On the New Uncertainty Relation Derived Geometrically from Aharonov's Weak Value

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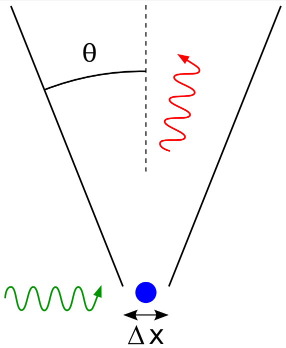
Heisenberg's originial idea (1927)

$$\Delta x \Delta p \gtrsim \hbar$$



Heisenberg's originial idea (1927)

$$\Delta x \Delta p \gtrsim \hbar$$

The Kennard inequality

$$\sigma(x)\sigma(p) \geq \frac{\hbar}{2}$$

The Robertson-Kennard inequality

$$\sigma(\mathsf{A})\sigma(\mathsf{B}) \geq \left| \frac{1}{2} \langle [\mathsf{A}, \mathsf{B}] \rangle \right|$$

The Schrödinger inequality

$$\sigma(\mathsf{A})^2\sigma(\mathsf{B})^2 \geq \left| \frac{1}{2} \langle [\mathsf{A}, \mathsf{B}] \rangle \right|^2 + \left| \frac{1}{2} \langle \{\mathsf{A}, \mathsf{B}\} \rangle - \langle \mathsf{A} \rangle \langle \mathsf{B} \rangle \right|^2$$

A new uncertainty relation

$$\sigma(\mathsf{A})^2\sigma(\mathsf{B})^2 \geq \left| \frac{1}{2} \langle [\mathsf{A}, \mathsf{B}] \rangle \right|^2 + \left| \frac{1}{2} \langle \{\mathsf{A}, \mathsf{B}\} \rangle - \langle \mathsf{A} \rangle \langle \mathsf{B} \rangle \right|^2 + \|\mathsf{A} - \mathsf{A}_{\mathsf{w}}(\mathsf{B})\|^2 \cdot \sigma(\mathsf{B})^2$$

Let B have a discrete spectrum and non-degenerated eigenstates.

Weak value operator

$|b_i\rangle$: an eigenstate of B, $A_w(b_i)$: a weak value (defined later)

$$A_w(B) := \sum_i A_w(b_i)\, |b_i\rangle\langle b_i|\,.$$

cf. the spectral decomposition of an operator function f(A).
(A: a self-adjoint operator, $|a_i\rangle$: an eigenstate of A, f_a: a real function)

$$f(A) := \sum_i f_a(a_i)\, |a_i\rangle\langle a_i|\,.$$

$A_w(B)$ is not necessarily self-adjoint. ($\because$ $A_w(b_i) \in \mathbb{C}$).

$$\mathrm{Re}\, A_w(B) := \frac{A_w(B) + A_w^\dagger(B)}{2}, \quad \mathrm{Im}\, A_w(B) := \frac{A_w(B) - A_w^\dagger(B)}{2i}.$$

$$A_w(B) = \mathrm{Re}\, A_w(B) + i\, \mathrm{Im}\, A_w(B),$$

Weak value

We define the weak value of A in the state $|\psi\rangle$ as

$$A_w(b_i) := \begin{cases} \dfrac{\langle b_i|A|\psi\rangle}{\langle b_i|\psi\rangle} & (\langle b_i|\psi\rangle \neq 0) \\ c_i & (\langle b_i|\psi\rangle = 0) \end{cases}.$$

Here, $|b_i\rangle$: an eigenstate of B, c_i: an arbitrary complex number.

The weak value was introduced in (Aharanov et al., 1988).

- two-state vector formalism
- the weak measurement

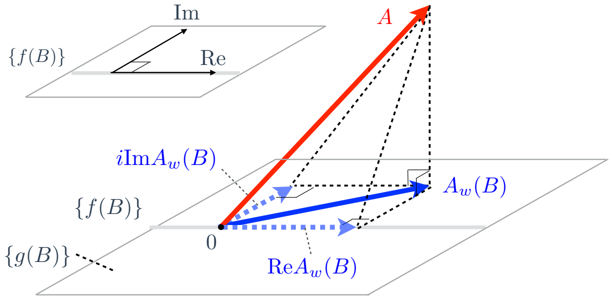
We can handle operators geometrically in the inner-product space, by defining the inner product as

$$\langle\!\langle X, Y \rangle\!\rangle := \langle X\psi | Y\psi \rangle = \langle X^\dagger Y \rangle \, .$$

$\{g(B)\}$: the space of the operators generated from B
$\{f(B)\}$: the space of the self-adjoint operators generated from B

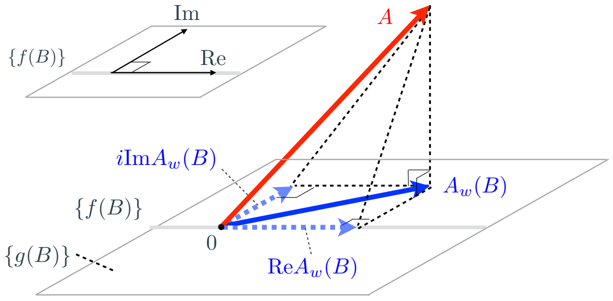
$$\langle\!\langle A, g(B) \rangle\!\rangle = \langle\!\langle A_w(B), g(B) \rangle\!\rangle .$$

$\longrightarrow$ The projection of A onto $\{g(B)\}$ corresponds to $A_w(B)$.



In the inner-product space of operators, we can use

- the Pythagorean identity,
- the Cauchy-Schwartz inequality.
- $\langle\!\langle A, g(B) \rangle\!\rangle = \langle\!\langle A_w(B), g(B) \rangle\!\rangle$.



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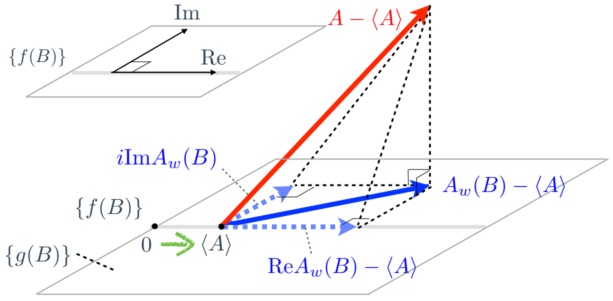
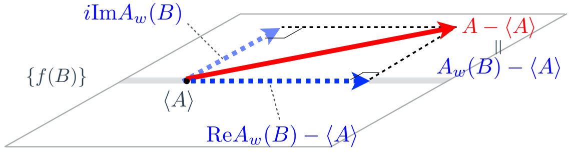


We can translate the origin by $\langle A \rangle \cdot \mathrm{Id} \in \{f(B)\}$ (writen as $\langle A \rangle$ in fig.).
($\because$ linearity of inner product)

From the Pythagorean identity, e.g.,

$$\|A - \langle A \rangle\|^2 = \|A - A_w(B)\|^2 + \|A_w(B) - \langle A \rangle\|^2.$$

When $\|A - A_w(B)\| = 0$,



$$\sigma(A)^2\sigma(B)^2$$

$$= \|A - \langle A \rangle\|^2 \cdot \sigma(B)^2$$

$$\stackrel{P}{=} (\underbrace{\|A - A_w(B)\|^2}_{=0} + \|A_w(B) - \langle A \rangle\|^2) \cdot \sigma(B)^2$$

$$\stackrel{P}{=} (\|\mathrm{Re}\, A_w(B) - \langle A \rangle\|^2 + \|\mathrm{Im}\, A_w(B)\|^2) \cdot \sigma(B)^2$$

$$= \boxed{\|\mathrm{Re}\, A_w(B) - \langle A \rangle\|^2 \cdot \sigma(B)^2} + \boxed{\|\mathrm{Im}\, A_w(B)\|^2 \cdot \sigma(B)^2}$$

$$\text{VI CS} \qquad\qquad \text{VI CS}$$

$$\geq \left|\frac{1}{2}\langle\{A, B\}\rangle - \langle A \rangle \langle B \rangle\right|^2 + \left|\frac{1}{2}\langle[A, B]\rangle\right|^2$$

Decomposition of the Schrödinger inequality

$$\|\mathrm{Im}\, A_w(B)\| \cdot \|B - \langle B \rangle\| \geq \left| \frac{1}{2} \langle [A, B] \rangle \right|,$$

$$\|\mathrm{Re}\, A_w(B) - \langle A \rangle\| \cdot \|B - \langle B \rangle\| \geq \left| \frac{1}{2} \langle \{A, B\} \rangle - \langle A \rangle \langle B \rangle \right|$$

Each equal in the inequalities holds if and only if

$$\exists \lambda \in \mathbb{R} \quad \text{s.t.} \quad \mathrm{Im}\, A_w(B) |\psi\rangle = \lambda (B - \langle B \rangle) |\psi\rangle ,$$

$$\exists \mu \in \mathbb{R} \quad \text{s.t.} \quad (\mathrm{Re}\, A_w(B) - \langle A \rangle) |\psi\rangle = \mu (B - \langle B \rangle) |\psi\rangle ,$$

respectively.
(Excluding the trivial case, $(B - \langle B \rangle) |\psi\rangle = 0$, where the left- and right-hand sides reduces to 0.)

The Schrödinger inequality

$$\sigma(A)^2\sigma(B)^2 \geq \left|\frac{1}{2}\langle[A,B]\rangle\right|^2 + \left|\frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle\right|^2$$

The equal holds if and only if

$$\exists \mu, \lambda \in \mathbb{R} \quad \text{s.t.} \quad (A - \langle A\rangle)|\psi\rangle = (\mu + i\lambda)(B - \langle B\rangle)|\psi\rangle,$$

or

$$(B - \langle B\rangle)|\psi\rangle = 0 \quad \text{(trivial)}$$

- For position and momentum ($A = \hat{p}, B = \hat{x}$), when the equal holds, the wave function $\psi(x)$ is,

$$\psi(x) = C\exp\left[i\left(\frac{\mu}{2\hbar}(x - \langle x\rangle)^2 + \frac{\langle p\rangle}{\hbar}x\right)\right]\exp\left[-\frac{\lambda}{2\hbar}(x - \langle x\rangle)^2\right].$$

(the minimum-uncertainty state)

For position and momentum ($A = \hat{p}, B = \hat{x}$),

- the minimum-uncertainty state of the Schrödinger inequality is

$$\psi(x) = C \exp\left[i\left(\frac{\mu}{2\hbar}(x - \langle x\rangle)^2 + \frac{\langle p\rangle}{\hbar}x\right)\right] \exp\left[-\frac{\lambda}{2\hbar}(x - \langle x\rangle)^2\right].$$

- each equal in the decomposed inequalities holds if and only if

$$\exists\lambda \in \mathbb{R} \quad \text{s.t.} \quad \operatorname{Im} A_w(B)\,|\psi\rangle = \lambda(B - \langle B\rangle)\,|\psi\rangle\,,$$

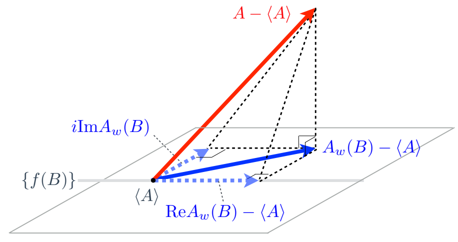
$$\exists\mu \in \mathbb{R} \quad \text{s.t.} \quad (\operatorname{Re} A_w(B) - \langle A\rangle)\,|\psi\rangle = \mu(B - \langle B\rangle)\,|\psi\rangle\,,$$

respectively. We can obtain $\psi(x)$ from each equation.

$$\psi(x) = C_0 \exp\left[i a(x)\right] \exp\left[-\frac{\lambda}{2\hbar}(x - \langle x\rangle)^2\right], \qquad (a(x) \in \mathbb{R})$$

$$\psi(x) = C_1 \exp\left[i\left(\frac{\mu}{2\hbar}(x - \langle x\rangle)^2 + \frac{\langle p\rangle}{\hbar}x\right)\right] \exp[b(x)]. \quad (b(x) \in \mathbb{R})$$

$\longrightarrow$ The condition of the minimum-uncertainty state is decomposed into distribution of $\psi(x)$ and phase of it.



$$\sigma(A)^2\sigma(B)^2$$

$$= \|A - \langle A \rangle\|^2 \cdot \sigma(B)^2$$

$$\stackrel{P}{=} (\|A - A_w(B)\|^2 + \|A_w(B) - \langle A \rangle\|^2) \cdot \sigma(B)^2$$

$$\stackrel{P}{=} (\|A - A_w(B)\|^2 + \|\mathrm{Re}\, A_w(B) - \langle A \rangle\|^2 + \|\mathrm{Im}\, A_w(B)\|^2) \cdot \sigma(B)^2$$

$$= \|A - A_w(B)\|^2 \cdot \sigma(B)^2 + \boxed{\|\mathrm{Re}\, A_w(B) - \langle A \rangle\|^2 \cdot \sigma(B)^2} + \boxed{\|\mathrm{Im}\, A_w(B)\|^2 \cdot \sigma(B)^2}$$

$$\text{VI CS} \qquad \text{VI CS}$$

$$\geq \underbrace{\|A - A_w(B)\|^2 \cdot \sigma(B)^2}_{\text{new term}} + \boxed{\left|\frac{1}{2}\langle\{A, B\}\rangle - \langle A \rangle \langle B \rangle\right|^2} + \boxed{\left|\frac{1}{2}\langle[A, B]\rangle\right|^2}$$

SUMMARY AND CONCLUSION

1. Geometrical interpretaiton:
   the inner-product space of operators
2. Decomposition of the Schrödinger inequality

$$\|\mathrm{Im}\, A_w(B)\| \cdot \|B - \langle B \rangle\| \geq \left|\tfrac{1}{2}\langle [A, B] \rangle\right|,$$
$$\|\mathrm{Re}\, A_w(B) - \langle A \rangle\| \cdot \|B - \langle B \rangle\| \geq \left|\tfrac{1}{2}\langle \{A, B\} \rangle - \langle A \rangle \langle B \rangle\right|.$$

3. The new uncertainty relation

the Schrödinger ineq.

the RK ineq.

$$\sigma(A)^2 \sigma(B)^2 \geq \left|\frac{1}{2}\langle [A, B] \rangle\right|^2 + \left|\frac{1}{2}\langle \{A, B\} \rangle - \langle A \rangle \langle B \rangle\right|^2$$
$$+ \|A - A_w(B)\|^2 \cdot \sigma(B)^2$$

Future Work

- Refining the new inequality
- Physical meaning / properties of the new term
- Relation to the other uncertainty relations
  (e.g., error-disturbance, time-energy, entropy)

APPENDIX

$|b_i\rangle$: an eigenvector of B,
$\Pi(b_i)$: an eigenprojection onto the eigenspace with eigenvalue b_i,
c_i: an arbitrary complex number.

Dfinition of the weak value

When B is degenerated (or non-degenerated),

$$A_w(b_i) := \begin{cases} \dfrac{\langle\psi|\Pi(b_i)A|\psi\rangle}{\|\Pi(b_i)\|^2} & (\langle\psi|\Pi(b_i)|\psi\rangle \neq 0), \\ c_i & (\langle\psi|\Pi(b_i)|\psi\rangle = 0). \end{cases}$$

- Even when B has no eigenvectors, the weak value can be defined by the Radon – Nikodým derivation.

The axioms of inner product

1. Linearity

$$\langle\langle X, c_1Y_1 + c_2Y_2\rangle\rangle = c_1\langle\langle X, Y_1\rangle\rangle + c_2\langle\langle X, Y_2\rangle\rangle,$$

2. Conjugate symmetry

$$\langle\langle X, Y\rangle\rangle = \langle\langle Y, X\rangle\rangle^*,$$

3. Positivity

$$\langle\langle X, X\rangle\rangle = \|X\|^2 \geq 0,$$

4. Positive-definiteness

$$\sqrt{\langle\langle X, X\rangle\rangle} = \|X\| = 0 \Rightarrow X = 0.$$

For all $X, Y \in \mathcal{O}$, the product

$$\langle\langle X, Y\rangle\rangle := \langle X\psi|Y\psi\rangle = \langle X^\dagger Y\rangle\,.$$

satisfies all above except for 4. Positive-definiteness.
$\|X\| = 0 \not\Rightarrow X = 0$ (What holds is $\|X\| = 0 \Rightarrow X|\psi\rangle = 0$.)

The inner-product space of operators

$\mathcal{O}$: the space of operators.
Define the equivalence relation between $X, Y \in \mathcal{O}$ as

$$X \sim Y \Leftrightarrow \|X - Y\| = 0,$$

and re-define the operator space as

$$\tilde{\mathcal{O}} := \mathcal{O}/\sim .$$

For all $X', Y' \in \tilde{\mathcal{O}}$,

$$\langle\langle X', Y' \rangle\rangle := \langle X'\psi | Y'\psi \rangle = \langle X'^{\dagger} Y' \rangle .$$

satisfies the axioms of inner product.

We use this re-defined inner-product space of operators,

$$\{\tilde{\mathcal{O}}, \langle\langle \cdot\,, \cdot \rangle\rangle\}.$$

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