Integrability
At Large Quantum Number

Simeon Hellerman
Kavli Institute for the Physics and Mathematics of the Universe
The University of Tokyo

Based on work in progress by
M. Dodelson, SH, M. Watanabe, and M. Yamazaki.

Berkeley Week at Kavli IPMU
Kashiwa, Chiba, Japan, January 17, 2020

The Large Quantum Number Expansion

This week you have heard from Susanne Reffert and Domenico Orlando about the simplification of otherwise-strongly-coupled quantum systems in the limit of large quantum number, which I'll refer to generically as "J".

The Large Quantum Number Expansion

The general setting for this subject is the simplification of otherwise-strongly-coupled quantum systems in the limit of large quantum number, which I'll refer to generically as "J". My particular focus will be the case where "J" is the weight of an $SO(N)$ representation in the $D = 2$ $O(N)$ model, but I will mention some other cases for context .

The Large Quantum Number Expansion

By "otherwise strongly coupled" I'll mean outside of any simplifying limit where the theory becomes semiclassical for other reasons or possibly in a simplifying limit but with the quantum number taken so large that the system behaves differently than you might have expected despite being weakly coupled.

The Large Quantum Number Expansion

The Large Quantum Number Expansion

The primary question in such a talk is, is this even a subject?

The Large Quantum Number Expansion

The Large Quantum Number Expansion

The answer is, yes, and in some sense it's an old one; many examples have appeared in the literature going far back into the past. Recently there have been a number of groups focusing on systematizing this point of view and applying it more broadly.

The Large Quantum Number Expansion

Pre-history:

- Atomic hypothesis [Democritus]
- Correspondence principle [Bohr]
- Large spin in hadron spectrum [Regge]
- Macroscopic limit [Deutsch] [Srednicki]

History:

- $\mathcal{N} = 4$ SYM at large R-charge [Bernstein, Maldacena, Nastase]
- and large spin [Belistsky, Basso, Korchemsky, Mueller], [Alday, Maldacena]
- Large-spin expansion in general CFT from light-cone bootstrap [Komargodski-Zhiboedov], [Fitzpatrick, Kaplan, Poland, Simmons-Duffin], [Alday 2016]
- Large-spin expansion in hadrons [SH, Swanson], [SH, Maeda, Maltz, Swanson], [Caron-Huot, Komargodski, Sever, Zhiboedov], [Sever, Zhiboedov]

The Large Quantum Number Expansion

Modern:

- Large-charge expansion in generic systems with abelian global symmetries: [SH, Orlando, Reffert, Watanabe 2015], [Monin 2016], [Monin, Pirtskhalava, Rattazzi, Seibold 2016], [Loukas 2016]
- Nonabelian symmetries: [Alvarez-Gaume, Loukas, Orlando, Reffert 2016], [Loukas, Orlando, Reffert 2016], [SH, Kobayashi, Maeda, Watanabe 2017], [Loukas 2017], [SH, Kobayashi, Maeda, Watanabe 2018]
- Charge AND spin: [Cuomo, de la Fuente, Monin, Pirtskhalava, Rattazzi 2017]
- Topological charge: [Pufu, Sachdev 2013] [Dyer, Mezei, Pufu, Sachdev 2015], [de la Fuente 2018]
- EFT connection with bootstrap: [Jafferis, Mukhametzhanov, Zhiboedov 2017]
- Large charge limit in gravity: [Nakayama, Nomura 2016], [Loukas, Orlando, Reffert, Sarkar 2018]

Vacuum manifolds $\Leftrightarrow$ chiral rings at large-R-charge:

- $D = 3$, $\mathcal{N} \geq 2$ theories : [SH, Maeda, Watanabe 2016]
- $D = 4$, $\mathcal{N} \geq 2$ theories : [SH, Maeda 2017], [SH, Maeda, Orlando, Reffert, Watanabe 2017]
- Double-scaling limit in lagrangian $\mathcal{N} \geq 2$ theories: [Bourget, Rodriguez-Gomez, Russo 2018]

Large-Scale Structure of Theory Space

- The goals of the LQNE are largely to answer the same questions as the conformal bootstrap:
- Learn to systematically and efficiently analyze QFT (in practice usually CFT) that have no exact solution in terms of explicit functions.

Large-Scale Structure of Theory Space

- We'd all like to know "what does theory space look like": Generic theories, generic amplitudes.
- This is a very consequential question for field theory, mathematics, quantum gravity, and cosmology.
- Most theories are not integrable, and we need to learn how to attack them in general circumstances.

Large charge J in the $O(2)$ model

- Simplest example: The conformal Wilson-Fisher $O(2)$ model at large O(2) charge J.
- Canonical question: What is the dimension Δ_J of the lowest operator $\mathcal{O}_J$ at large J?
- Translated via radial quantization: Energy of lowest state of charge J on unit S^2?
- Renormalization-group analysis reveals the low-lying large-charge sector is described by an EFT of a single compact scalar χ, which can be thought of as the phase variable of the complex scalar ϕ in the canonical UV completion of the $O(2)$ model.

Large charge J in the $O(2)$ model

- The leading-order Lagrangian of the EFT is remarkably simple:

$$\mathcal{L}_{\mathrm{leading-order}} = b|\partial\chi|^3$$

- The coefficient b is not something we know how to compute analytically; nonetheless the simple structure of this EFT has sharp and unexpected consequences.
- The immediate consequence of the structure of the EFT is that the lowest operator is a scalar, of dimension

$$\Delta_J \simeq c_{\frac{3}{2}} J^{\frac{3}{2}} ,$$

where $c_{\frac{3}{2}}$ has a simple expression in terms of b.

Large charge J in the $O(2)$ model

- The leading-order EFT predicts more than just the leading power law, because quantum loop effects in the EFT are suppressed at large J, so the EFT can be quantized as a weakly-coupled effective action with effective loop-counting parameter $J^{-\frac{3}{2}}$.
- For instance we can compute the entire spectrum of low-lying excited primaries.
- The dimensions, spins, and degeneracies of the excited primaries, are those of a Fock space of oscillators of spin ℓ, with $\ell \geq 2$.

Large charge J in the $O(2)$ model

- The propagation speed of the χ-field is equal to $\frac{1}{\sqrt{2}}$ times the speed of light.
- So the frequencies of the oscillators are

$$\omega_\ell = \frac{1}{\sqrt{2}} \sqrt{\ell(\ell+1)}\ , \qquad\qquad \ell \geq 1\ .$$

- The $\ell = 1$ oscillator is also present, but exciting it only gives descendants; the leading-order condition for a state to be a primary is that there be no $\ell = 1$ oscillators excited.
- So for instance, the first excited primary of charge J always has spin $\ell = 2$ and dimension $\Delta_J^{(1)} = \Delta_J + \sqrt{3}$.

Large charge J in the $O(2)$ model

- Subleading terms can be computed as well.
- These depend on higher-derivative terms in the effective action with powers of $|\partial\chi|$ in the denominator.
- These counterterms have a natural hierarchical organization in J :

Large charge J in the $O(2)$ model

- At any given order in derivatives, there are only a finite number of such terms.
- As a result, at a given order in the large-J expansion, only a finite number of these terms contribute.
- Since there are far more observables than effective terms, there are an infinite number of theory-independent relations among terms in the asymptotic expansions of various observables.

Large charge J in the $O(2)$ model

- Our gradient-cubed term is the only term allowed by the symmetries at order $J^{\frac{3}{2}}$, and there is only one other term contributing with a nonnegative power of J, namely

$$\mathcal{L}_{J^{+\frac{1}{2}}} = b_{\frac{1}{2}} \left[|\partial\chi| \, \mathtt{Ric}_3 + 2 \, \frac{(\partial\,|\partial\chi|)^2}{|\partial\chi|} \right]$$

- In particular, there are no terms in the EFT of order J^0, with the result that the J^0 term in the expansion of Δ_J is calculable, independent of the unknown coefficients in the effective lagrangian.

Large charge J in the $O(2)$ model

- Specifically, the formula for Δ_J takes the form

$$\Delta_J = c_{\frac{3}{2}} \, J^{+\frac{3}{2}} \quad + \quad c_{\frac{1}{2}} \, J^{+\frac{1}{2}} \quad -0.0937256 \cdots$$

up to terms vanishing at large J.

Large charge J in the $O(2)$ model

- Specifically, the formula for Δ_J takes the form

$$\Delta_J = c_{\frac{3}{2}}\, J^{+\frac{3}{2}} \quad + \quad c_{\frac{1}{2}}\, J^{+\frac{1}{2}}$$

up to terms vanishing at large J.

Large charge J in the $O(2)$ model

- Specifically, the formula for Δ_J takes the form

$$\Delta_J = c_{\frac{3}{2}} \, J^{+\frac{3}{2}} \quad + \quad c_{\frac{1}{2}} \, J^{+\frac{1}{2}} \quad -0.0937256 \cdots$$

up to terms vanishing at large J.

Large charge J in the $O(2)$ model

- Specifically, the formula for Δ_J takes the form

$$\Delta_J = c_{\frac{3}{2}} \, J^{+\frac{3}{2}} \quad + \quad c_{\frac{1}{2}} \, J^{+\frac{1}{2}}$$

up to terms vanishing at large J.

Large charge J in the $O(2)$ model

- Specifically, the formula for Δ_J takes the form

$$\Delta_J = c_{\frac{3}{2}}\, J^{+\frac{3}{2}} \quad + \quad c_{\frac{1}{2}}\, J^{+\frac{1}{2}} \quad -0.0937256\cdots$$

up to terms vanishing at large J.

Large charge J in the $O(2)$ model

- Specifically, the formula for Δ_J takes the form

$$\Delta_J = c_{\frac{3}{2}}\, J^{+\frac{3}{2}} \quad + \quad c_{\frac{1}{2}}\, J^{+\frac{1}{2}}$$

up to terms vanishing at large J.

Large charge J in the $O(2)$ model

- Specifically, the formula for Δ_J takes the form

$$\Delta_J = c_{\frac{3}{2}} \, J^{+\frac{3}{2}} \quad + \quad c_{\frac{1}{2}} \, J^{+\frac{1}{2}} \quad -0.0937256 \cdots$$

up to terms vanishing at large J.

Large charge J in the $O(2)$ model

- Specifically, the formula for Δ_J takes the form

$$\Delta_J = c_{\frac{3}{2}}\, J^{+\frac{3}{2}} \quad + \quad c_{\frac{1}{2}}\, J^{+\frac{1}{2}}$$

up to terms vanishing at large J.

Large charge J in the $O(2)$ model

- This universal term and the other universal large-J relations in the $O(2)$ model don't have any fudge factors or adjustable parameters;
- Given the identification of the universality class, these values and relations are universal and absolute;
- Similar predictions have been made for OPE coefficients [Monin, Pirtskhalava, Rattazzi, Seibold 2016]

FAQ

- You might think that there is something "*weird*" or "*inconsistent*" or "*uncontrolled*" about a Lagrangian like $\mathcal{L} = |\partial\chi|^3$.
- So, let me anticipate some frequently asked questions:

FAQ

- Q: Isn't this Lagrangian singular?? It is a nonanalytic functional of the fields, so when you expand it around $\chi = 0$, you will get ill-defined amplitudes.
- A: Yes, but you aren't supposed to use the Lagrangian there. It is only meant to be expanded around the large charge vacuum , which at large J is the classical solution

$$\chi = \mu t,$$

with

$$\mu = O(\sqrt{\rho}) = O(J^{\frac{1}{2}}) \ .$$

- The expansion into vev and fluctuations carries a suppression of μ^{-1} or more for each fluctuation.

FAQ

- Q: Isn't this effective theory ultraviolet-divergent ? That means that loop corrections are incalculable and observables are meainingless beyond leading order.
- A: No. The EFT is quantized in a limit where loop corrections are small . Our UV cutoff Λ for the EFT is taken to satisfy

$$E_{\rm IR} = R_{\rm S^2}^{-1} \quad \ll \quad \Lambda \quad \ll E_{\rm UV} = \sqrt{\rho} \propto J^{+\frac{1}{2}}\, R_{\rm S^2}^{-1}$$

- Loop divergences go as powers of $\Lambda^3/\rho^{\frac{3}{2}} \ll 1$, and are proportional to nonconformal local terms which are to be subtracted off to maintain conformal invariance of the EFT.

FAQ

- Q: OK but then don't the counterterms ruin everything? Don't they render the theory incalculable?
- A: No. As usual in EFT the counterterm ambiguities of subtraction correspond one-to-one with terms in the original action allowed by symmetries;
- As we've mentioned there are only a finite and small number of those contributing at any given order in the expansion, and at some orders there are no ambiguities at all.

FAQ

- Q: You're saying that every CFT with a conserved global charge has this exact same asymptotic expansion. But here's a counterexample! ⟨ `describes theory SH didn't say anything about` ⟩ Doesn't that mean you're a crackpot?
- A: No. I didn't make any claim that broad. Our RG analysis applies to many but not all CFT with a conserved global charge. More generally, CFT can be organized into large-charge universality classes.
- For instance, free complex fermions as well as free complex scalars in $D = 3$ are in different large-J universality classes.
- The large-J universality class of the O(2) model contains many other interesting theories, such as
  - The $\mathbb{CP}(n)$ models at large topological charge;
  - The $D = 3, \mathcal{N} = 2$ superconformal fixed point for a chiral superfield with $W = \Phi^3$ superpotential, at large R-charge;
  - Probably others $\circ \circ \circ$

Other large-J universality classes

- Many other interesting universality classes in $D = 3$:
- Large Noether charge in the higher Wilson-Fisher $O(N)$ [Alvarez-Gaumé , Loukas, Reffert, Orlando 2016] and $U(N)$ models;
- Also the $\mathbb{CP}(n)$ [de la Fuente] and higher Grassmanian models real and complex ; [Loukas, Reffert, Orlando 2017]
- Large baryon charge in the $SU(N)$ Chern-Simons-matter theories;
- Large monopole charge in the $U(N)$ Chern-Simons-matter theories;
- Of course these last two are dual to one another and would be interesting to investigate.

Confirmation of the large-$\mathcal{J}$ expansion

- Though precise bootstrap results only exist up to $J = 2$, note that the values of the EFT parameters calculated from Monte Carlo calculation give

  $$\Delta_{J=2} = 1.236(1) \qquad\qquad [\text{Monte Carlo } + \text{ large} - \text{J}]$$

  which one can compare to the bootstrap result

  $$\Delta_{J=2} = 1.236(3) \qquad\qquad [\text{bootstrap}]\ .$$

- There are other high-precision agreements between large-J theory and MC simulation in [Banerjee, Chandrasekharan, Orlando 2017].

Confirmation of the large-$\mathcal{J}$ expansion

- Moving beyond the $O(2)$ case to other models in the same large- J universality class, one can look at dimensions of operators carrying topological charge J in the $\mathbb{CP}(n)$ models.
- This analysis was done by [de la Fuente 2018], using a combination of large-N methods and numerical methods, with the result

$$\Delta_J^{\mathbb{CP}(n)} = c_{\frac{3}{2}}(n)\, J^{\frac{3}{2}} + c_{\frac{1}{2}}(n)\, J^{\frac{1}{2}} + c_0 + O(J^{-\frac{1}{2}}) \ ,$$

  where the first two coefficients depend on the n of the model, but the J^0 term does not; in particular he finds

$$c_0 = -0.0935 \pm 0.0003 \ ,$$

  as compared to the EFT prediction

$$c_0 = -0.0937 \cdots \ .$$

- So the error bars are less than one percent , and the EFT prediction sits inside of them.

Basics of the $O(N)$ model at large quantum number

- So now I'll describe some work in progress by me together with M. Dodelson, M. Watanabe and M. Yamazaki on the $O(N)$ model in $D = 2$.
- This model is nonconformal and quantum mechanically integrable .
- For context le me start with some review of the O(N) models in general.

Basics of the $O(N)$ model at large quantum number

- You've heard from Susanne Reffert and Domenico Orlando about some basics of the large quantum number expansion and in particular the case of the conformal point of the $O(N)$ models in $D = 3$.
- In contrast to the $O(2)$ model, the symmetry group is nonabelian so there is no unique charge.
- The large quantum number limit is most naturally described by taking the lowest-energy state in a given representation of the symmetry group with large weights of the representation.
- Describing the LQN limit in these terms we find some striking things:

Basics of the $O(N)$ model at large quantum number

- First of all, in contrast to the case of $O(2)$ model, a generic large-weight representation of the $O(N)$ model does not have a have a homogeneous ground state for $N \geq 4$.
- A fully homogeneous ground state corresponds only to the traceless totally symmetric tensor representation of $O(N)$.
- All other representations have ground states that are interpreted either as inhomogeneous semiclassical states or quantum excitations on top of a homogeneous ground state, depending whether the weights are taken large in fixed ratio or taken to be small deviations from the weights of a large-order symmetric tensor representation.

Comments on the derivative expansion in the O(N) model

- Let me say a little bit more about the derivative expansion in these theories and its organization in the large quantum number limit .
- In the case of the $O(2)$ model, the natural organization is in terms of the phase variable χ and there is a parametric suppression of higher derivative terms in low-lying states of large $O(2)$ charge.
- In the case of the $O(N)$ model, analogous statements apply but the systematics of the derivative expansion is more involved because there are more degrees of freedom and there is no canonical parametrization of the coset S^{N-1}.

Comments on the derivative expansion in the O(N) model

- The explicit demonstration of the parametric suppression of higher-derivative terms is not as simple, but there is a simple argument to show that there is always a controlled derivative expansion.
- The symmetric tensor ground state can be realized as the overall ground state of the Hamiltonian with a chemical potential added.
- Then, the conventional low-energy expansion of the $O(N)$ model with chemical potential is equivalent to the derivative expansion of the large-charge EFT about the symmetric tensor ground state .

Comments on the derivative expansion in the O(N) model

- We will mostly avoid the description in terms of chemical potentials however.
- It is completely equivalent to the description in terms of a time-dependent solution of the unmodified Hamiltonian, but the description in terms of chemical potentials obscures the underlying Lorentz invariance, full nonabelian symmetry, and background independence, by which we mean that the infrared-inhomogeneous ground states of the other representations are described by the same effective action as the symmetric tensor ground state.

Nonconformality and integrability in the $O(N)$ model in D=2

- Now we turn to the case of $D = 2$, where the O(N) model is asymptotically free and does not have a conformal fixed point.
- Instead, the model flows to a theory with a mass gap .

Nonconformality and integrability in the $O(N)$ model in D=2

- Despite the absence of a conformal point, the $D = 2$ case of the $O(N)$ model is still tractable to a large quantum-number analysis because it has the remarkable simplifying property of integrability .
- So now let me tell you some basics about integrability in the $O(N)$ model in $D = 2$.

Nonconformality and integrability in the $O(N)$ model in D=2

- Our most convenient description is in terms of N real fields ϕ^a with a constraint $\phi^a\phi^a = 1$.
- Then the Lagrangian is simply

$$\mathcal{L} = \frac{1}{g^2}\,(\partial_+\phi^a)(\partial_-\phi^a) + \lambda\,(\phi^2 - 1)\ ,$$

where $x^\pm$ are light-cone coordinates and λ is a Lagrange multiplier enforcing the constraint.

Nonconformality and integrability in the $O(N)$ model in D=2

- In these variables the Nother currents are given simply as $J_{\pm}^{[ab]} = \phi^a \phi^b_{,\pm} - [ab]$
- These currents are of course present in the $O(N)$ models in any dimension, but in the special case of $D = 2$ we can use them to construct an infinite dimensional symmetry algebra that constrains the theory to the point of making it completely integrable.

Nonconformality and integrability in the $O(N)$ model in D=2

- The construction of the symmetry algebra is given in terms of a one-parameter family of connections $\mathcal{A}^{[ab]} = \mathcal{A}_{+}^{[ab]} dx^{+} + \mathcal{A}_{-}^{[ab]} dx^{-}$.
- The connection $\mathcal{A}_{\pm}^{[ab]}$ is not a fixed background field nor an independent dynamical variable but rather a composite field constructed from the dynamical fields ϕ and their derivatives.
- This connection, called a Lax connection, is flat for on-shell configurations, that is,

$$d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0 \qquad \text{on shell}$$

Nonconformality and integrability in the $O(N)$ model in D=2

- Explicitly, the formula for the Lax connection can be decomposed conveniently into parity-even and parity-odd pieces:

$$\mathcal{A}^{[ab]}_{\pm} = \mathcal{A}^{[ab]}_{\pm\ (\mathrm{ev})} + \mathcal{A}^{[ab]}_{\pm\ (\mathrm{od})} \ ,$$

with

$$\mathcal{A}^{[ab]}_{\pm\ (\mathrm{ev})} \equiv c_{\mathrm{ev}} \left(\phi^a \phi^b_{,\pm} - [ab]\right) \ , \qquad \mathcal{A}^{[ab]}_{\pm\ (\mathrm{od})} \equiv \pm c_{\mathrm{od}} \left(\phi^a \Pi^b_{,\pm} - [ab]\right) \ ,$$

$$\Pi_{\pm} \equiv \frac{\delta \mathcal{L}}{\delta \phi_{,\mp}} = \frac{1}{g^2}\, \phi_{,\pm}$$

and with the two parameters $c_{\mathrm{ev}}, c_{\mathrm{od}}$ parametrized by

$$c_{\mathrm{ev}} = 1 + \cosh[\lambda] \ , \qquad c_{\mathrm{od}} = g^2 \sinh[\lambda] \ .$$

Nonconformality and integrability in the $O(N)$ model in D=2

- The existence of this flat connection implies the existence of an infinite hierarchy of nonlocal conserved charges .
- In infinite volume the algebra of conserved charges is called the Yangian.
- In finite volume, which we will focus on, the conserved charges are a subalgebra of the Yangian called the Bethe subalgebra.

Nonconformality and integrability in the $O(N)$ model in D=2

- The explicit form of the Bethe subalgebra is

$$Q_k \equiv \frac{d^k}{d\lambda^k}\,\mathrm{tr}\left[\mathrm{P}\,\exp\int dx_1\,\widehat{\mathcal{A}}_1\right],$$

where:

  - λ is the spectral parameter parametrizing the family of Lax connections;
  - the notation P exp denotes the path-ordered exponential; and
  - $\widehat{A}_\mu = \widehat{A}_\mu(\lambda)$ is the antihermitean matrix-valued connection component.

Nonconformality and integrability in the $O(N)$ model in D=2

- The facts I have told you all refer to the classical two-derivative action for the $O(N)$ model.
- At the quantum level it is clear that the story must change to some extent.
- For instance, the inverse coupling g^{-2} multiplying the action is no longer a constant but runs logarithmically with energy at short distances,

$$g^{-2}(M^2) \simeq g_0^{-2} + \beta_0 \log[M^2/M_0^2]$$

and runs strongly in the infrared.

Nonconformality and integrability in the $O(N)$ model in D=2

- Despite the running, the integrability is known to persist at the quantum level, and decades of study have uncovered many interesting facts about the $O(N)$ model at the quantum level.
- The quantum integrability has been used to solve many observables exactly at the quantum level.
- For the most part, an S-matrix for massive particles is used as the primary "exact" object at the quantum level, rather than directly replacing the classical Lax connection with a quantum version of itself.

Nonconformality and integrability in the $O(N)$ model in D=2

- However we will be exploring the large quantum-number regime in which we expect physics to be semiclassical for low-lying states in finite volume.
- In this regime we will encounter a quantum-corrected version of the Lax connection.

Nonconformality and integrability in the $O(N)$ model in D=2

- So we will now describe the large-quantum-number limit of the $O(N)$ model in the same way one describes the $O(2)$ model model at large charge;
- That is, we describe it in terms of an effective Lagrangian with Wilsonian cutoff Λ, in the limit where the gradients $\partial\phi$ are much larger than Λ, and the covariant higher derivatives $\nabla^k\phi$ are small in units of $\partial\phi$.
- As in the $O(2)$ model, we expect higher-derivative corrections and quantum loops to make parametrically suppressed contributions at large charge.

Nonconformality and integrability in the $O(N)$ model in D=2

- Again, this limit is really equivalent to a conventional low-energy limit for the system in terms of a chemical potential or a more general flat background gauge field, so there is no real issue of principle in terms of controlling the derivative expansion;
- but we avoid this way of describing the system because it obscures the underlying Lorentz and nonabelian global symmetries as well as the background-independence among ground states in large-weight representations.

Nonconformality and integrability in the $O(N)$ model in D=2

- Since we anticipate that higher-derivative corrections and quantum loops will make suppressed contributions to low-lying states at large quantum number, it is useful to separate the effective Lagrangian into pure-gradient terms independent of the cutoff, and other terms, which contain positive powers of the cutoff Λ and higher covariant derivatives $\nabla^k \phi$ of the dynamical fields:

$$\mathcal{L} = \mathcal{L}\Big|_{\text{gradient only, cutoff-independent}} + \mathcal{L}\Big|_{\text{other}} ,$$

with all terms in "other"making parametrically suppressed contributions at large quantum number.

Nonconformality and integrability in the $O(N)$ model in D=2

- The dominance of the pure gradient terms at large quantum number surely simplifies the form of the effective Lagrangian a great deal; but the most general pure-gradient lagrangian is still considerably more complicated than in the conformal $O(2)$ model in $D = 3$.
- In any dimension, the conformal $O(2)$ model has only one pure-gradient term, namely $|\partial\chi|^D$.
- But even in the conformal case, the O(N) model has more invariants for $N \geq 4$.

Nonconformality and integrability in the $O(N)$ model in D=2

- Defining

$$K \equiv \phi^a_{,\mu}\phi^{a,\mu}, \qquad \tilde{U} \equiv (\phi^a_{,\mu}\phi^b_{,\nu} - [ab])^2 \ ,$$

$$U \equiv \tilde{U}/K^2 \ ,$$

it is easy to see the invariants K and U are independent for $N \geq 4$ and so the most general conformal effective Lagrangian at the pure-gradient level is of the form

$$\mathcal{L} = K^{D/2} f(U) \ ,$$

for some arbitrary function f.

Nonconformality and integrability in the $O(N)$ model in D=2

- Dropping the requirement of conformal invariance allows further generality at the pure-gradient level, allowing an effective action of the form

$$\mathcal{L} = K^{D/2}\mathcal{F}[K, U] \ .$$

- Depending on the dimension D there may be still other invariants and a pure-gradient effective lagrangian depending on three or more variables.

Nonconformality and integrability in the $O(N)$ model in D=2

- In $D = 2$ on the other hand one can show that K, U are the only independent invariants one can construct out of the gradient of ϕ .
- So the most general pure-gradient effective lagrangian one can construct, is of the form

$$\mathcal{L} = \mathcal{F}[K, U] \ ,$$

where $\mathcal{L}$ can depend on the running coupling g^2 or equivalently on the dynamical scale $M_{\rm dyn}$ controlling the mass gap of the system.

Nonconformality and integrability in the $O(N)$ model in D=2

- As in the case of the conformal $O(2)$ model in $D = 3$ this pure-gradient Lagrangian contains a great deal of information about leading-order properties of the system at large quantum number.
- For instance, the free energy of the ground state in infinite volume at fixed chemical potential μ is given by the functional form of $\mathcal{L} = \mathcal{F}[K, U]$ evaluated at $U = 0$ and $K = \mu^2$:

$$\mathcal{F}[\mu] = \mathcal{L}\Big|_{U=0,\ K=\mu^2} .$$

- This quantity can be directly legendre transformed to the energy density at finite charge density in infinite volume:

$$\rho = \mathcal{F}'[\mu] , \qquad \mathcal{E}(\rho) = \rho\mu - \mathcal{F} .$$

Nonconformality and integrability in the $O(N)$ model in D=2

- The quantity $\mathcal{E}(\rho)$ in turn, gives the leading large- k limit of the energy of the rank- k symmetric tensor ground state in finite volume v, with $\rho \equiv k/v$ held fixed:

$$E\Big|_{\text{rank k}} \propto \frac{k^2}{v} + (\text{subleading in } k) \ .$$

- Note that this identification is nonperturbative as a function of g^2 or equivalently as a function of $M^2_{\rm dyn}/\rho^2$.
- Just as in the conformal $O(2)$ model in $D = 3$, the leading-order classical EFT action resums an infinite series of quantum corrections and in particular all those that contribute to leading-order large volume quantities at fixed density or fixed chemical potential.

Nonconformality and integrability in the $O(N)$ model in D=2

- So you can ask, then what do the quantum effects in the EFT compute?
- They contribute subleading large-quantum-number corrections to observables in finite volume at fixed density.

Nonconformality and integrability in the $O(N)$ model in D=2

- So for instance the one-loop correction to the rank- k symmetric tensor ground state, gives the first subleading term in the large-k expansion of the ground state energy.
- This term is straightforwardly computable as a Casimir energy and scales as k^0 at large k:

$$E\Big|_{subleading} = -\frac{\pi c_s}{6v} \ ,$$

where c_s is the speed of sound,

$$c_s^2 = \frac{\mathcal{F}_{,K}}{\mathcal{F}_{,K} + 2K\mathcal{F}_{,KK}}\Big|_{K=\mu^2,\ U=0}$$

Nonconformality and integrability in the $O(N)$ model in D=2

- All the leading-order and much of the first subleading-order large-k physics of the symmetric tensor ground state depends only on $\mathcal{F}[K, U]$ at $U = 0$, or equivalently the free energy $\mathcal{F}[\mu]$ in infinite volume at fixed chemical potential.
- Remarkably, the full form of $\mathcal{F}[\mu]$ can be determined algorithmically using integrability via the thermodynamic Bethe ansatz starting from the exact quantum S-matrix originally worked out by Polyakov in the $1970s$.

Nonconformality and integrability in the $O(N)$ model in D=2

- The form of $\mathcal{F}[\mu]$ has been worked out to several orders in the in recent work of Marino and Reis, and in the asymptotically free regime takes the form of a series in inverse powers of $\ell \equiv \log[\mu/M_{\rm dyn}]$.
- The leading behavior of $\mathcal{F}$ is

$$\mathcal{F} = \beta_0 \log[\mu/M_{\rm dyn}] + \frac{1}{g_0^2} + O[(\log[\mu/M_{\rm dyn}])^{-1}] \ .$$

  where β_0 is the one-loop beta function coefficient.
- This expression agrees with perturbation theory and with the Ward identity for broken scale invariance .

Integrability and the dependence on U

- In order to compute large-quantum-number behaviors of representations beyond the symmetric tensor, even at leading order in the large-quantum-number expansion, we need to know some information about the dependence of $\mathcal{F}[K, U]$ on U away from the locus $U = 0$.
- This information is not contained directly in the free energy $\mathcal{F}[\mu]$.
- But remarkably, we will be able to use integrability in a different way, to calculate the U-dependence of $\mathcal{F}[K, U]$ given the functional form $\mathcal{F}[\mu]$.

Integrability and the dependence on U

- This is where we use the infinite dimensional Yangian symmetry generated by the holonomies of the Lax connection $\mathcal{A}^{[ab]}$.
- The quantum integrability of the $O(N)$ model means that the Yangian symmetry is preserved quantum mechanically rather than merely classically.
- And therefore the Yangian symmetry must be present in the Wilsonian effective action .

Integrability and the dependence on U

- As we have said earlier, all observables for low-lying states above the symmetric tensor ground state are computed at leading order by the pure gradient, cutoff-independent piece of the Lagrangian $\mathcal{F}[K, U]$.
- So, the Yangian symmetry must be present already at the classical level in the $\mathcal{F}[K, U]$ Lagrangian.
- We will now see that the Yangian symmetry is absent for a generic $\mathcal{F}[K, U]$ Lagrangian.
- Therefore, quantum integrability will impose a nontrivial constraint on the functional form of $\mathcal{F}[K, U]$.

Integrability and the dependence on U

- A sufficient and necessary$^{(*)}$ condition for preservation of the Yangian symmetry, is the existence of a one-parameter family of Lax connections $\mathcal{A}_{\pm}^{[ab]}$.
- So we can write the most general possible form of a connection $\mathcal{A}_{\pm}^{[ab]}$ constructed from the dynamical fields ϕ^a in the $O(N)$ model, which is flat for any configuration ϕ satisfying the classical equations of motion of a $\mathcal{F}[K, U]$ Lagrangian.

Integrability and the dependence on U

- Analyzing the most general possible form one can write for $\mathcal{A}^{[ab]}$ with the correct symmetry properties, we find again

$$A_{\pm}^{[ab]} = A_{\pm}^{[ab](\mathrm{ev})} + A_{\pm}^{[ab](\mathrm{od})} \ ,$$

with

$$A_{\pm}^{[ab](\mathrm{od})} \equiv \pm c_{\mathrm{od}}\, J_{\pm}^{[ab]} \ ,$$

$$A_{\pm}^{[ab(\mathrm{ev})]} \equiv c_{\mathrm{ev}}\, \phi^{a} \phi^{b}_{,\pm} - [ab]$$

for some constants $c_{\mathrm{ev}}, c_{\mathrm{od}},$ where

$$J_{\pm}^{[ab]} \equiv \phi^{a} \Pi_{\pm}^{b} - [ab] \ , \qquad \Pi \pm^{b} \equiv \frac{\delta \mathcal{L}}{\delta \phi^{b}_{,\mp}} \ .$$

Integrability and the dependence on U

- This form of the Lax connection is formally the same as in the microscopic theory with only the form of the Noether current depending on the form of the Lagrangian.
- Note that this is not some random ansatz for the Lax connection; it is provably the most general form of the Lax connection for a classical Lagrangian depending on gradients only.

Integrability and the dependence on U

- This form is necessary but not sufficient for flatness of $\mathcal{A}$ on shell.
- In particular, this form is equivalent to the cancellation of the second-derivative term in the curvature of the Lax connection.
- The curvature of the Lax connection also contains a pure-gradient term, whose vanishing imposes the additional conditions

$$c_{\mathrm{od}} = \frac{\sinh[\lambda]}{c_F} \ , \qquad\qquad c_{\mathrm{ev}} = \cosh[\lambda] - 1 \ .$$

and

$$\mathcal{F}_{,K}^2 - \frac{4\,(1+U)}{K}\,\mathcal{F}_{,U}\,\mathcal{F}_{,K} + 4\,\frac{U(1+U)}{K^2}\,\mathcal{F}_{,U}^2 = c_F^2 \ .$$

Integrability and the dependence on U

- This first-order nonlinear ODE for $\mathcal{F}$ can be evolved straightforwardly in the U-direction given a "boundary condition" at $U = 0$.
- So given a functional form of $\mathcal{F}[K, 0]$ we can straightforwardly write a series solution in U:

$$\mathcal{F}[K, U] = \sum_{m \geq 0} \mathcal{F}_{(m)}[K]\, U^m \ ,$$

with

$$\mathcal{F}_{(1)}[K] = \frac{1}{4}\, K\, \mathcal{F}_{(0)}{}'[K] \ ,$$

$$\mathcal{F}_{(2)}[K] = -\frac{1}{16}\, K\, \mathcal{F}_{(0)}{}'[K] + \frac{1}{32}\, K^2\, \mathcal{F}''_{(0)}[K] \ ,$$

and so forth.

Integrability and the dependence on U

- Since the semiclassical energies of the non-symmetric-tensor ground states depend on the Taylor expansion at $U = 0$, these coefficients are physically meaningful and can be checked in principle in the full $O(N)$ model at the quantum level.
- Furthermore, the large- K expansion corresponds to the asymptotically free regime of the $O(N)$ model, so we can check these predictions directly in perturbation theory.

Integrability and the dependence on U

- So in particular, the U^1 term contributes to the leading-order energies of first-excited states above the symmetric tensor ground state, such as representations with k boxes in the first column and one box in the second column of the Young tableau.
- These energies can be checked against perturbation theory at leading order at large μ^2/M_{dyn}^2.
- The calculation is still in progress!
- But I am still giving it publicly in a conference talk, which I hope convinces you I am relatively confident in the consistency of the LQN methods I and Reffert and Orlando have been telling you about this week.

Conclusions

- The LQN expansion gives an analytically controlled way to compute QFT data outside of any other sort of simplifying limit but can be checked in known limits and against other methods.
- For integrable theories such as the D=2 O(N) model we have tools to constrain the LQN action .
- These constraints give sharp predictions for physical observables that can be checked directly in various limits.
- Analysis of more examples is sure to yield further interesting surprises about the large-scale structure of theory space .
- Thank you.