On the infinite gradient-flow for the domain-wall formulation of chiral lattice gauge theories

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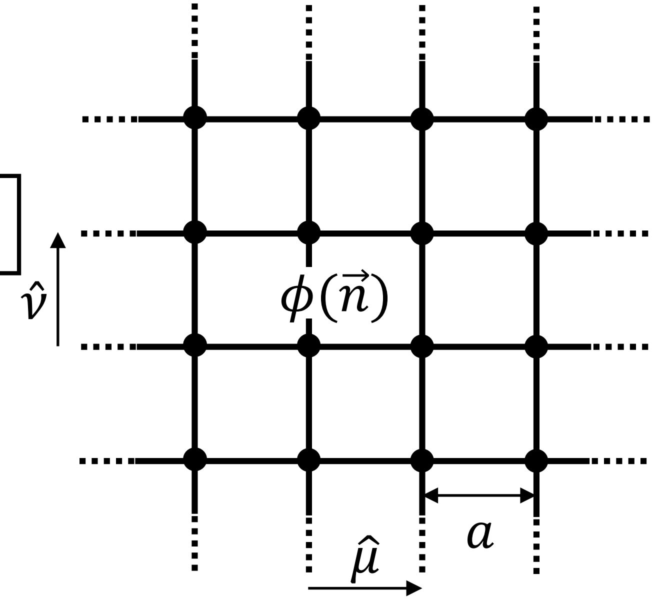
1. Introduction

Background



Lattice theory: discretization of a spacetime

expectation value = integration w.r.t. $\phi(\vec{n})$

$$\langle \mathcal{O} \rangle = \int \left(\prod d\phi(\vec{n})\right) \mathcal{O}(\phi) \exp(-S(\phi))$$

approximated by the finite DOF

→ **numerical simulation**

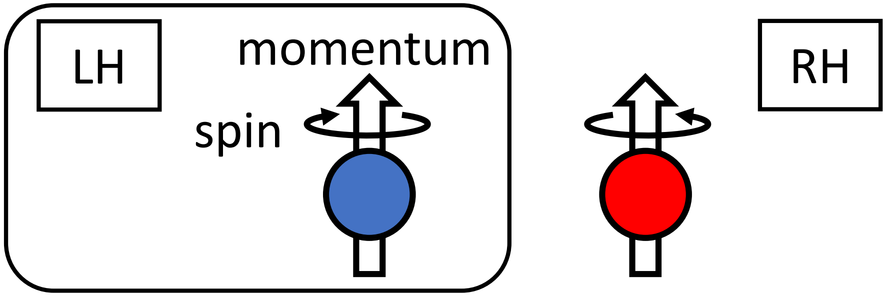
Big problem in lattice gauge theories

(Non-Abelian) chiral gauge theories are not formulated on the lattice

- Standard model is a chiral gauge theory

Chiral gauge theories

LH- and RH-fermions couple to gauge field differently.

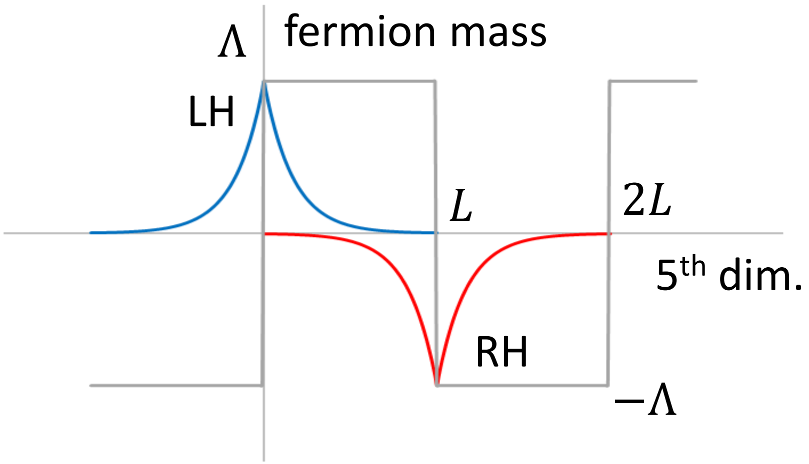


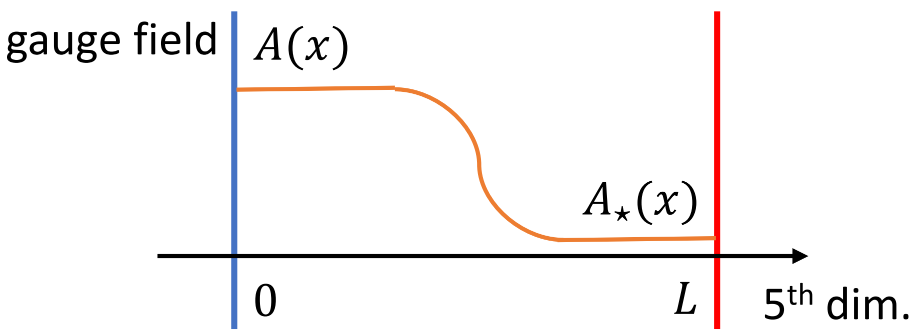
Grabowska-Kaplan's formulation

Proposal to formulate chiral gauge theories on the lattice

[Grabowska-Kaplan, 2015]

Domain-wall [Kaplan, 1992] $+$ Gradient flow [Luscher, 2010]





- LH and RH are localized around defects
- gauge fields are damped along 5^{th} dim.

→ LH fermions couple to the original gauge field $A_\mu(x)$

Plan of this talk

Problems of GK's proposal

- Formulation and properties of gradient flow on the lattice ?
- Infinite gradient-flow maps gauge fields non-locally

⇓ $\mathrm{U}(1)$ gauge theory

We examine

- The formulation of gradient flow which satisfies "admissibility"
- Relation of GK's formulation and Luscher's one
- Locality of GK's formulation

<u>Outline</u>

1. Introduction
2. Luscher's formulation
3. Gradient flow
4. Locality
5. Summary

2. Luscher's formulation

Luscher's formulation

U(1) chiral lattice gauge theories with exact gauge invariance

[Luscher, 1999]

Ginsparg-Wilson relation ("chiral symmetry" on the lattice)

$$\gamma_5 D + D\hat{\gamma}_5 = 0, \qquad \hat{\gamma}_5 = \gamma_5(1-2D)$$

Projection operators: $P_\pm = \frac{1 \pm \gamma_5}{2}, \qquad \hat{P}_\pm = \frac{1 \pm \hat{\gamma}_5}{2}$

LH-fermions: $\psi_-(x) = \hat{P}_- \psi_-(x), \qquad \bar{\psi}_-(x) = \bar{\psi}_-(x) P_+$

Action of LH-fermions: $S_F = \sum_x \bar{\psi}_-(x) D \psi_-(x)$

Measure: $D[\psi_-]D[\bar{\psi}_-] = \prod_j dc_j \prod_k d\bar{c}_k$

$$\psi_-(x) = \sum_j v_j(x) c_j\,, \qquad \bar{\psi}_-(x) = \sum_k \bar{c}_k \bar{v}_k(x)$$

($v_j(x)$, $\bar{v}_k(x)$: basis vectors)

Effective action: $\Gamma[U] = \log \det M, \qquad M_{jk} = \bar{v}_j D v_k$

Luscher's formulation

Variation of a link field: $\delta_\eta U(x,\mu) = i\eta_\mu(x)U(x,\mu),\qquad U(x,\mu) = e^{iA_\mu(x)}$

$$\delta_\eta \Gamma[U] = \mathrm{Tr}\, P_+ \delta_\eta D D^{-1} + \underline{\sum\nolimits_j (v_j, \delta_\eta v_j)}$$

$\equiv -i\mathcal{L}_\eta$ (measure term)

$\Gamma[U]$ depends on the choice of basis vectors v_j

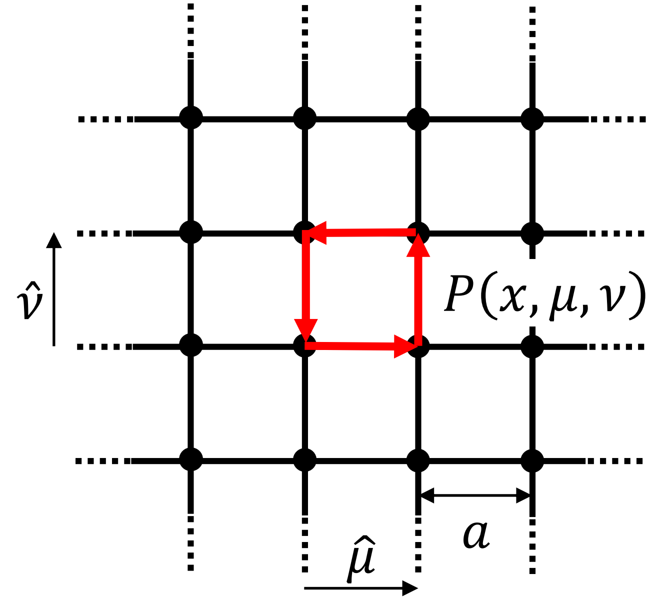
Suppose $\mathcal{L}_\eta = \sum_x \eta_\mu(x) j_\mu(x)$ satisfies the following condition

Then there exist a smooth fermion measure

1. $j_\mu(x)$ depends smoothly on the "admissible" link fields $U(x,\mu)$
2. $j_\mu(x)$ is gauge-invariant and transforms as an axial vector
3. Integrability condition: $\delta_\eta \mathcal{L}_\zeta - \delta_\zeta \mathcal{L}_\eta = i\mathrm{Tr}\hat{P}_-[\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-]$
4. anomalous conservation law: $\partial^*_\mu j_\mu(x) = \mathrm{tr}\,\gamma_5 T(1-D)(x,x)$

Such a $j_\mu(x)$ can be explicitly constructed
if the anomaly cancellation condition is satisfied

Admissibility



- Admissibility condition

$$\left|F_{\mu\nu}(x)\right| < \epsilon \qquad \text{for all } x, \mu, \nu$$

$$F_{\mu\nu}(x) = i^{-1}\log P(x,\mu,\nu), \qquad -\pi < F_{\mu\nu} < \pi,$$
$$P(x,\mu,\nu) = U(x,\mu)U(x+\mu,\nu)U(x+\nu,\mu)^{-1}U(x,\nu)^{-1}$$

Admissibility is the constraint on the gauge fields

- Existence of local Dirac operator $\|D(x,y)\| \leq \kappa e^{-\|x-y\|_1/\rho}$
- Topological structure $\sum_x -\mathrm{tr}\,\gamma_5 D(x,x) = (\mathrm{integer})$

Gradient flow should satisfy the admissibility condition

3. Gradient flow

Gradient flow

$$\partial_t \bar{A}_\mu(x,t) = -g_0^2 \frac{\delta S_{\mathcal{G}}}{\delta \bar{A}_\mu} = -D_\nu \bar{F}_{\mu\nu}(x,t)$$

lattice ⇒

$$\frac{d}{dt} U_t(x,\mu) = -g_0^2 [\partial_{x,\mu} S(U_t)] U_t(x,\mu)$$

Choice of $S(U)$: $S(U) = 1/(4g_0^2) \sum_x F_{\mu\nu}(x)^2$

⇓ Assume admissibility
→Bianchi identity: $\epsilon_{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$

"diffusion eq.": $\frac{d}{dt} F_{\mu\nu}(x,t) = \partial_\rho^* \partial_\rho F_{\mu\nu}(x,t)$

$$\partial_\mu f(x) = f(x+\hat{\mu}) - f(x)$$
$$\partial_\mu^* f(x) = f(x) - f(x-\hat{\mu})$$

Gradient flow

diffusion equation

$$\frac{d}{dt}F_{\mu\nu}(x,t) = \partial_\rho^* \partial_\rho F_{\mu\nu}(x,t)$$

Maximum principle

Suppose $f(x,t)$ is periodic w.r.t. x, and satisfies

$$\frac{d}{dt}f(x,t) = \partial_\mu^* \partial_\mu f(x,t)$$

Then $f(x,t)$ has a maximum at $t = 0$,

$$\max_{x\in\Gamma, t\in[0,T]} f(x,t) = \max_{x\in\Gamma} f(x,0)$$

if $F_{\mu\nu}(x,0)$ satisfies admissibility, $F_{\mu\nu}(x,t)$ respects this condition

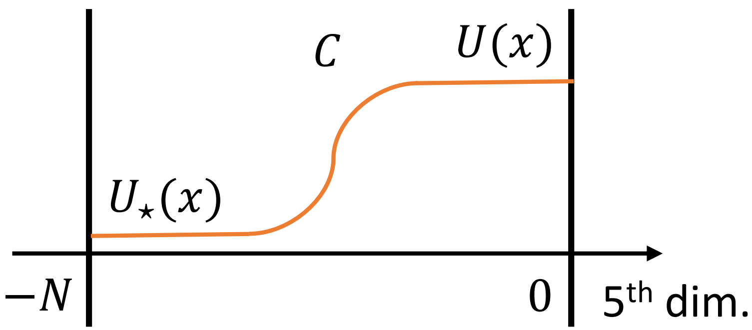
4. Locality

Relation of the two formulations

$$\exp(\Gamma_{\mathrm{GK}}) \equiv \frac{\det(D_{5\mathrm{w}} - m_0)|_{\mathrm{Dir.}}^{C}}{\left|\det(D_{5\mathrm{w}} - m_0)|_{\mathrm{AP}}^{C+(-C)}\right|^{1/2}}$$

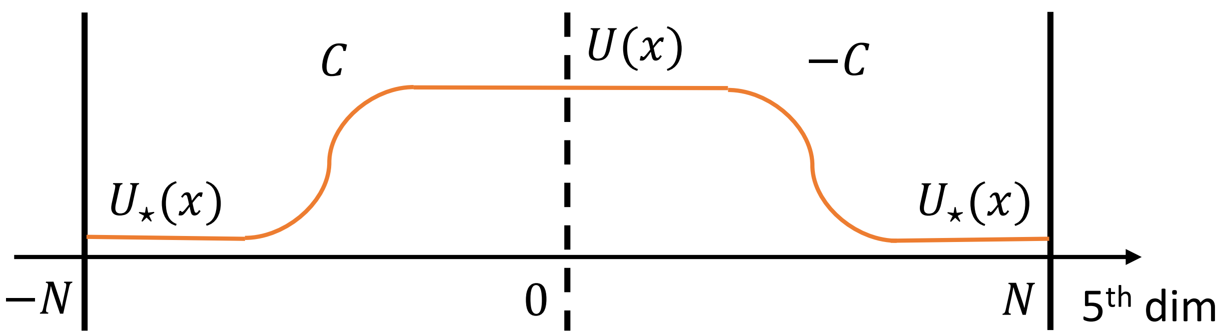
cf. [Kikukawa, 2002]

Domain wall fermion



LH + RH + massive modes

Pauli-Villars fermion



massive modes

Relation of the two formulations

$$\exp(\Gamma_{\mathrm{GK}}) \equiv \frac{\det(D_{5\mathrm{w}} - m_0)|^{C}_{\mathrm{Dir.}}}{\left|\det(D_{5\mathrm{w}} - m_0)|^{C+(-C)}_{\mathrm{AP}}\right|^{1/2}}$$

cf. [Kikukawa, 2002]

$$\partial_s \Gamma_{\mathrm{GK}}[U] \to \mathrm{Tr}\, P_+ \partial_s D D^{-1}|_{t=0} + (\mathrm{Tr}\, P_+ \partial_s D D^{-1})^*|_{t=-\infty} + \int_{-\infty}^{0} dt\, \mathrm{Tr} \hat{P}_-[\partial_t \hat{P}_-, \partial_s \hat{P}_-]$$

LH RH Bulk

Luscher's formulation: $\partial_t \Gamma_{\mathrm{L}}[U] = \mathrm{Tr}\, P_+ \delta_\eta D D^{-1} - i\mathcal{L}_\eta$

$$\Gamma_{\mathrm{GK}}[U] = \Gamma_{\mathrm{L}}[U] + \Gamma_{\mathrm{L}}[U_\star]^* + i\int_{-\infty}^{0} dt \mathcal{L}_\eta$$

$$\eta_\mu = i^{-1}\partial_t U_t(x,\mu) U_t(x,\mu)^{-1}$$

Locality

$$\mathcal{L}_\eta = \sum_x \left(e^{-\Box t} \partial_\nu^* F_{\nu\mu}(x) \right) j_\mu(x) \Big|_{U=U_t}$$

$j_\mu(x)$: gauge-invariant local field

- Perturbative analysis in the infinite volume limit

$$j_\mu(p) = \sum_{k\geq 4} \int V^{(k)}_{\mu\mu_1\nu_1\cdots\mu_k\nu_k}(p; p_1, \dots, p_{k-1}) (2\pi)^4 \delta^{(4)}(\textstyle\sum p_i) \displaystyle\prod_{i=1}^{k} e^{\hat{p}_i^2 t} \tilde{F}_{\mu_i\nu_i}(p_i) \frac{dp_i^4}{(2\pi)^4}$$

$V^{(k)}_{\mu\mu_1\nu_1\cdots\mu_k\nu_k}(p; p_1, \dots, p_{k-1})$: analytic function

$$\Rightarrow \int_{-\infty}^{0} dt \mathcal{L}_\eta = \sum_{k\geq 5} \int \frac{\tilde{V}^{(k)}_{\mu_1\cdots\mu_k}(p_1, \dots, p_{k-1})}{\sum \hat{p}_i^2} (2\pi)^4 \delta^{(4)}(\textstyle\sum p_i) \displaystyle\prod_{i=1}^{k} \tilde{A}_{\mu_i}(p_i) \frac{dp_i^4}{(2\pi)^4}$$

$$\equiv \sum_{k\geq 5} \int \tilde{\Gamma}^{(k)}_{\mu_1\cdots\mu_k}(p_1, \dots, p_{k-1}) (2\pi)^4 \delta^{(4)}(\textstyle\sum p_i) \displaystyle\prod_{i=1}^{k} \tilde{A}_{\mu_i}(p_i) \frac{dp_i^4}{(2\pi)^4}$$

$$\hat{p}^2 = \sum_\mu \left[2\sin(p_\mu/2) \right]^2$$

Locality

- $\tilde{\Gamma}^{(k)}_{\mu_1\cdots\mu_k}(p_1,\ldots,p_{k-1})$ are not analytic function
- For every positive integer N, there exist $n > N$ and $x_1,\cdots,x_{k-1}$ such that

$$\left|\Gamma^{(k)}_{\mu_1\cdots\mu_k}(x_1,\ldots,x_{k-1},0)\right| > \frac{c}{n^{5k-4}}, \qquad n = \|x_1\|_1 + \cdots + \|x_{k-1}\|_1$$

- There may be some effect on the critical behavior and the continuum limit of the lattice model
- (Numerical) approach to the dynamics is necessary to solve this question

5. Summary

Summary

- We considered U(1) chiral gauge theories
- the gradient flow is formulated for the admissible U(1) link fields
- GK's effective action is the sum of Luscher's effective actions and the measure term
- The measure term is non-local (not suppressed exponentially)
- There may be some effect on the critical behavior and the continuum limit of the lattice model