) = \begin{cases} A_\zeta (k/k_{\rm min})^{n-1}, & k_{\rm max} \geq k \geq k_{\rm min}, \\ 0. & \text{otherwise}, \end{cases}$$

(top-hat: n = 1)

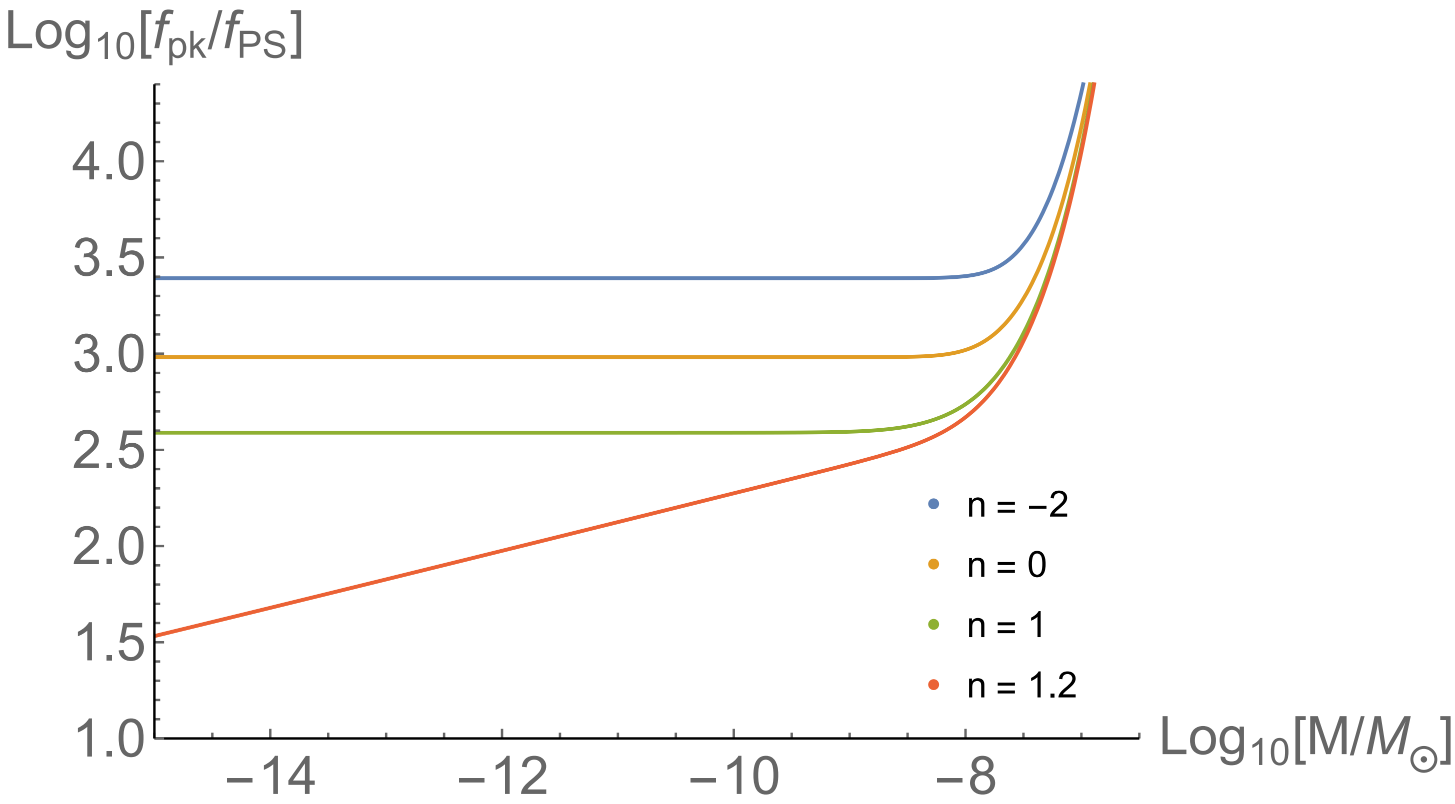
$$\sigma^2_\Delta(R) = \frac{4}{81}\frac{A_\zeta}{2}(k_{\rm min}R)^{1-n} \times \left[\Gamma\left(\frac{3+n}{2}, k^2_{\rm min}R^2\right) - \Gamma\left(\frac{3+n}{2}, k^2_{\rm max}R^2\right)\right],$$




red-tilted
Log10[f(M)]
Press–Schechter
peak theory
Log10[M/M⊙]
blue-tilted
Log10[f(M)]
Press–Schechter
peak theory
Log10[M/M⊙]
top-hat
Log10[f(M)]
Press–Schechter
peak theory
Log10[M/M⊙]
trapezoid (blue)
Log10[f(M)]
Press–Schechter
peak theory
Log10[M/M⊙]

Summary

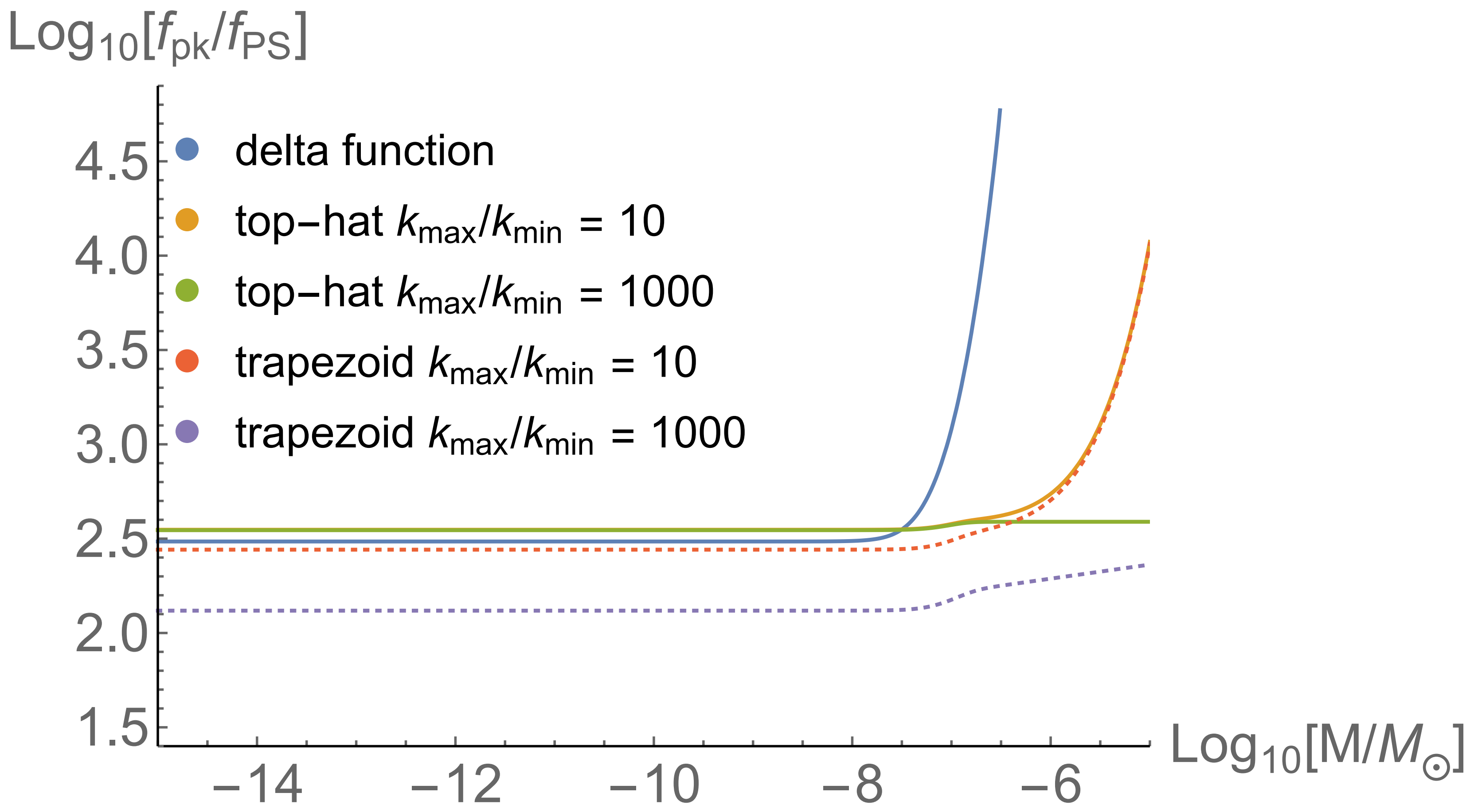
(broken) power-law



$$\frac{f_{\rm pk}}{f_{\rm PS}} \approx \left.\frac{\beta_{\rm pk}}{\beta_{\rm PS}}\right|_{M_H=M_{\rm min}} \simeq \nu_c^3,$$

Summary

blue trapezoid (top-hat)



Messages

- The peak statistics reveals an universal enhancement to the PBH mass function obtained from the Press-Schechter method (as PBHs must form at high-peak values).

- This enhancement is due to cross-correlation of density peaks with their "peak-shape" spatial configurations.

- For inflationary spectrum in the narrow-spike class, the enhancement is at least by a factor of $10^{2.5}$.

Implication to GW cosmology

Induced (second-order) GWs from inflationary spectrum: Saito & Yokoyama 2009

PBHs as all DM and induced GWs: Cai, Pi & Sasaki 2018; Bartolo et. al. 2018

$$\Omega_{\rm GW} \sim A_\zeta^2$$