Amplification of gravitational motion via quantum weak measurement

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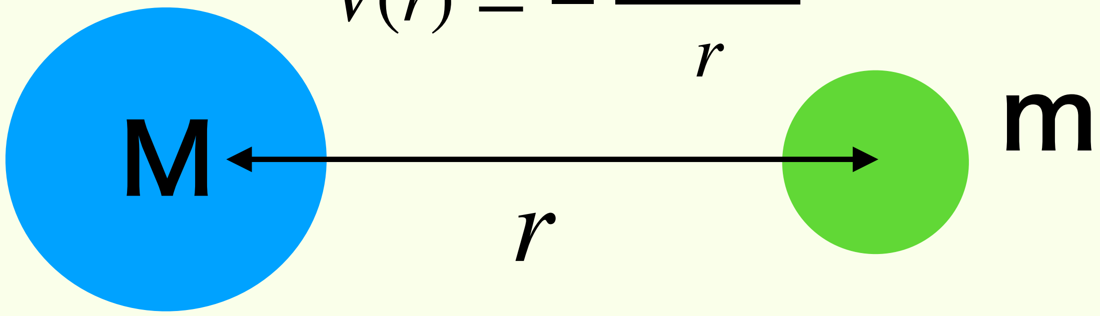
Gravitational motion

$$V(r) = -\frac{GMm}{r}$$



gravitational source

Test particle

EOM: $$m\frac{d^2r}{dt^2} = -G\frac{mM}{r^2}$$

Initial condition: $r(t) \equiv R + x(t), \quad x(t=0)=0, \quad \dot{x}(t=0)=0$

$$x(t) = -\frac{GMt^2}{2R^2} = -3.3\times10^{-4}\left(\frac{t}{1\mathrm{s}}\right)^2\ [\mathrm{mm}]$$ e.g. Pt $M = 100\ [\mathrm{kg}],\ R = 10^2\ [\mathrm{mm}]$

Can we amplify gravitational motion by weak measurement?

Plan of Talk

1. Review of Weak Measurement

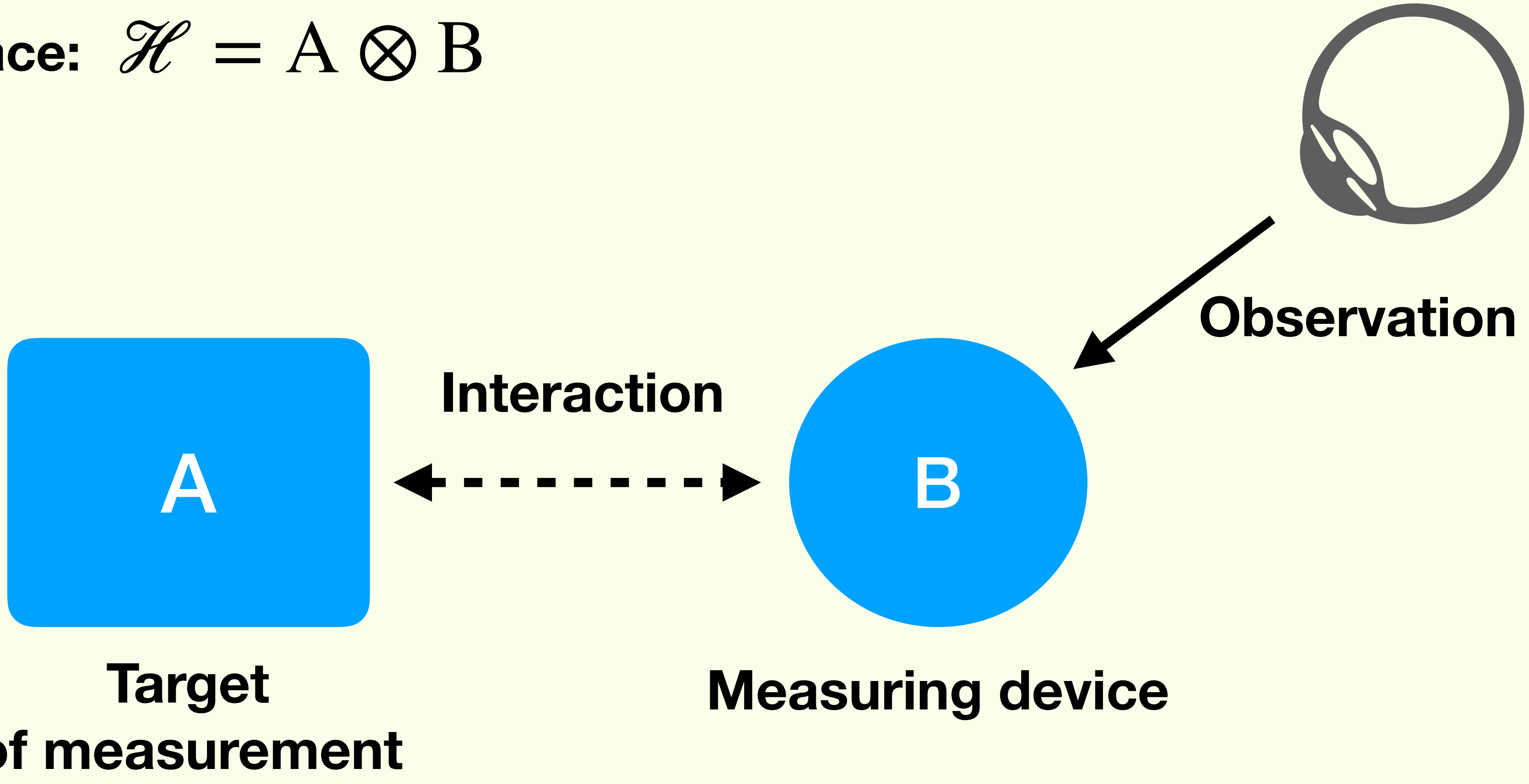
2. Weak measurement of gravitational force by using atomic system

3. Conclusion

Weak measurement is one of indirect measurement

- **Indirect measurement**

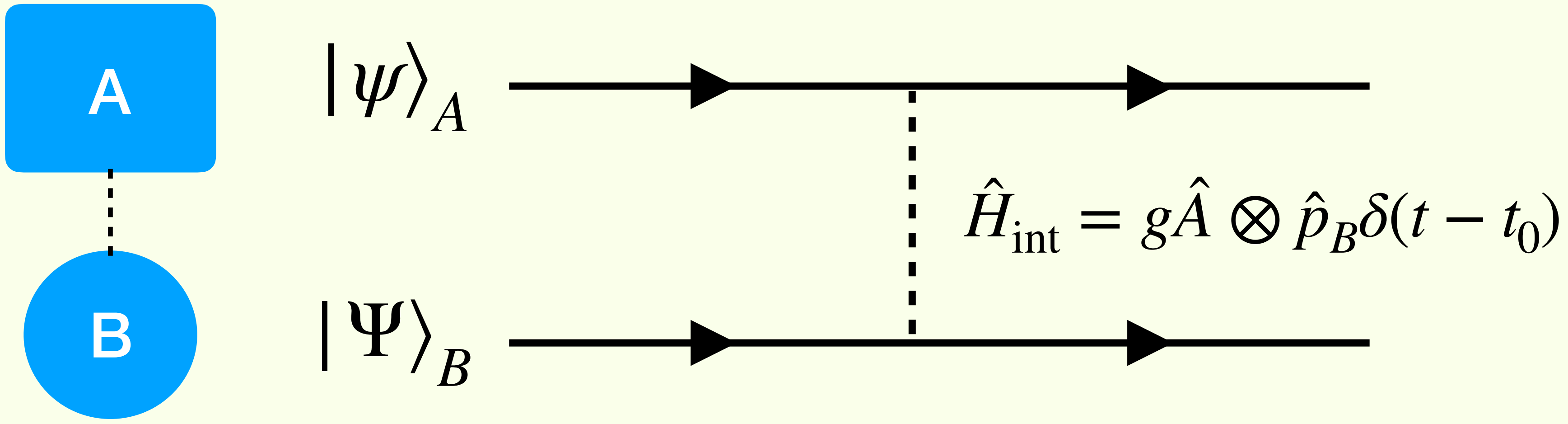
Hilbert space: $\mathscr{H} = \mathrm{A} \otimes \mathrm{B}$



Getting informations of A by observing B

Indirect measurement

e.g.



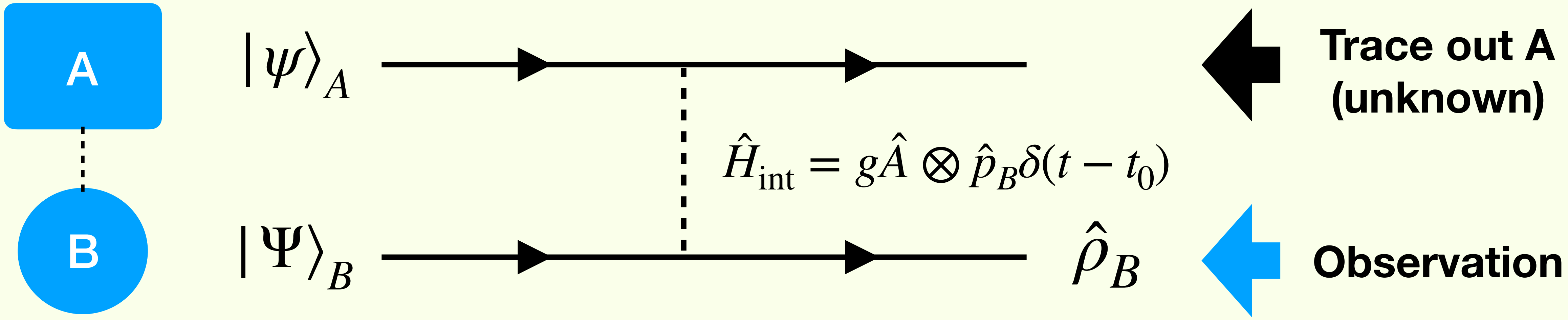
$\hat{A}$: observable of **A** $\hat{p}_B$: momentum of **B**

$$e^{-i\int \hat{H}_{\text{int}} dt} \underbrace{|\psi \otimes \Psi\rangle}_{\textbf{Initial state}} = e^{-ig\hat{A}\otimes\hat{p}_B} |\psi \otimes \Psi\rangle = \sum_i |a_i\rangle\langle a_i|\psi\rangle_A \underbrace{e^{-iga_i\hat{p}_B}}_{\textbf{Translation operator}} |\Psi\rangle_B$$

※ $\hat{A}|a_i\rangle_A = a_i|a_i\rangle_A$

Indirect measurement

e.g.



Trace out A (unknown)

Observation

$\hat{A}$: observable of **A** $\hat{p}_B$: momentum of **B**

$$e^{-i\int\hat{H}_{\rm int}dt}|\psi\otimes\Psi\rangle = e^{-ig\hat{A}\otimes\hat{p}_B}|\psi\otimes\Psi\rangle = \sum_i |a_i\rangle\langle a_i|\psi\rangle_A \underbrace{e^{-iga_i\hat{p}_B}}_{\text{Translation operator}}|\Psi\rangle_B$$

※ $\hat{A}|a_i\rangle_A = a_i|a_i\rangle_A$

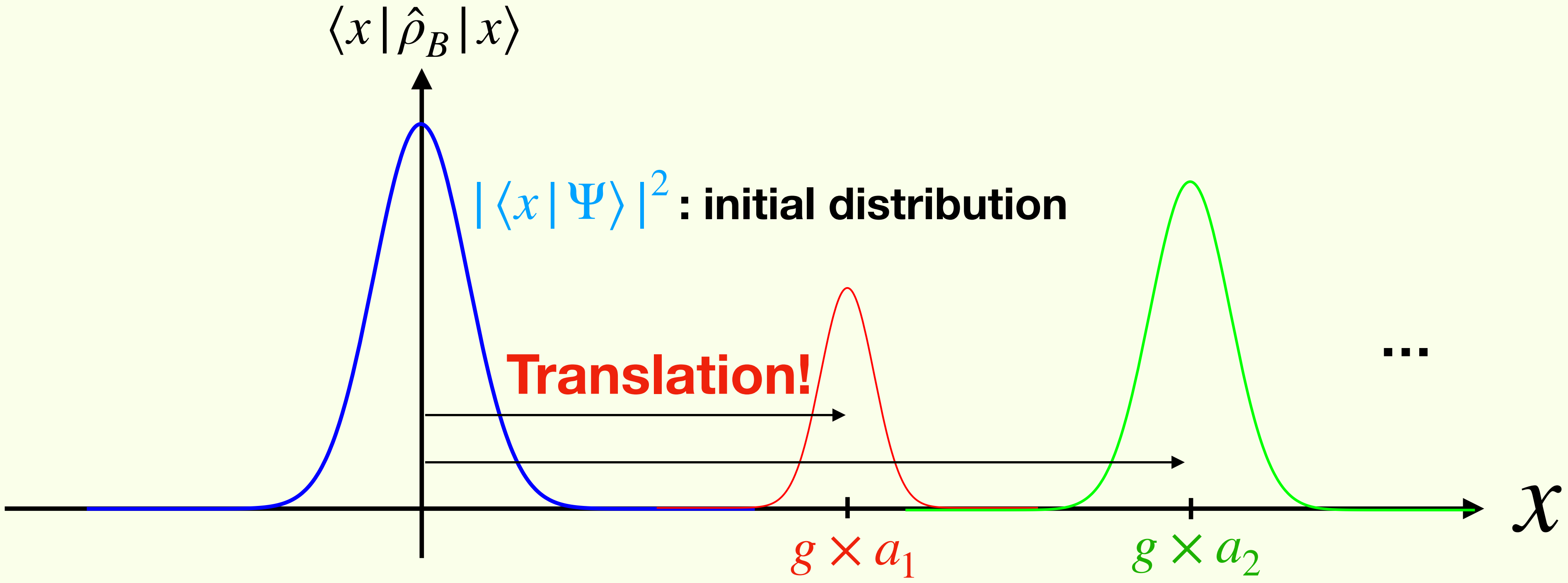
$$\hat{\rho}_B \equiv \mathrm{Tr}_A\left[e^{-i\int\hat{H}_{\rm int}dt}|\psi\otimes\Psi\rangle\langle\psi\otimes\Psi|e^{+i\int\hat{H}_{\rm int}dt}\right]$$

Indirect measurement

$$\hat{\rho}_B \equiv \mathrm{Tr}_A \left[e^{-i\int \hat{H}_{\mathrm{int}} dt} | \psi \otimes \Psi \rangle \langle \psi \otimes \Psi | e^{+i\int \hat{H}_{\mathrm{int}} dt} \right]$$

$$e^{-i\int \hat{H}_{\mathrm{int}} dt} | \psi \otimes \Psi \rangle = \sum_i | a_i \rangle \langle a_i | \psi \rangle_A \underline{e^{-iga_i \hat{p}_B} | \Psi \rangle_B}$$

$$\langle x | \hat{\rho}_B | x \rangle = \sum_i \left| \langle a_i | \psi \rangle \right|^2 \times \underline{\left| \langle x - ga_i | \Psi \rangle \right|^2}$$



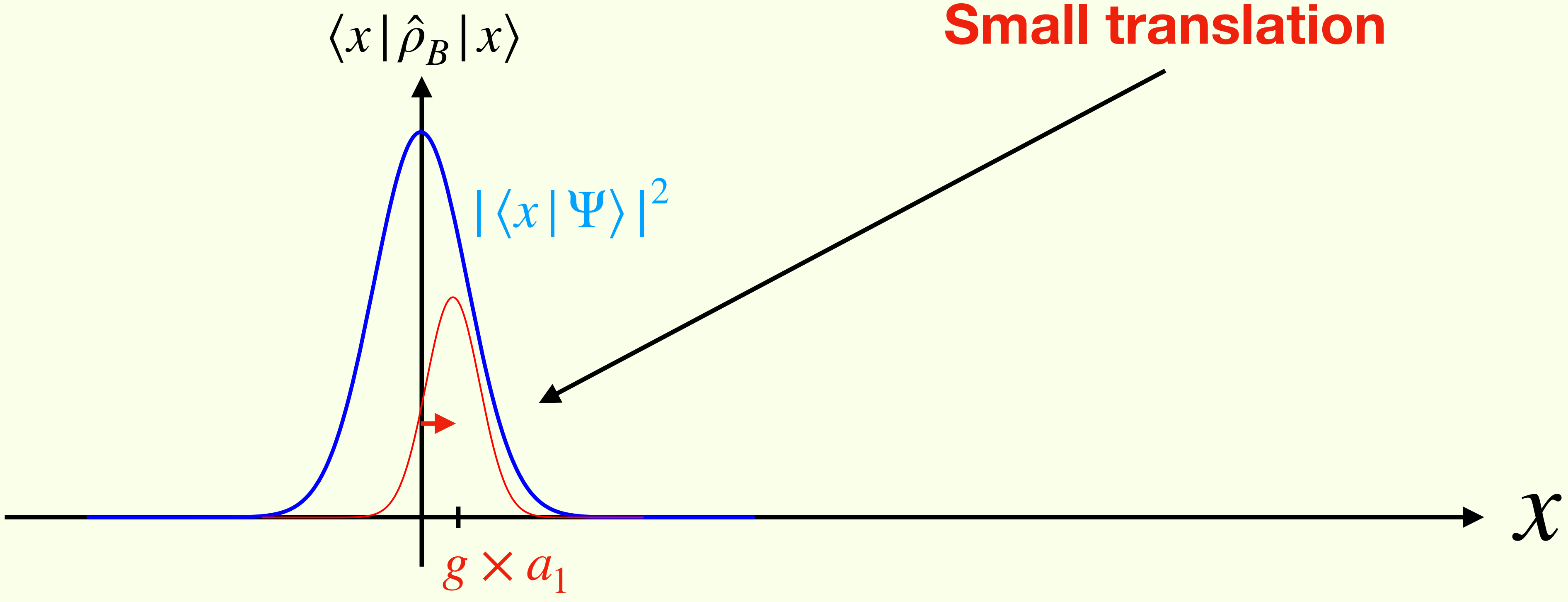
By observing B (ρ$_B$), we can measure A (a$_1$, a$_2$, …)

Weak measurement

Weak measurement: g << 1

$$\langle x|\hat{\rho}_B|x\rangle = \sum_i \left|\langle a_i|\psi\rangle\right|^2 \times \underline{\left|\langle x - g a_i|\Psi\rangle\right|^2}$$

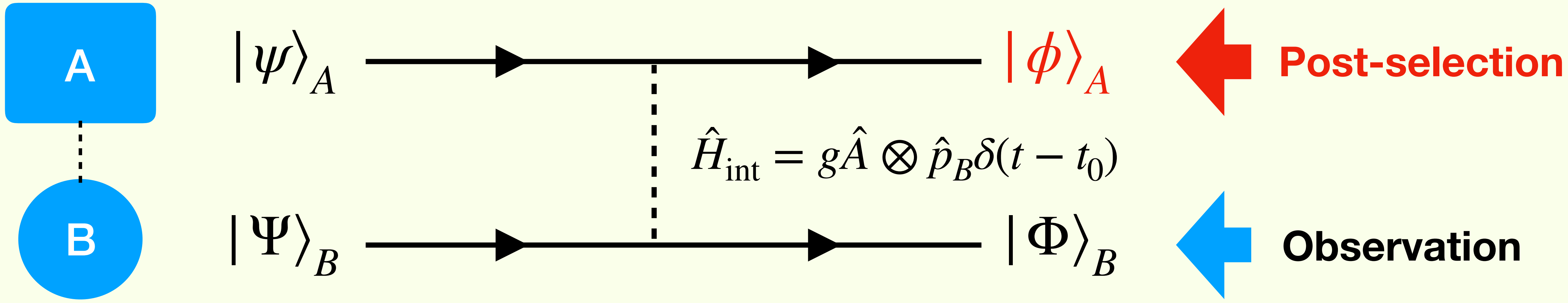
Small translation



Measuring peak of distribution becomes difficult

Weak measurement with post-selection

- **Restricting the final state of A : post-selection**



$$\hat{H}_{\text{int}} = g\hat{A}\otimes\hat{p}_B\delta(t-t_0)$$

Post-selection

Observation

$$|\Phi\rangle_B = \langle\underline{\phi}|e^{-ig\hat{A}\otimes\hat{p}_B}|\psi\otimes\Psi\rangle \simeq \langle\phi|(1-ig\hat{A}\otimes\hat{p}_B)|\psi\otimes\Psi\rangle$$

$$= \langle\phi|\psi\rangle\left(1-ig\underline{\frac{\langle\phi|\hat{A}|\psi\rangle}{\langle\phi|\psi\rangle}}\hat{p}_B\right)|\Psi\rangle_B$$

Weak value

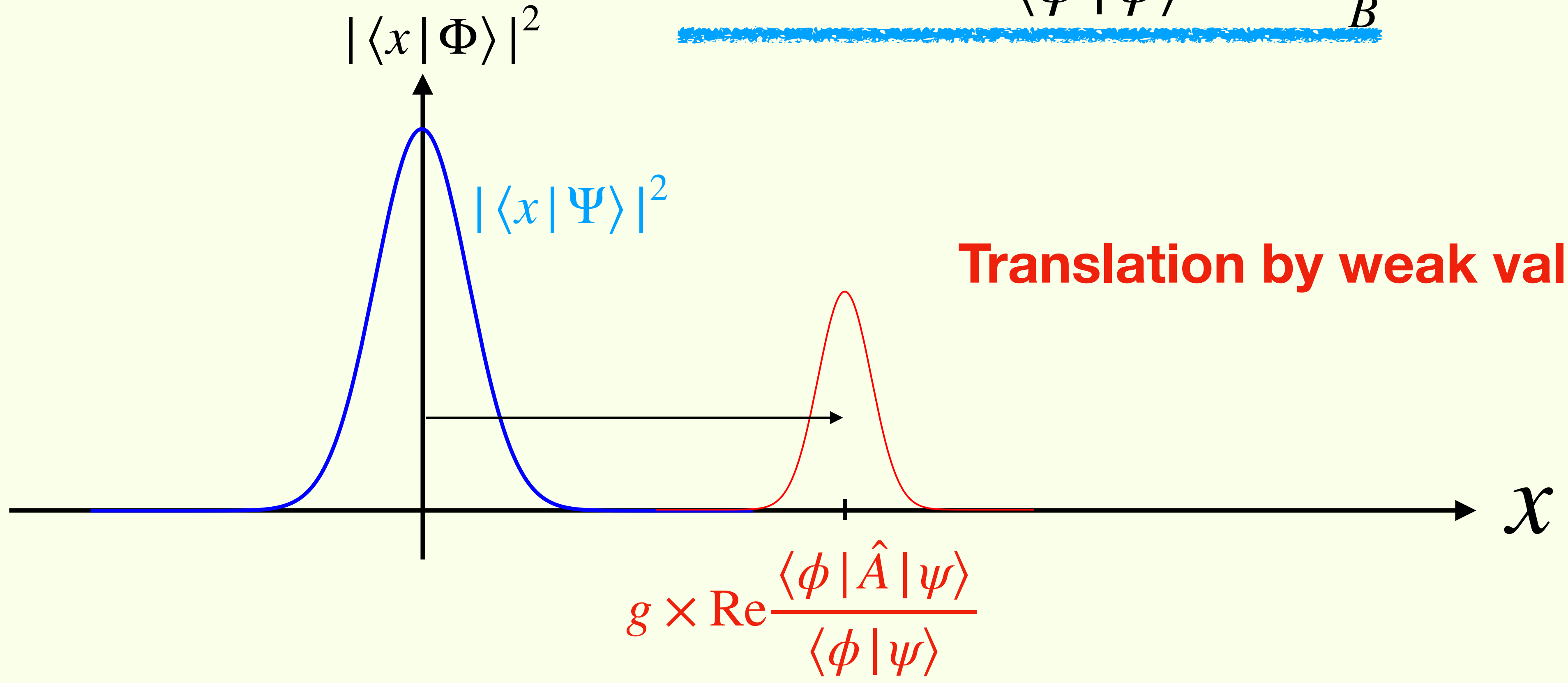
$$\simeq \langle\phi|\psi\rangle\underline{e^{-ig\frac{\langle\phi|\hat{A}|\psi\rangle}{\langle\phi|\psi\rangle}\hat{p}_B}}|\Psi\rangle_B$$

Translation by weak value

Translation by weak value

$$|\Phi\rangle_B = \langle\phi|e^{-ig\hat{A}\otimes\hat{p}_B}|\psi\otimes\Psi\rangle \simeq \langle\phi|\psi\rangle e^{-ig\frac{\langle\phi|\hat{A}|\psi\rangle}{\langle\phi|\psi\rangle}\hat{p}_B}|\Psi\rangle_B$$

$$\langle x|\Phi\rangle_B \sim \langle\phi|\psi\rangle\langle x - g\cdot\mathrm{Re}\frac{\langle\phi|\hat{A}|\psi\rangle}{\langle\phi|\psi\rangle}|\Psi\rangle_B$$



Translation by weak value!

Weak value can become large

Weak value: $$\frac{\langle\phi|\hat{A}|\psi\rangle}{\langle\phi|\psi\rangle}$$

- **Large weak value = small transition amplitude** $\langle\phi|\psi\rangle \sim 0$

- **Trade off in weak value:**

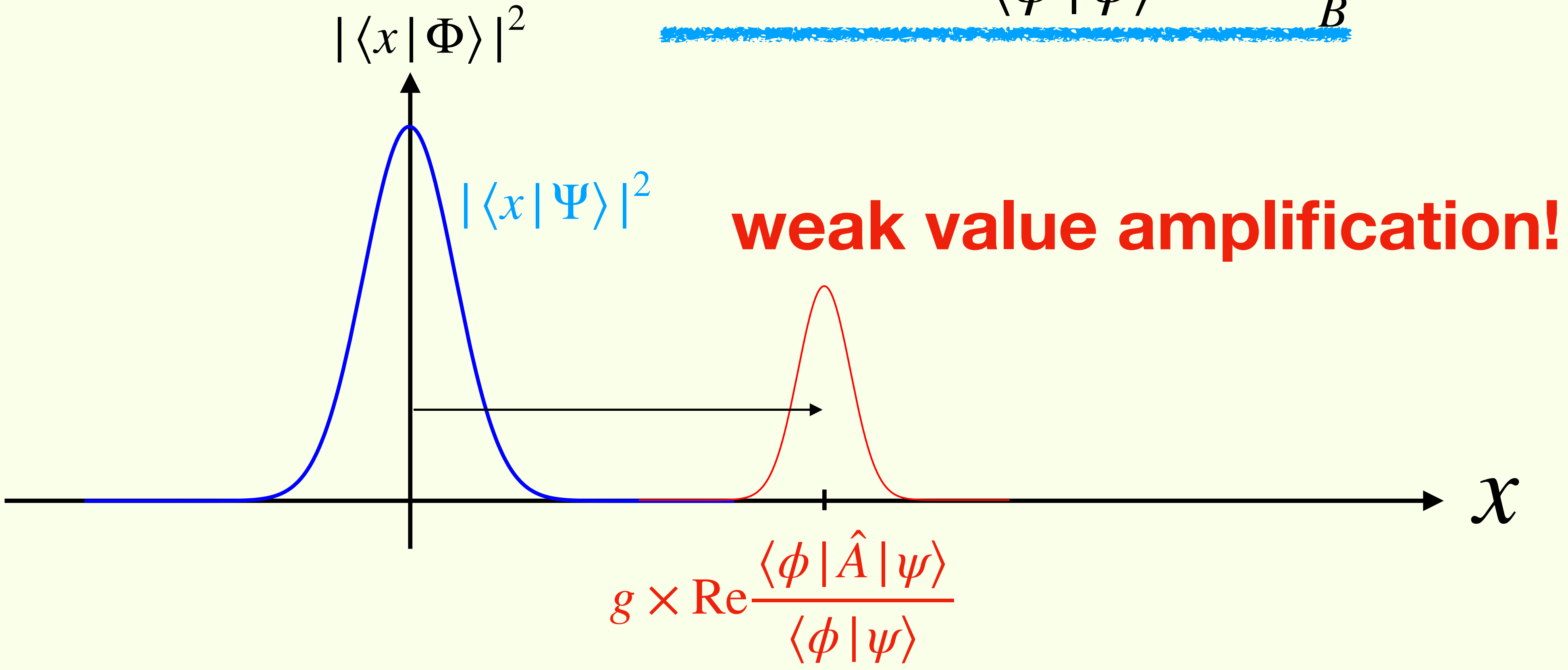
Large weak value vs Statistics

Weak value amplification

$$|\Phi\rangle_B = \langle\phi|e^{-ig\hat{A}\otimes\hat{p}_B}|\psi\otimes\Psi\rangle \simeq \langle\phi|\psi\rangle e^{-ig\frac{\langle\phi|\hat{A}|\psi\rangle}{\langle\phi|\psi\rangle}\hat{p}_B}|\Psi\rangle_B$$

$$\Downarrow$$

$$\langle x|\Phi\rangle_B \sim \langle\phi|\psi\rangle\langle x - g\cdot\mathrm{Re}\frac{\langle\phi|\hat{A}|\psi\rangle}{\langle\phi|\psi\rangle}|\Psi\rangle_B$$



weak value amplification!

Even if g is small, large shift by large weak value

Plan of Talk

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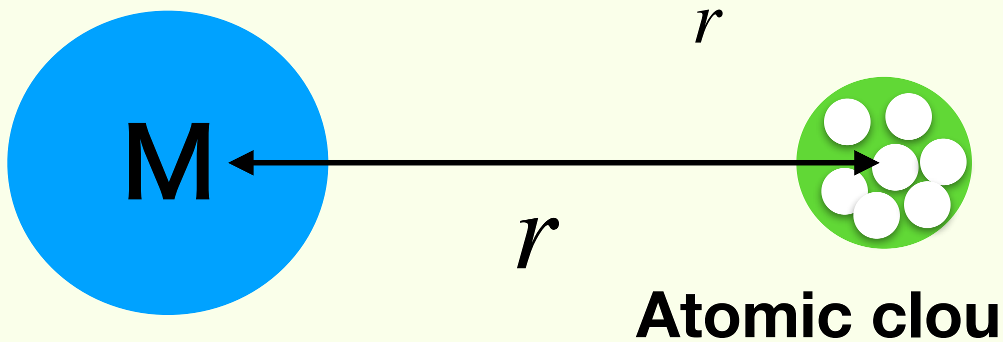
Atomic cloud in gravitational source

$$V(r) = -\frac{GMm}{r}$$



gravitational source

Atomic cloud (BEC) as test particle ← **Large weak value** vs **Statistics**

e.g. ^{87}Rb atoms

- Hilbert space of this system:

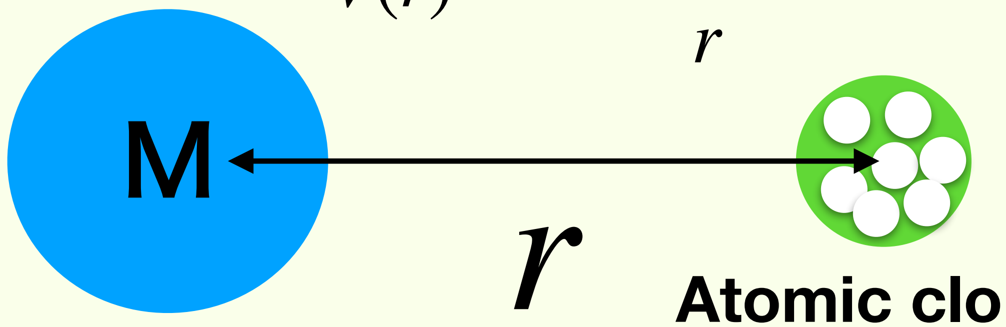
$$\mathscr{H} = \underbrace{\mathscr{H}_A}_{\text{Internal state of atom}} \otimes \underbrace{\mathscr{H}_B}_{\text{Relative motion: } \Psi(r)}$$

e.g. $\underbrace{\hat{a}_1^\dagger|0\rangle}_{\text{ground state}}, \underbrace{\hat{a}_2^\dagger|0\rangle}_{\text{1st excited state (Hyper fine splitting)}}, \cdots$

$\hat{a}_1^{\dagger 2}|0\rangle, \hat{a}_1^\dagger\hat{a}_2^\dagger|0\rangle, \cdots$

Atomic cloud in gravitational source

$$V(r) = -\frac{GMm}{r}$$



gravitational source

Atomic cloud

e.g. ^{87}Rb atoms

- **Hamiltonian:**

$$\hat{H} = \hat{H}_{\text{atom}} + \frac{\hat{p}_r^2}{2\hat{m}} - \underline{\frac{GM\hat{m}}{\hat{r}}}$$

Interaction

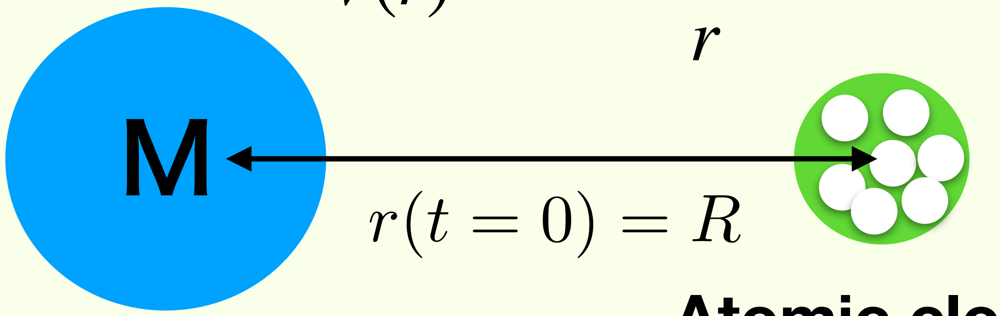
b/w (Internal state) and (Relative motion state)

$$\hat{H}_{\text{atom}} = E_1\underline{\hat{a}_1^\dagger\hat{a}_1} + E_2\underline{\hat{a}_2^\dagger\hat{a}_2},\ [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij},\ [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0$$

ground state 1st excited state (Hyper fine splitting)

$\hat{m} = m_1\hat{a}_1^\dagger\hat{a}_1 + m_2\hat{a}_2^\dagger\hat{a}_2$ **: mass operator of atomic cloud**

Approximation



$$V(r) = -\frac{GMm}{r}$$

$r(t=0) = R$

gravitational source

Atomic cloud

e.g. ^{87}Rb atoms

- Hamiltonian:

$$\hat{H} = \hat{H}_{\text{atom}} + \frac{\hat{p}_r^2}{2\hat{m}} - \frac{GM\hat{m}}{\hat{r}} \simeq \hat{H}_{\text{atom}} + \frac{\hat{p}_r^2}{2\hat{m}} \underline{- \frac{GM\hat{m}}{R}\left(1 - \frac{\hat{x}}{R}\right)}$$

※ expand r around the initial distance R

$$\frac{1}{\hat{r}} \simeq \frac{1}{R}\left(1 - \frac{\hat{x}}{R}\right), \quad \hat{r} \equiv R + \hat{x}$$

Time evolution

$$\hat{H} = \underbrace{\hat{H}_{\mathrm{atom}} - \frac{GM\hat{m}}{R}}_{\equiv \hat{H}_0} + \underbrace{\frac{\hat{p}^2}{2\hat{m}} + \frac{GM\hat{m}\otimes\hat{x}}{R^2}}_{\equiv \hat{H}_1}$$

$$e^{-i\hat{H}_1 T} e^{-i\hat{H}_0 T} \underline{|\psi_i\rangle}\,\underline{|\phi_i\rangle} \simeq e^{-i\frac{GMT\hat{m}\otimes\hat{x}}{R^2}} e^{-i\hat{H}_0 T}|\psi_i\rangle e^{-i\frac{\hat{p}^2}{2\hat{m}}T}|\phi_i\rangle$$

$$\equiv e^{-i\frac{GMt\hat{m}\otimes\hat{x}}{R^2}} |\psi(t)\rangle\, |\phi(t)\rangle$$

Initial state

||

Internal state × Relative motion state

Post selection

$$\hat{H} = \underbrace{\hat{H}_{\rm atom} - \frac{GM\hat{m}}{R}}_{\equiv \hat{H}_0} + \underbrace{\frac{\hat{p}^2}{2\hat{m}} + \frac{GM\hat{m}\otimes\hat{x}}{R^2}}_{\equiv \hat{H}_1}$$

$$e^{-i\hat{H}_1 T} e^{-i\hat{H}_0 T} |\psi_i\rangle |\phi_i\rangle \simeq e^{-i\frac{GMT\hat{m}\otimes\hat{x}}{R^2}} e^{-i\hat{H}_0 T} |\psi_i\rangle e^{-i\frac{\hat{p}^2}{2\hat{m}}T} |\phi_i\rangle$$

$$\equiv e^{-i\frac{GMT\hat{m}\otimes\hat{x}}{R^2}} |\psi(T)\rangle |\phi(T)\rangle$$

$\times \langle\psi_f|$: **post-selection of internal state** at $t = T$

$$|\phi_f(T)\rangle \equiv \underline{\langle\psi_f|} e^{-i\frac{GMT\hat{m}\otimes\hat{x}}{R^2}} |\psi(T)\rangle |\phi(T)\rangle$$

$$= \langle\psi_f| \left(1 - i\frac{GMt\hat{m}\otimes\hat{x}}{R^2}\right) |\psi(T)\rangle |\phi(T)\rangle + \mathcal{O}(G^2)$$

$$\simeq \langle\psi_f|\psi(T)\rangle \left(1 - i\frac{GMT}{R^2} \underbrace{\frac{\langle\psi_f|\hat{m}|\psi(T)\rangle}{\langle\psi_f|\psi(T)\rangle}}_{\text{Weak value}} \hat{x}\right) |\phi(T)\rangle$$

Expectation value of position

- relative motion state after post-selection:

$$|\phi_f(T)\rangle = \langle\psi_f|\psi(T)\rangle\left(1 - i\frac{GMT}{R^2}\frac{\langle\psi_f|\hat{m}|\psi(T)\rangle}{\langle\psi_f|\psi(T)\rangle}\hat{x}\right)|\phi(T)\rangle$$

Initial condition:

$$\langle x|\phi_i\rangle = \frac{1}{(\pi d^2)^{1/4}}e^{-\frac{x^2}{2d^2}}$$

$$\frac{\langle\phi_f(T)|\,\hat{x}\,|\phi_f(T)\rangle}{\langle\phi_f(T)|\phi_f(T)\rangle} \simeq x_{\mathrm{cl}}(T) + d^2\frac{GMT}{R^2}\underbrace{\mathrm{Im}\left[\frac{\langle\psi_f|\,\hat{m}\,|\psi(T)\rangle}{\langle\psi_f|\psi(T)\rangle}\right]}_{\text{Translation by weak value}}$$

classical motion $x_{\mathrm{cl}}(T) = -\frac{GMT^2}{2R^2}$

Expectation value of position

- Expectation value of position:

$$\frac{\langle \phi_f(T)|\,\hat{x}\,|\phi_f(T)\rangle}{\langle \phi_f(T)|\phi_f(T)\rangle} = x_{\rm cl}(T) + d^2\frac{GMT}{R^2}{\rm Im}\left[\frac{\langle \psi_f|\,\hat{m}\,|\psi(T)\rangle}{\langle \psi_f|\psi(T)\rangle}\right] + \mathcal{O}(G^2)$$

Internal state

$|\psi_i\rangle = \frac{1}{\sqrt{2}}(|1;N\rangle + |2;N\rangle)$(initial), $|\psi_f\rangle = \frac{1}{\sqrt{2}}(|1;N\rangle - |2;N\rangle)$(finial)

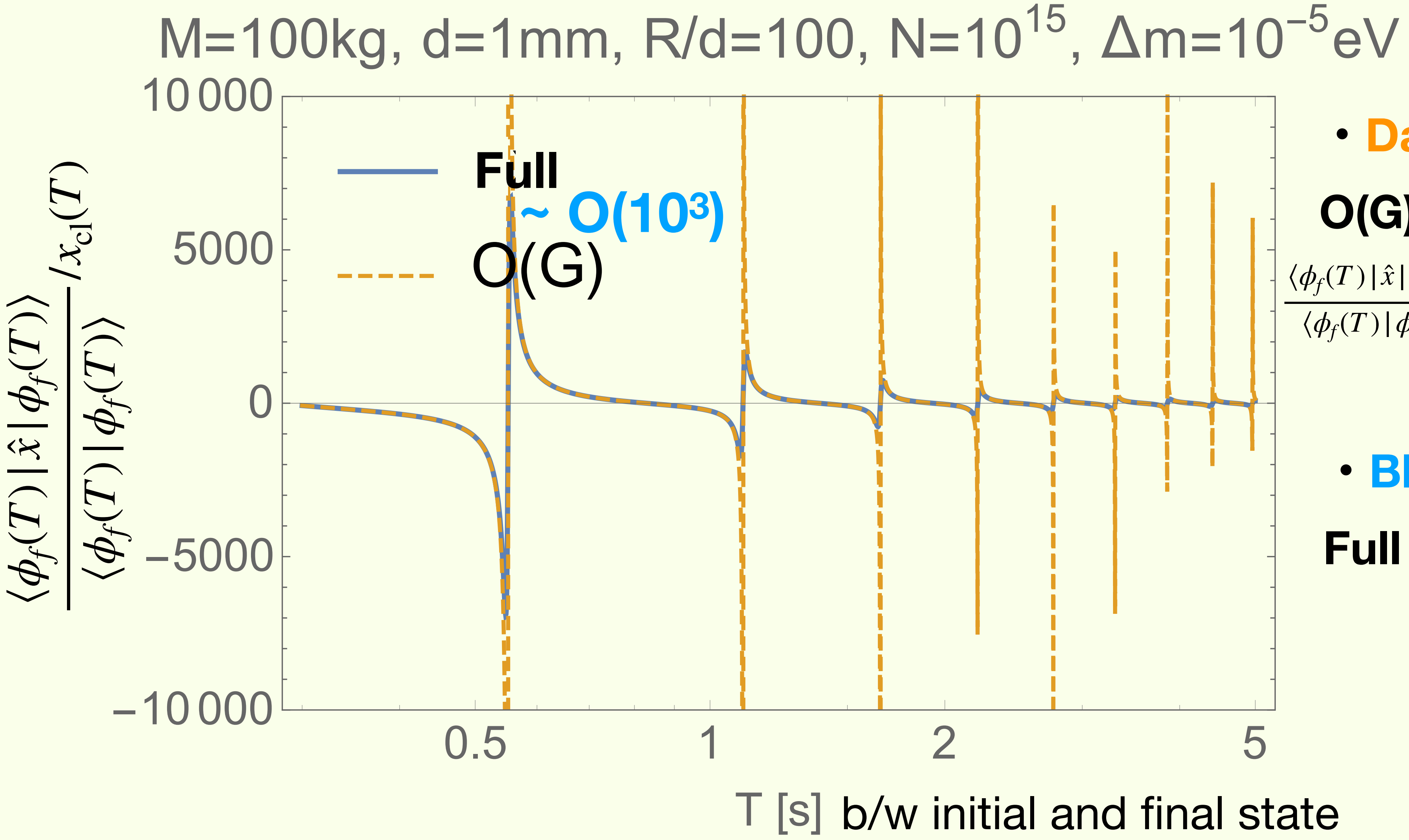
Coherent states (BEC): $\hat{a}_1|1;N\rangle = \sqrt{N}|1;N\rangle$, $\hat{a}_2|2;N\rangle = \sqrt{N}|2;N\rangle$

$$= x_{\rm cl}(T)\left(1 + \underline{\frac{N\Delta m d^2}{T}\frac{\sin f(T)}{1-\cos f(T)}}\right)$$

f(T) ~ 2π × (integer): Weak value is amplified!

$$\Delta m \equiv m_2 - m_1 \quad f(T) \equiv GM\Delta m NT/R \quad x_{\rm cl}(T) \equiv -GMT^2/2R^2$$

Result



b/w initial and final state

- **Dashed orange line**:

O(G) calculation

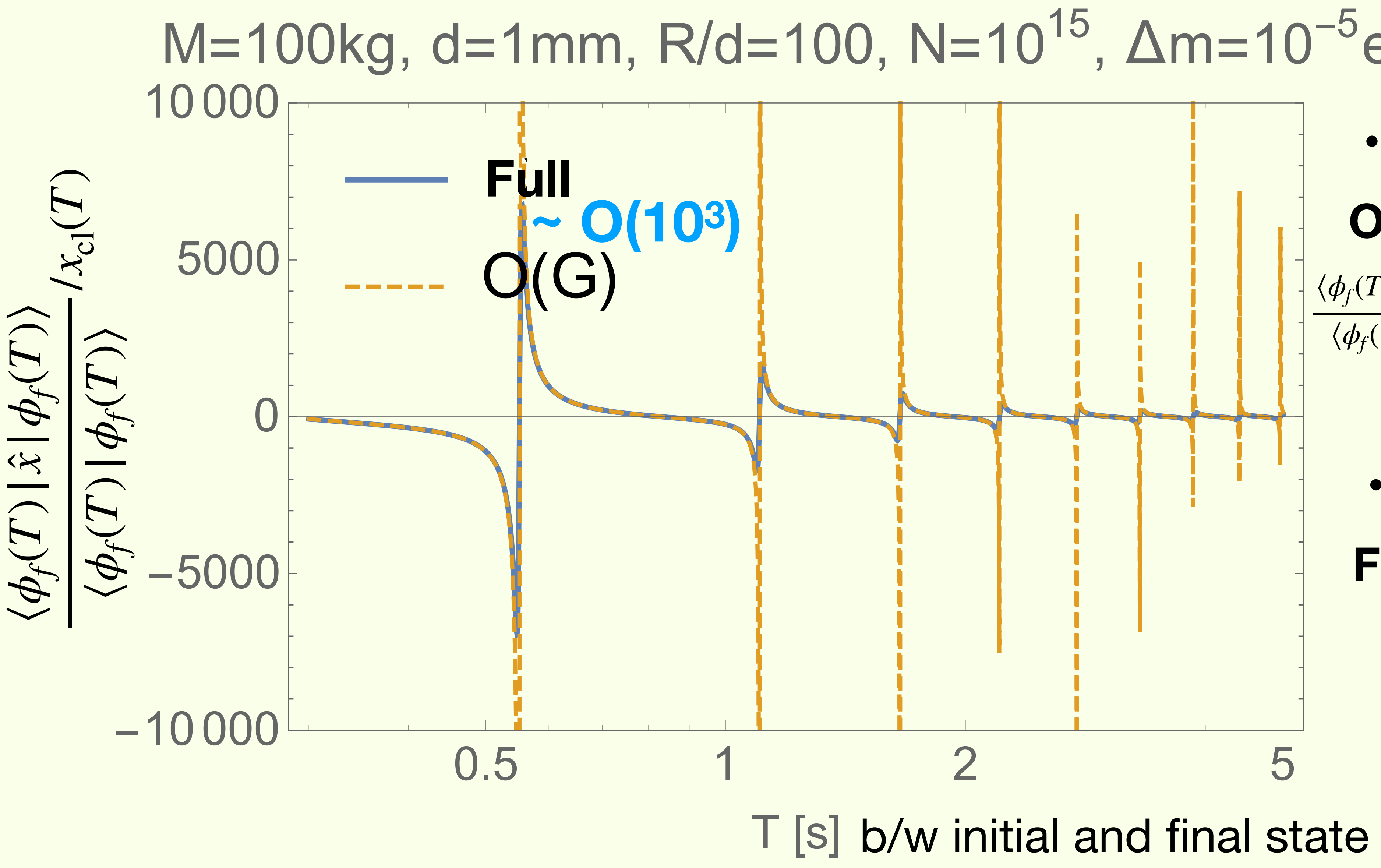
$$\frac{\langle\phi_f(T)|\hat{x}|\phi_f(T)\rangle}{\langle\phi_f(T)|\phi_f(T)\rangle}/x_{\rm cl}(T) \simeq 1 + \underline{\frac{N\Delta m d^2}{T}\frac{\sin f(T)}{1-\cos f(T)}}$$

can be negative

- **Blue line**:

Full order calculation

Result



- **Dashed orange line**:

O(G) calculation

$$\frac{\langle\phi_f(T)|\hat{x}|\phi_f(T)\rangle}{\langle\phi_f(T)|\phi_f(T)\rangle}/x_{\rm cl}(T) \simeq 1 + \underline{\frac{N\Delta m d^2}{T}\frac{\sin f(T)}{1-\cos f(T)}}$$

can be negative

- **Blue line**:

Full order calculation

- **Gravity is amplified by O(10^3) when N =10^{15}**
- **behave as repulsive force because of negative weak value**

Conclusion and future work

- We proposed a thought experiment by using atoms
- In the weak measurement, gravity is amplified by $O(10^3)$ compared with classical motion when $N = 10^{15}$
- Gravity behaves as repulsive force by the negative weak value
- Application to measuring the Newtonian constant

Selection

- pre-selection, post-selection:

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}\left(|1;N\rangle + |2;N\rangle\right) \quad |\psi_f\rangle = \frac{1}{\sqrt{2}}\left(|1;N\rangle - |2;N\rangle\right)$$

$|n;N\rangle \equiv e^{-N/2}e^{\sqrt{N}\hat{a}_n^\dagger}|0\rangle$**: coherent state**

- BEC = Coherent state:

$$|n;N\rangle \equiv e^{-N/2}e^{\sqrt{N}\hat{a}_n^\dagger}|0\rangle$$

$$\hat{a}_n|n;N\rangle = \sqrt{N}|n;N\rangle \longrightarrow \langle n;N|\hat{a}_n^\dagger\hat{a}_n|n;N\rangle = N$$

How to select

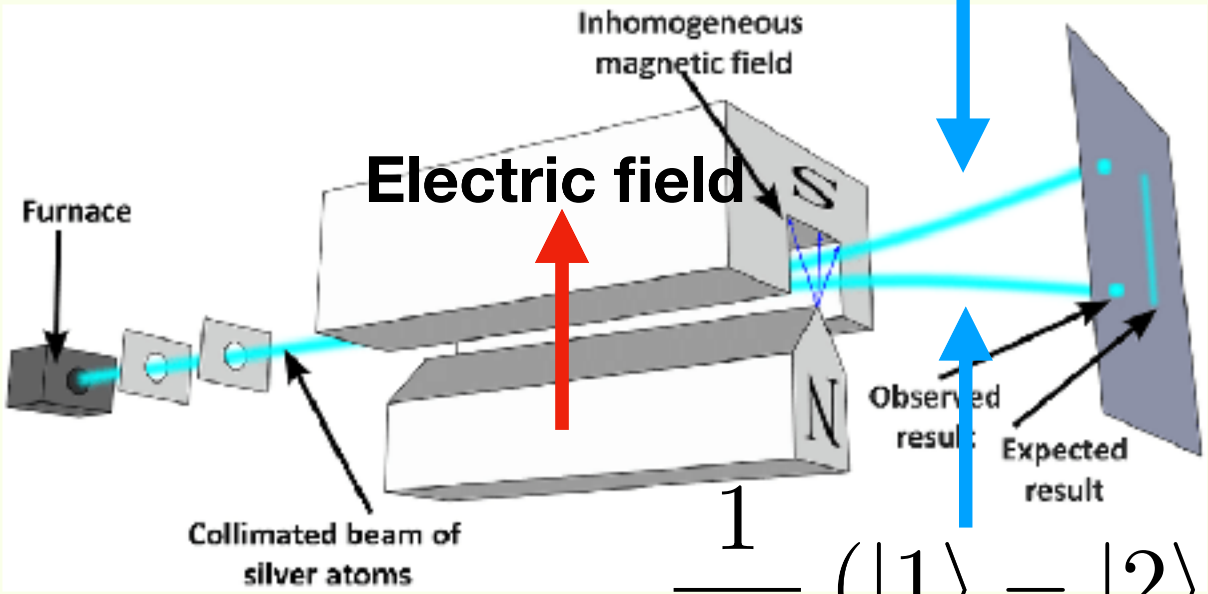
- Cold atoms in electric field

$$\hat{V} = -q\hat{x} \cdot \hat{E} \quad \text{(EDM)}$$

$$= -q \sum_{i,j=1}^{2} \underbrace{|i\rangle \langle i|}_{\text{energy state of atom}} \hat{x} \, |j\rangle \langle j| \cdot \hat{E} = -q \langle 2| \, \hat{x} \, |1\rangle \cdot \hat{E} \left(|1\rangle \langle 2| + |2\rangle \langle 1| \right)$$

$$\langle 1| \, \hat{x} \, |1\rangle = \langle 2| \, \hat{x} \, |2\rangle = 0 \quad \textbf{(Parity symmetry)}$$

$$\frac{1}{\sqrt{2}} \left(|1\rangle + |2\rangle \right)$$



$$\frac{1}{\sqrt{2}} \left(|1\rangle - |2\rangle \right)$$

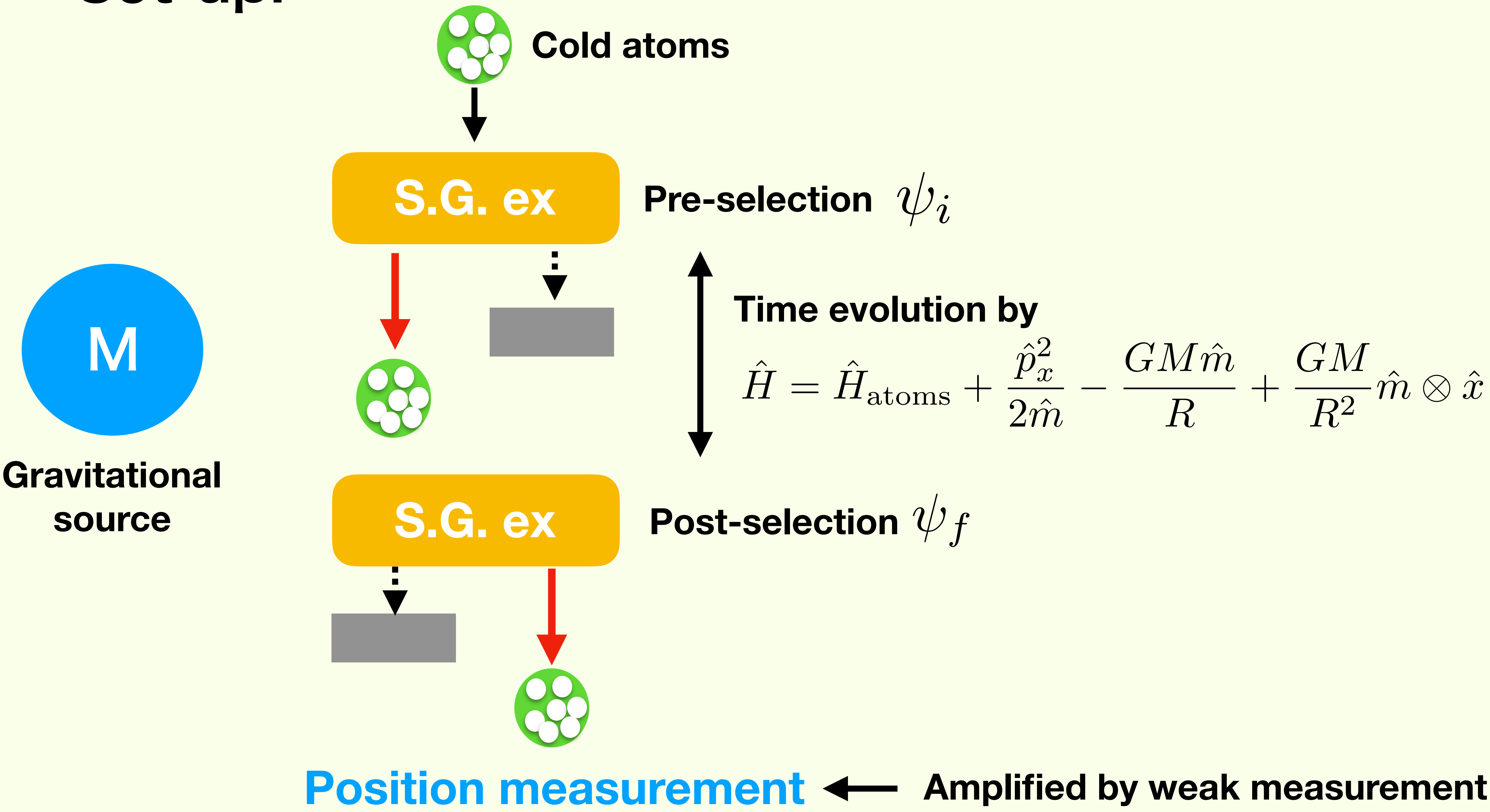
-Eigenstate of EDM:

$$\frac{1}{\sqrt{2}} \left(|1\rangle \pm |2\rangle \right)$$

-S.G. ex is possible

Weak measurement

- Set-up:



Cold atoms

S.G. ex

Pre-selection ψ_i

M

Gravitational source

Time evolution by

$$\hat{H} = \hat{H}_{\mathrm{atoms}} + \frac{\hat{p}_x^2}{2\hat{m}} - \frac{GM\hat{m}}{R} + \frac{GM}{R^2}\hat{m}\otimes\hat{x}$$

S.G. ex

Post-selection ψ_f

Position measurement ← Amplified by weak measurement

Full order calculation

- Leading order

$$\frac{\langle\phi_f(T)|\,\hat{x}\,|\phi_f(T)\rangle}{\langle\phi_f(T)|\phi_f(T)\rangle} = x_{\rm cl}(T) + d^2\frac{GMT}{R^2}\mathrm{Im}\left[\frac{\langle\psi_f|\,\hat{m}\,|\psi(T)\rangle}{\langle\psi_f|\psi(T)\rangle}\right] + \mathcal{O}(G^2)$$

$$= x_{\rm cl}(T)\left(1 + \frac{N\Delta m d^2}{T}\frac{\sin f(T)}{1-\cos f(T)}\right)$$

$$\Delta m \equiv m_2 - m_1 \quad f(T) \equiv GM\Delta m NT/R \quad x_{\rm cl}(T) \equiv -GMT^2/2R^2$$

- Full order

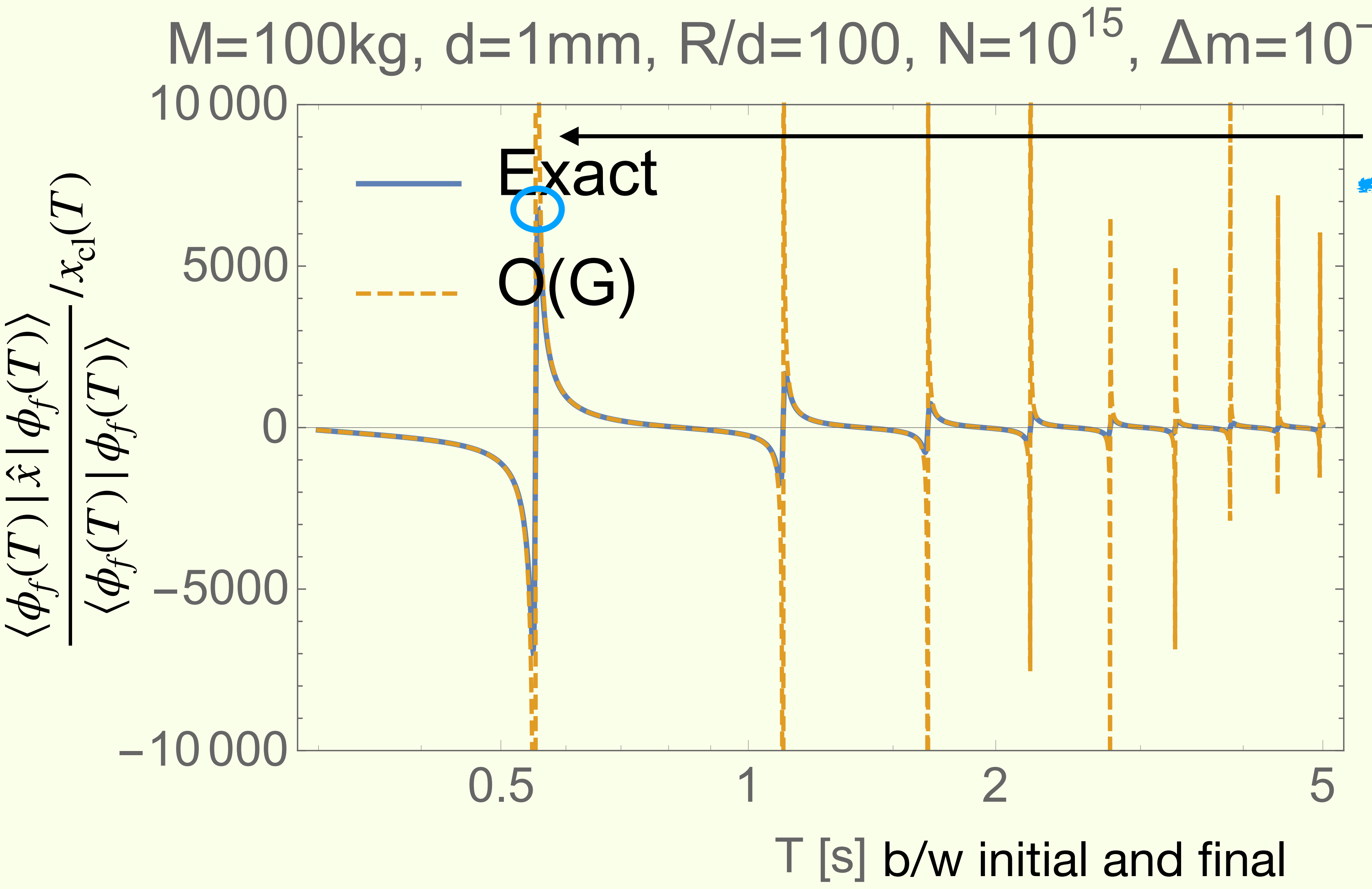
$$\frac{\langle\phi_f(T)|\,\hat{x}\,|\phi_f(T)\rangle}{\langle\phi_f(T)|\phi_f(T)\rangle} = x_{\rm cl}(T)\left(1 + \frac{N\Delta m d^2}{T}\frac{e^{-g(T)}\sin f(T)}{1-e^{-g(T)}\cos f(T)}\right)$$

Damping factor appears

$$g(T) = \frac{N^2 d^2 \Delta m^2}{T^2}x_{\rm cl}(T)^2$$

Weak measurement

- Plot(Full-order vs Leading-order):



$N\,\langle \phi_f(T)|\phi_f(T)\rangle \sim 10^{-4}N$

Survived atoms after selection

- Gravity is amplified by $O(10^3)$ when $N = 10^{15}$
- behave as repulsive force

Possibility of realization

- Large number of atoms:

  -In current technology $N \sim \mathcal{O}(10^9)$

- Time scale of post-selection is very small:

  -we must do selection with accuracy 10^{-10} s

$$\frac{\langle \phi_f(T)|\, \hat{x}\, |\phi_f(T)\rangle}{\langle \phi_f(T)|\phi_f(T)\rangle} = x_{\mathrm{cl}}(T)\left(1 + \frac{N\Delta m d^2}{T}\frac{e^{-g(T)}\sin f(T)}{1 - e^{-g(T)}\cos f(T)}\right)$$

$$f(T) = \underline{N\Delta m T} + \frac{GM\Delta m N T}{R}$$

$$\hat{H} = \boxed{\hat{H}_{\mathrm{atoms}}} + \frac{\hat{p}_x^2}{2\hat{m}} - \frac{GM\hat{m}}{R} + \frac{GM}{R^2}\hat{m}\otimes\hat{x}$$

Selected state

- coherent state is not special in our calculation

$$|1; N\rangle \longrightarrow \hat{a}_1^{\dagger N} \left|0\right\rangle / \sqrt{N}$$

$$|2; N\rangle \longrightarrow \hat{a}_2^{\dagger N} \left|0\right\rangle / \sqrt{N}$$

Our results don’t change