Introduction to the large charge expansion

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INFN | Torino

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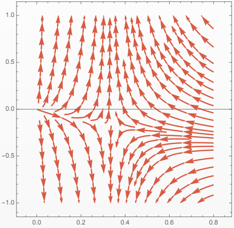
arXiv:1505.01537, arXiv:1610.04495, arXiv:1707.00711, arXiv:1804.01535, arXiv:1902.09542, arXiv:1905.00026,arXiv:1909.02571, arXiv:1909.08642
and more to come…

Who’s who

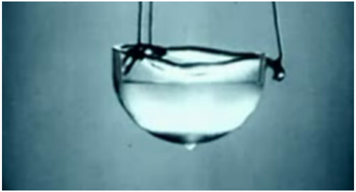
S. Reffert (AEC Bern);
L. Alvarez Gaumé (CERN and SCGP);
F. Sannino (CP3-Origins);
D. Banerjee (DESY);
S. Chandrasekharan (Duke);
S. Hellerman (IPMU);
M. Watanabe (Weizmann).

Why are we here? Conformal field theories

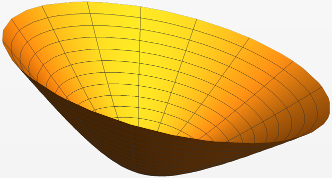
extrema of the RG flow



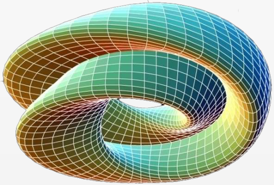
critical phenomena



quantum gravity



string theory



Why are we here? Conformal field theories are hard

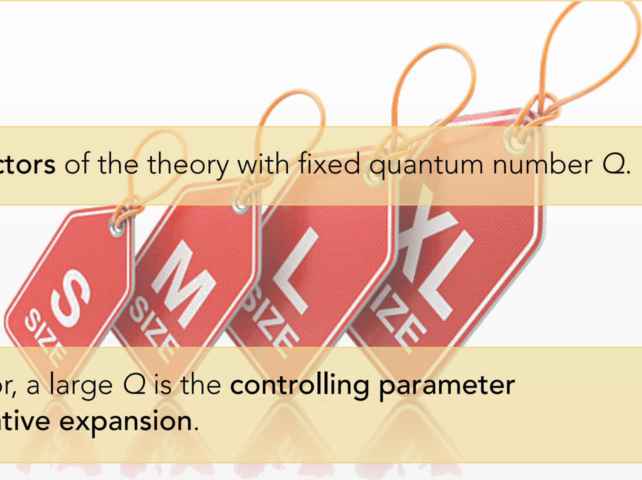
Most conformal field theories (CFTs) lack nice limits where they become simple and solvable.

No parameter of the theory can be dialed to a simplifying limit.

Why are we here? Conformal field theories are hard

In presence of a symmetry there can be sectors of the theory where anomalous dimension and OPE coefficients simplify.

The idea



Study **subsectors** of the theory with fixed quantum number Q.

In each sector, a large Q is the **controlling parameter** in a **perturbative expansion**.

no bootstrap here!

This approach is orthogonal to bootstrap.
We will use an effective action.
We will access sectors that are difficult to reach with bootstrap.
(However, arXiv:1710.11161).

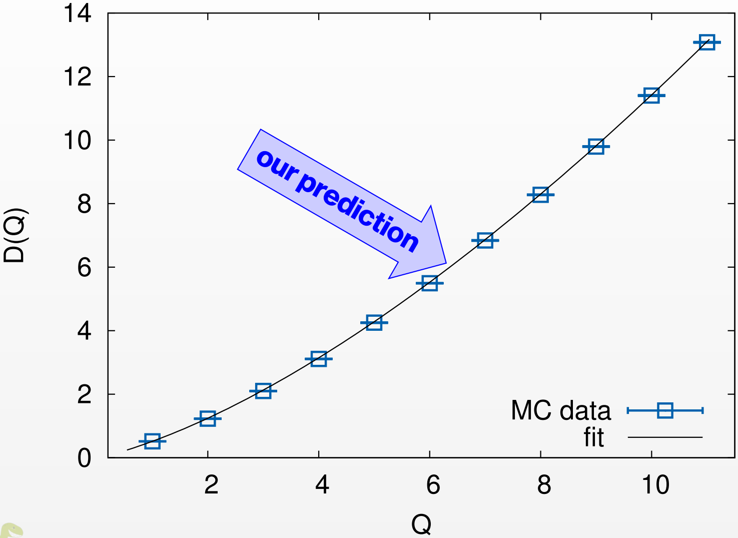
Concrete results

We consider the $O(N)$ vector model in three dimensions. In the IR it flows to a conformal fixed point Wilson & Fisher.

We find an explicit formula for the dimension of the lowest primary at fixed charge:

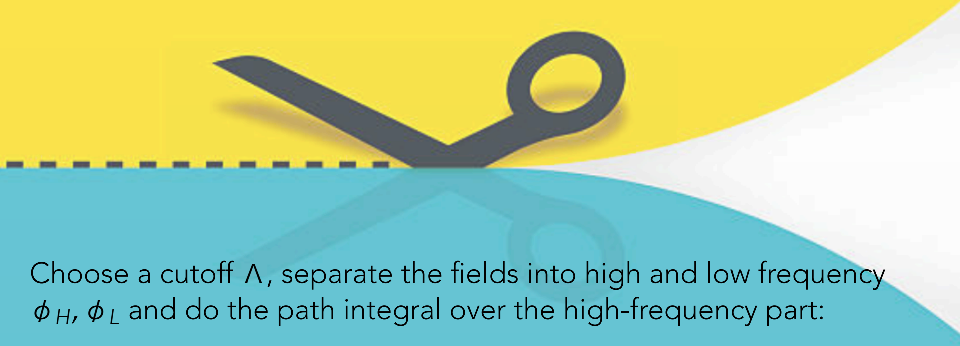
$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi}\, c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}\left(Q^{-1/2}\right)$$

Summary of the results: O(2)



Scales

We want to write a Wilsonian effective action.



Choose a cutoff Λ, separate the fields into high and low frequency ϕ_H, ϕ_L and do the path integral over the high-frequency part:

$$e^{iS_\Lambda(\phi_L)} = \int \mathscr{D}\phi_H \, e^{iS(\phi_H, \phi_L)}$$

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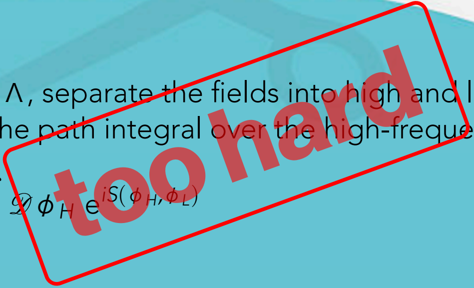
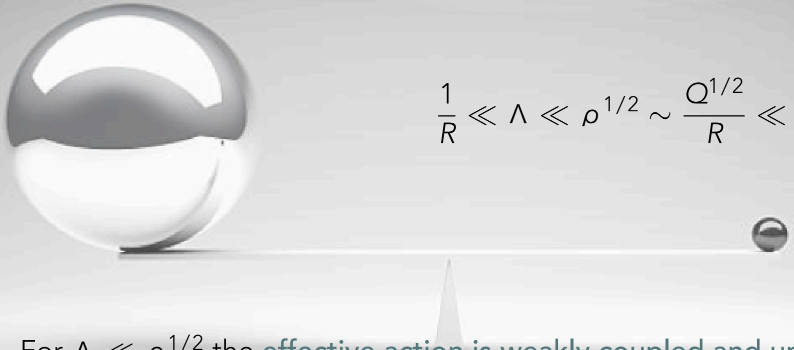


Scales

- We look at a finite box of typical length R
- The $U(1)$ charge Q fixes a second scale $\rho^{1/2} \sim Q^{1/2}/R$



$$\frac{1}{R} \ll \Lambda \ll \rho^{1/2} \sim \frac{Q^{1/2}}{R} \ll \Lambda_{UV}$$

For $\Lambda \ll \rho^{1/2}$ the effective action is weakly coupled and under perturbative control in powers of ρ^{-1}.

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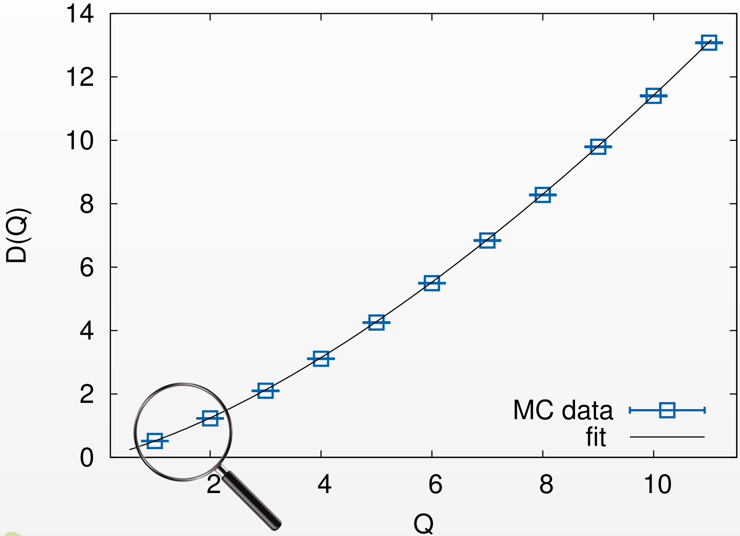
superstition

Too good to be true?

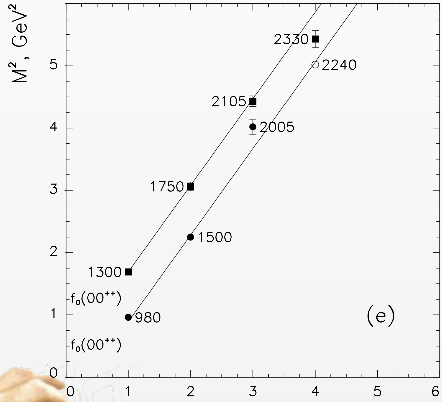


Too good to be true?

Think of Regge trajectories.
The prediction of the theory is

$$m^2 \propto J\left(1 + \mathcal{O}\left(J^{-1}\right)\right)$$

but *experimentally* everything works so well at small J that String Theory was invented.



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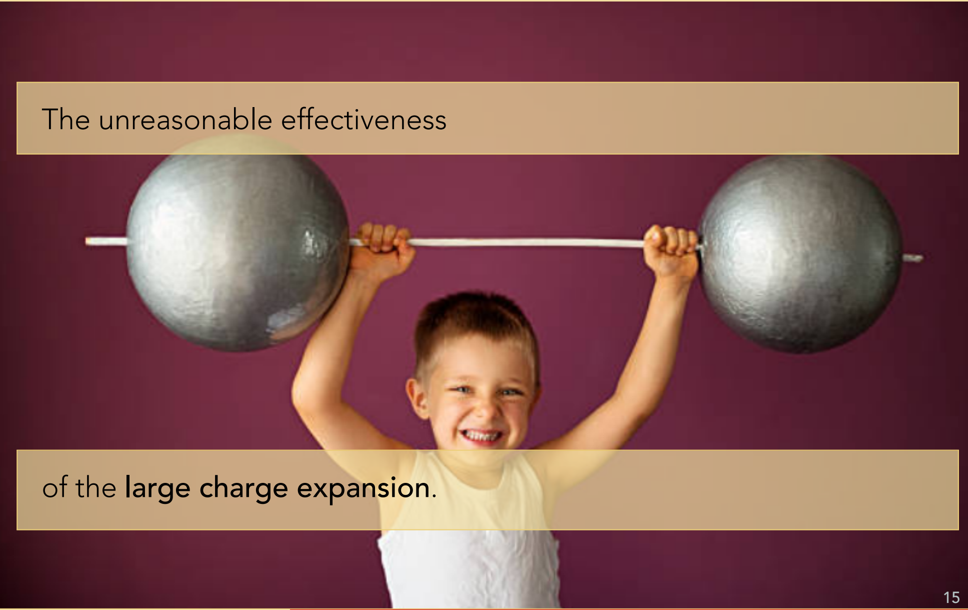

The unreasonable effectiveness

of the **large charge expansion**.

Today's talk

The effective field theory (EFT) for the $O(2)$ model in $d = 3$

- An EFT for a CFT.
- The physics at the saddle.
- State/operator correspondence for anomalous dimensions.

PARENTAL ADVISORY

EXPLICIT CONTENT

An EFT for a CFT

The $O(2)$ model

The simplest example is the Wilson–Fisher (WF) point of the $O(2)$ model in three dimensions.

- Non-trivial fixed point of the ϕ^4 action

$$L_{UV} = \partial_\mu \phi^* \partial_\mu \phi - u(\phi^* \phi)^2$$

- Strongly coupled
- In nature: ^4He.
- Simplest example of spontaneous symmetry breaking.
- Not accessible in perturbation theory. Not accessible in $4 - \varepsilon$. Not accessible in large N.
- Lattice. Bootstrap.

Charge fixing

We assume that the $O(2)$ symmetry is not accidental.

We consider a **subsector of fixed charge** Q.
Generically, the classical solution at fixed charge **breaks spontaneously** $U(1) \to \emptyset$.

We have one **Goldstone boson** χ.

An action for χ

Start with two derivatives:

$$L[\chi] = \frac{f_\pi}{2} \partial_\mu \chi \, \partial_\mu \chi - C^3$$

(χ is a Goldstone so it is dimensionless.)

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We want to describe a CFT: we can dress with a dilaton

$$L[\sigma, \chi] = \frac{f_\pi e^{-2f\sigma}}{2} \partial_\mu \chi \, \partial_\mu \chi - e^{-6f\sigma} C^3 + \frac{e^{-2f\sigma}}{2} \left(\partial_\mu \sigma \, \partial_\mu \sigma - \frac{\xi R}{f^2} \right)$$

The fluctuations of χ give the Goldstone for the broken $U(1)$, the fluctuations of σ give the (massive) Goldstone for the broken conformal invariance.

Linear sigma model

We can put together the two fields as

$$\Sigma = \sigma + i f_\pi \chi$$

and rewrite the action in terms of a complex scalar

$$\varphi = \frac{1}{\sqrt{2f}} e^{-f\Sigma}$$

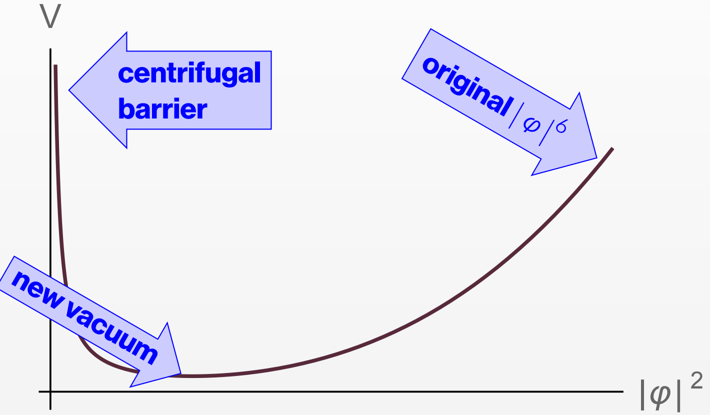
We get

$$L[\varphi] = \partial_\mu \varphi^* \partial^\mu \varphi - \xi R \varphi^* \varphi - u(\varphi^* \varphi)^3$$

Only depends on dimensionless quantities $b = f^2 f_\pi$ and $u = 3(Cf^2)^3$.
Scale invariance is manifest.
The field φ is some complicated function of the original ϕ.

Centrifugal barrier

The $O(2)$ symmetry acts as a shift on χ.
Fixing the charge is the same as adding a centrifugal term $\propto \frac{1}{|\varphi|^2}$.



Ground state

We can find a fixed-charge solution of the type

$$\chi(t,x) = \mu t \qquad \sigma(t,x) = \frac{1}{f}\log(v) = \text{const.},$$

where

$$\mu \propto Q^{1/2} + \dots \qquad v \propto \frac{1}{Q^{1/2}}$$

The classical energy is

$$E = c_{3/2} V Q^{3/2} + c_{1/2} R V Q^{1/2} + \mathcal{O}\left(Q^{-1/2}\right)$$

Fluctuations

The fluctuations over this ground state are described by two modes.

- A universal "conformal Goldstone". It comes from the breaking of the $U(1)$.

$$\omega = \frac{1}{\sqrt{2}} p$$

- The massive dilaton. It controls the magnitude of the quantum fluctuations. All quantum effects are controled by $1/Q$.

$$\omega = 2\mu + \frac{p^2}{2\mu}$$

(This is a heavy fluctuation around the semiclassical state. It has nothing to do with a light dilaton in the full theory)

Non-linear sigma model

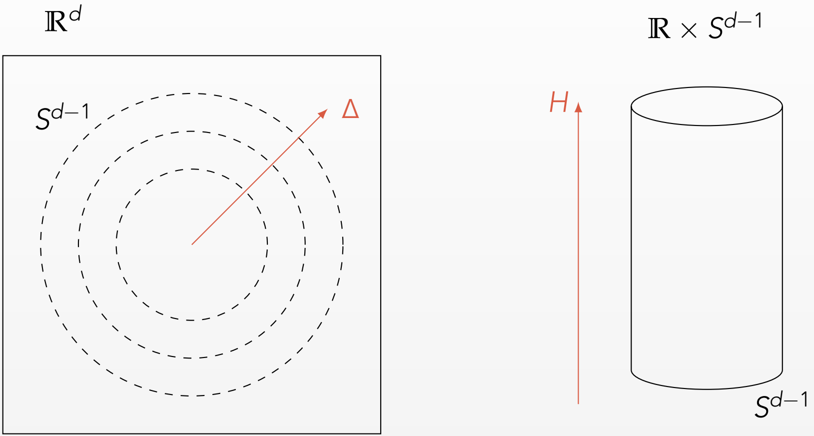
Since σ is heavy we can integrate it out and write a non-linear sigma model (NLSM) for χ alone.

$$L[\chi] = k_{3/2}(\partial_\mu \chi \, \partial^\mu \chi)^{3/2} + k_{1/2} R (\partial_\mu \chi \, \partial^\mu \chi)^{1/2} + \dots$$

These are the leading terms in the expansion around the classical solution $\chi = \mu t$. All other terms are suppressed by powers of $1/Q$.

State-operator correspondence

The anomalous dimension on $\mathbb{R}^d$ is the energy in the cylinder frame.



Protected by conformal invariance: a well-defined quantity.

Conformal dimensions

We know the energy of the ground state.
The leading quantum effect is the Casimir energy of the conformal Goldstone.

$$E_G = \frac{1}{2\sqrt{2}} \zeta\left(-\tfrac{1}{2} | S^2\right) = -0.0937\dots$$

This is the unique contribution of order Q^0.

Final result: the conformal dimension of the lowest operator of charge Q in the $O(2)$ model has the form

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi}\, c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}\left(Q^{-1/2}\right)$$

The $O(2N)$ model

Next step: $O(2N)$.
N charges can be fixed.

Again, homogeneous ground state.
The ground-state energy only depends on the sum of the charges

$$Q = Q_1 + \cdots + Q_N$$

and takes the same form

$$E = \frac{c_{3/2}(N)}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi}\, c_{1/2}(N) Q^{1/2} + \mathcal{O}\left(Q^{-1/2}\right)$$

The coefficients depend on N and cannot be computed in the EFT (but *e.g.* in large-N).

Fluctuations

The symmetry breaking pattern is

$$O(2N) \xrightarrow{\text{exp.}} U(N) \xrightarrow{\text{spont.}} U(N-1)$$

and there are $\dim(U(N)/U(N-1)) = 2N-1$ degrees of freedom (DOF).

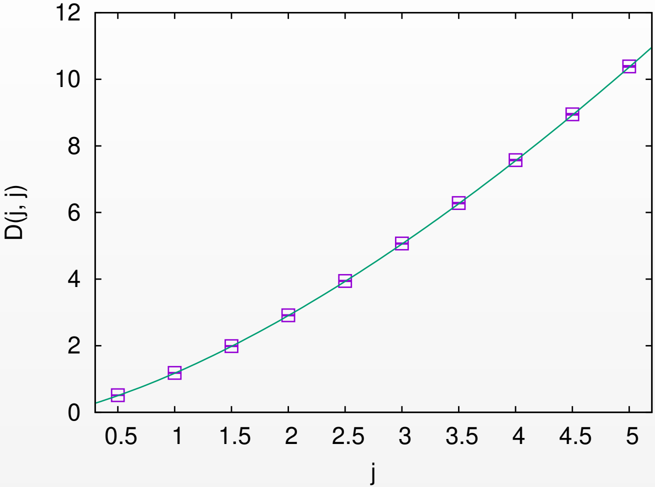
- One singlet, the universal conformal Goldstone $\omega = \frac{1}{\sqrt{2}}p$
- One vector of $U(N-1)$, with quadratic dispersion $\omega = \frac{p^2}{2\mu}$

Each type-II Goldstone counts for two DOF:

$$1 + 2 \times (N-1) = 2N-1.$$

Only the type-I has a Q^0 contribution: it is universal.

$O(4)$ on the lattice



$$\Delta_j = \frac{c_{3/2}}{2\sqrt{\pi}}(2j)^{3/2} + 2\sqrt{\pi}\, c_{1/2}(2j)^{1/2} - 0.094 + \mathcal{O}\left(j^{-1/2}\right)$$

What happened?

We started from a CFT.
There is no mass gap, there are no particles, there is no Lagrangian.

We picked a sector.

In this sector the physics is described by a semiclassical configuration plus massless fluctuations.

The full theory has no small parameters but we can study this sector with a simple EFT.
We are in a strongly coupled regime but we can compute physical observables using perturbation theory.

In conclusion

- With the large-charge approach we can study strongly-coupled systems perturbatively.
- Select a sector and we write a controllable effective theory.
- The strongly-coupled physics is (for the most part) subsumed in a semiclassical state.
- Compute the CFT data.
- Very good agreement with lattice (supersymmetry, large N).
- Precise and testable predictions.