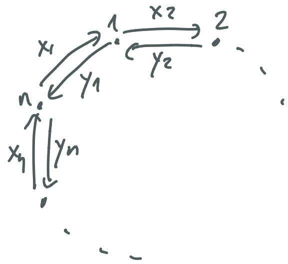
14.8.2020

Cluster categories for Grassmannians

$1 < k < n \quad ; \quad k \leq n/2$

Q:



$B := B_{k,n} := \mathbb{C}Q \Big/ \left\langle \begin{matrix} xy - yx \\ x^k - y^{n-k} \end{matrix} \right\rangle$

complete

$Z(B) = \mathbb{C}[[t]]$ for $t := \sum_{i=1}^{n} x_i y_i$

Jensen-King-Su '16: $\mathcal{F}_{k,n} := \mathrm{mCM}(B) = \{ M \text{ B-module} \mid M|_Z \text{ is free} \}$

modules: $\{(M_i)_i\}_{1 \leq i \leq n}$: M_i free as a Z-module, of same rank

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Thm (JKS '16) $\mathcal{F}_{k,n}$ is an add. categorification of Scott's cluster alg. structure of the homog. coord. ring of $Gr(k,n)$

Rank 1 modules JKS: $\left\{ \begin{matrix} \text{rank } 1 \\ \text{modules} \end{matrix} \right\} / \sim \xleftarrow{1:1} \left\{ \begin{matrix} k\text{-subsets} \\ \text{of } \{1, \dots, n\} \end{matrix} \right\}$. Take $I \in \binom{[n]}{k}$

Define $L_I = \left((U_i)_{1 \leq i \leq n} ; x_i, y_i, 1 \leq i \leq n \right)$

* $U_i := \mathbb{C}[[t]]$
* $x_i : U_{i-1} \to U_i$ mult. by $\begin{matrix} 1 \text{ if } i \in I \\ t \text{ if } i \notin I \end{matrix}$
* $y_i : U_i \to U_{i-1}$ mult. by $\begin{matrix} t \text{ if } i \in I \\ 1 \text{ if } i \notin I \end{matrix}$

view as a lattice diagram:

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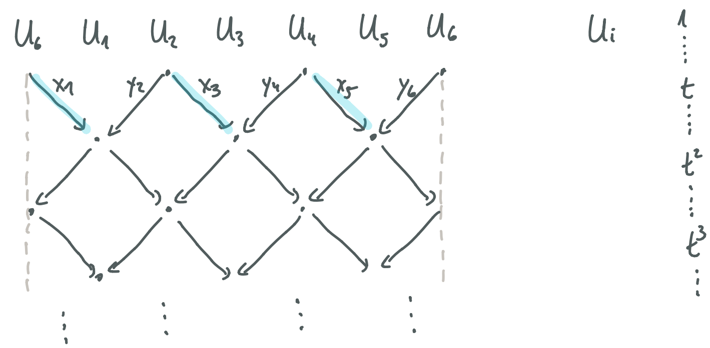
Ex.:

$I=\{1,3,5\}$

$n=6$

$k=3$



see page 3

$U=\{2,4,6\}$

$V=\{1,3,5\}$

Category $\mathcal{F}_{k,n}$?

- modules ∞-dim
- $\mathrm{Hom}(L_I, L_J) \cong \mathbb{C}[[t]] \quad \forall\, I,J$
- fin. type (fin. many indec.) : $k=2$; $k=3$, $n\in\{6,7,8\}$

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- Higher (rank) modules have filtrations by rank 1 modules (unique if rigid)

  $\rightarrow$ compute extensions between rank 1 modules [B-Bogdanic '16]

- Periodicity : objects are periodic under the Auslander-Reiten translate

  $\mathcal{F}_{3,9}$ and $\mathcal{F}_{4,8}$ are tubular (indec. objects form tubes)
  - Demonet-Luo
  - B-Bogd.
  - B-B-Garcia Elsener '19

$M \sim$ filtr. $L_{I_1}|\ldots|L_{I_s}$ $\rightsquigarrow$ add k to each element in I_j's

$\rightsquigarrow$ determine AR-sequences ("short ex. sequences") with rank 2 middle terms B-B-GE '19

* construct indec. rank 2 modules, some rank 3 module BBGE '20

* ranks 2 and 3 : more information, e.g. necessary cond. on indecomposability BBGE-Li '20

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Towards ind. rank 2 modules $I,J \in \binom{[n]}{k}$ are r-interlacing if there exist

$\{a_1, a_2, \dots, a_r\} \subset I \setminus J$, $\{b_1, \dots, b_r\} \subset J \setminus I$ s.t. $a_1 < b_1 < \dots < a_r < b_r$ and r is max^l

(I, J r-interlacing for some $r > 0$: I, J are crossing)

Thm (BBogd.'16) I, J r-interlacing. $\mathbb{C}[t]$

$$\Rightarrow \quad \mathrm{Ext}^1(L_I, L_J) \cong \mathbb{Z}/(t^{m_1}) \times \dots \times \mathbb{Z}/(t^{m_{r-1}}) \qquad 0 \le m_1 \le \dots \le m_{r-1} \text{ in } \mathbb{Z}$$

$$\bigoplus_{v \in V} P_v \xrightarrow{D} \bigoplus_{u \in U} P_u \to L_I \qquad U := \{u \notin I \mid u+1 \in I\}, \quad V := \{v \in I \mid v+1 \notin I\}$$

apply $\mathrm{Hom}(-, L_J)$ $\rightsquigarrow$ $\mathrm{Ext}^1(L_I, L_J)$

[look at offsets of lattice pictures for L_I and L_J]

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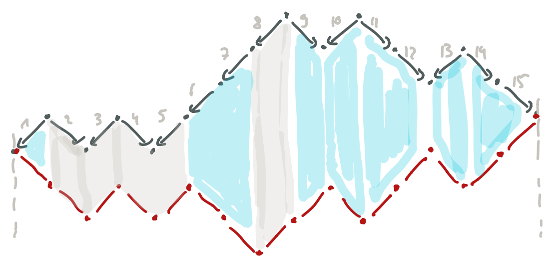

L_I

L_J

$U = \{1, 3, 8, 10, 13\}$

$V = \{2, 4, 9, 12, 15\}$

grey: "parallel"

$|I \cap J| = k-3$

Cor. $k=3$ • $\mathrm{Ext}^1(L_I, L_J) \cong \mathbb{C} \times \mathbb{C} \iff I$ and J are 3-interlacing

• If I and J are crossing but not 3-interlacing, $\mathrm{Ext}^1(L_I, L_J) \cong \mathbb{C}$.
(1- or 2-interlacing)

Rem.: $M \in \mathcal{F}_{k,n}$ is rk 2, indec., with filtration $L_I \mid L_J$ $\Rightarrow$ I & J are r-interl. for $r \geq 3$.

Prop.: (1) $M \in \mathcal{F}_{3,9}$ or $\mathcal{F}_{4,8}$ rigid, indec., rk 2 $\Rightarrow$ $M \cong L_I \mid L_J$ w. I & J 3-interlacing

(2) $M \in \mathcal{F}_{k,n}$, $M \cong L_I \mid L_J$, I & J tightly 3-interlacing $\Rightarrow$ M is indec.

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Construction of rank 2 modules $I, J \in \binom{[n]}{k}$, $|I \setminus J| = |J \setminus I| = 3$, I, J 3-interlacing

$\dots$, $A_1 := \begin{pmatrix} t & -2 \\ 0 & 1 \end{pmatrix}$ $\quad B_1 := \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ $\quad C_1 := \begin{pmatrix} t & -1 \\ 0 & 1 \end{pmatrix}$ $\quad D_1 = \mathrm{Id}$

let $\{$ $A_2 := \begin{pmatrix} 1 & 2 \\ & t \end{pmatrix}$ $\quad B_2 := \begin{pmatrix} 1 & 0 \\ & t \end{pmatrix}$ $\quad C_2 := \begin{pmatrix} 1 & 1 \\ & t \end{pmatrix}$ $\quad D_2 = t\cdot Id$

$V_i := \mathbb{C}[[t]] \oplus \mathbb{C}[[t]]$ for $1 \le i \le n$ $\quad x_i, y_i$ using these matrices. $\quad$ arbitrary k,n

Ex.: $n = 6$, $I = \{1,3,5\}$, $J = \{2,4,6\}$

$M(I,J)$ $\quad x_i : V_{i-1} \to V_i$ acts as $\quad y_i : V_i \to V_{i-1}$

x_i		y_i	arbitrary k,n
A_1	if $i=1$	A_2	$i = i_1$
B_2	$i=2$	B_1	$i = j_1$
B_1	$i=3$	B_2	$i = i_2$
C_2	$i=4$	C_1	$i = j_2$
C_1	$i=5$	C_2	$i = i_3$
A_2	$i=6$	A_1	$i = j_3$
D_1		D_2	$i \in I \cap J$
D_2		D_1	$i \in I^c \cap J^c$

$\Rightarrow xy = t\cdot Id = yx,\ x^k = y^{n-k}$ $\Big\}$ $M(I,J) \in \mathcal{F}_{k,n}$

$M(I,J)|_Z$ is free

$$6 \underset{A_2}{\overset{A_1}{\rightleftarrows}} 1 \underset{B_1}{\overset{B_2}{\rightleftarrows}} 2 \underset{B_2}{\overset{B_1}{\rightleftarrows}} 3 \rightleftarrows 4 \rightleftarrows 5 \underset{A_1}{\overset{A_2}{\rightleftarrows}} 6$$

$I \setminus J = \{i_1, i_2, i_3\}$ $\quad i_1 < j_1 < i_2 < j_2 < i_3 < j_3$

$J \setminus I = \{j_1, j_2, j_3\}$

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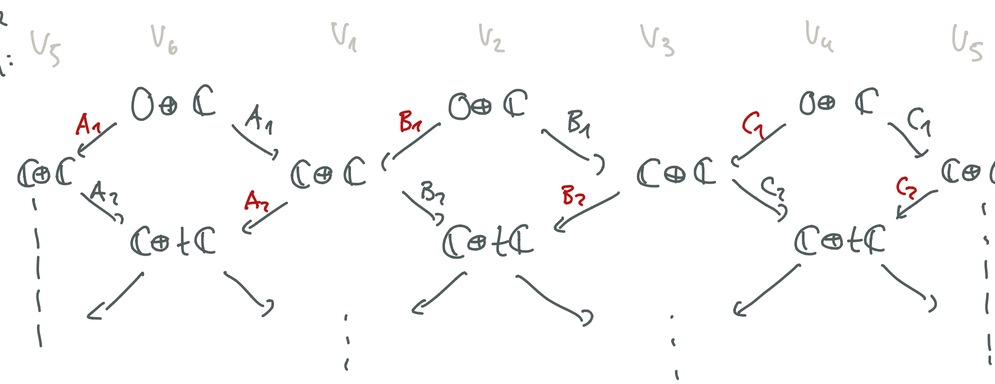
as lattice diagram:



Prop (BBGE'20) I, J tightly 3-interlacing $\Rightarrow M(I,J)$ is indec.

(Show that the only idempotent endom. of $M(I,J)$ are trivial)

(can take $b_{i_1}, \dots, b_{i_3}$, $b_{j_1}, b_{j_2}, b_{j_3}$ for the A_i's, B_i's, C_i's : $t \nmid b_{i_s} + b_{j_s}$ $\quad s = 1,2,3$)

Rem.: Have constr. for $M(I,J)$ for r-interlacing, $r \ge 4$: (not rigid) there are ∞ many non-isom. indec.'s w. this filtration.

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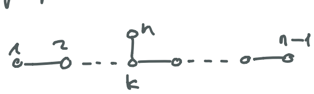
Roots combinatorics $\quad$ graph $J_{k,n}$:



$(k \le n/2)$

$k = 2$: Dynkin, D_n. $\quad k = 3$: E_n $(n = 6,7,8)$.

Identify root lattice of $J_{k,n}$ w. $\mathbb{Z}^n(k) := \{(x_i) \in \mathbb{Z}^n \mid k \mid \sum x_i\}$

with quadr. form $q(x) := \sum x_i^2 + \frac{2-k}{k^2}(\sum x_i)^2$. x is a real root if $q(x)=2$ and imaginary if $q(x)<2$.

$\alpha_i := -e_i + e_{i+1}$, $i=1,\dots,n-1$ $\beta := e_1 + \dots + e_k$: $\{\alpha_1,\dots,\alpha_{n-1},\beta\}$ system of simple roots

JKS: map from $\mathcal{F}_{k,n}$ to $\Phi(J_{k,n})$: $\overset{\text{indec.}}{M}$ w. filtration $L_{I_1}|\dots|L_{I_s}$: $x_i := x_i(M)$:= mult. of i in $I_1 \cup \dots \cup I_s$

$\Rightarrow (x_1,\dots,x_n) \in \mathbb{Z}^n(k)$. Degree of a root : coeff. of β in the root.

\# rk 1-modules = \# of roots of deg. 1

in finite types $\{$ 2-modules = 2· 2 } not true in general

3-modules = 3· 3

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Thm (BBGE Li '20) M is indec. in $\mathcal{F}_{k,n} \Rightarrow q(M) \in \{2,0,\dots,8-2k\}$

If $M = L_I | L_J$ is rigid, then I & J are 3-tightly interlacing and there are at most $N_{k,n}$ such modules.

$$N_{k,n} = \sum_{r=3}^{k} \left(\frac{2r}{3} p_1(r) + 2r p_2(r) + 4r\, p_3(r) \right) \binom{n}{2r} \binom{n-2r}{k-r}$$

$p_i(r) = \#\{\text{partitions } r = r_1+r_2+r_3 \mid r_i \in \mathbb{Z}_{\geq 1} \;\; |\{r_1,r_2,r_3\}| = i\}$

- use roots
- use the two rims of L_I, L_J : 3 boxes; look at sizes in x-direction. BBGE Li 20

Rank 3: $\{$ constructions
conditions for $M = L_I | L_J | L_K$ to be indec. (interlacing, boxes).

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