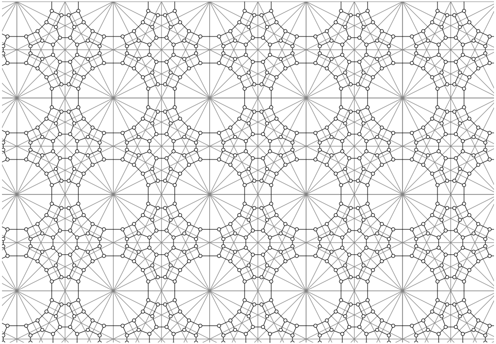
Tits Cone Intersections & Applications



Michael Wemyss

www.maths.gla.ac.uk/∼mwemyss

Plan of Talk

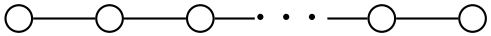
1. ADE Dynkin diagrams, and their extended versions. Hyperplane arrangements and friends.

2. Application 1: Kleinian singularities, and partial McKay.

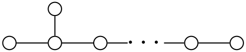
3. Application 2: Flopping contractions, mutation, and stability conditions. (plus: what is the picture on the first slide?)

ADE Dynkin Diagrams

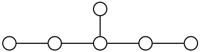
A_n
$n \geq 1$



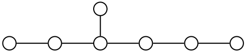
D_n
$n \geq 4$



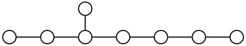
E_6



E_7



E_8



ADE Dynkin Diagrams + choice of nodes

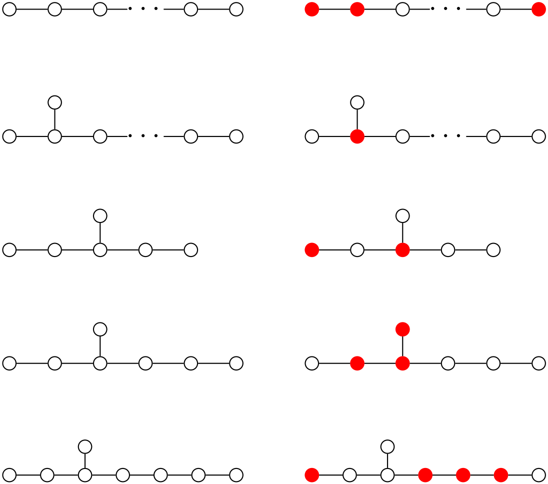
A_n, $n \geq 1$

D_n, $n \geq 4$

E_6

E_7

E_8



Construction

Input

- Any choice of ADE Dynkin diagram Δ,
- and any choice of nodes $\mathcal{J} \subseteq \Delta$.

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Now, each such Δ has an associated *root system*. This is just a real vector space $\mathbb{R}^{|\Delta|}$, with basis given by the nodes, together with some *reflecting hyperplanes*.

This does not depend on the choice $\mathcal{J}$.

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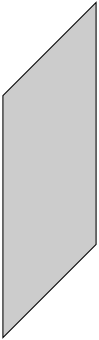
Aim

Want something similar, but which also depends on $\mathcal{J}$.

Intersection arrangements

The root system has a basis given by the nodes. Thus, the choice $\mathcal{J}$ gives *some* of these, so a *subspace* $\mathbb{R}^{|\mathcal{J}|}$. Picture for $|\mathcal{J}| = 2$ is:



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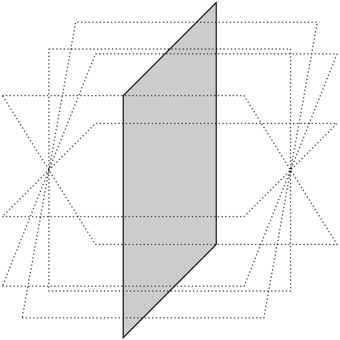

The reflecting hyperplanes slice the subspace

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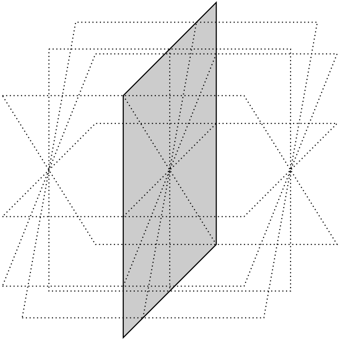

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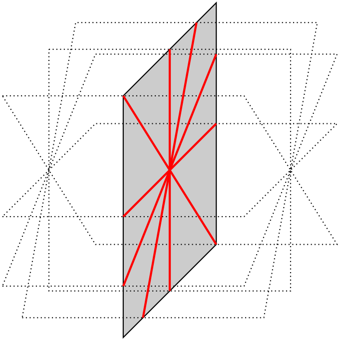
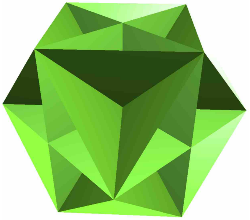


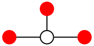
We *intersect* the reflecting hyperplanes with the subspace

Output

A finite collection of (red) hyperplanes, written $\mathrm{Cone}(\mathcal{J})$.

Some Examples





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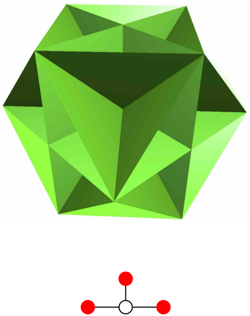
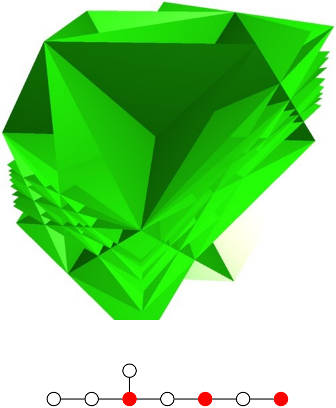
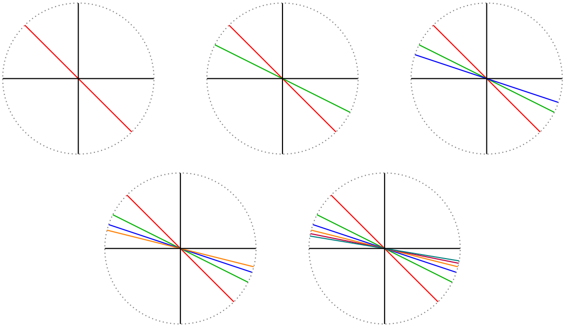




Theorem (Iyama–W)

Consider any $\mathcal{J} \subseteq \Delta$ with Δ ADE Dynkin and $|\mathcal{J}| = 2$. Then, up to changing the slopes of the lines, $\mathrm{Cone}(\mathcal{J})$ *is one of:*



The number of chambers is $6, 8, 10, 12$ *and* 16 *respectively.*

Better: extended ADE Dynkin Diagrams

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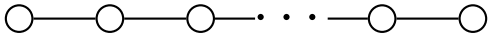
$n \geq 1$



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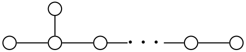

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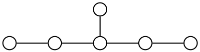

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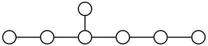

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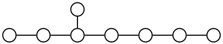

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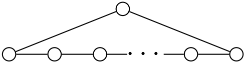

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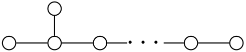

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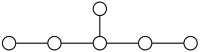

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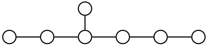

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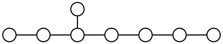

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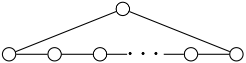

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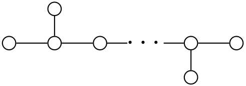

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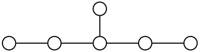

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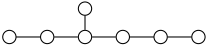

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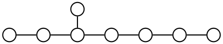

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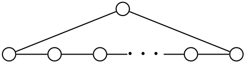

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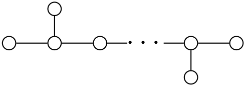

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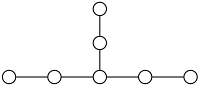

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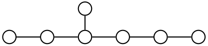

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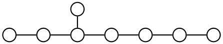

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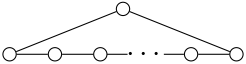

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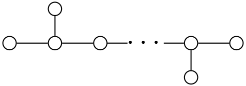

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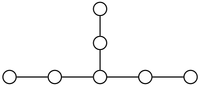

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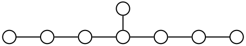

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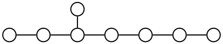

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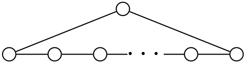

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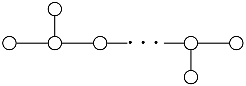

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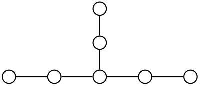

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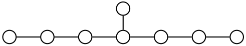

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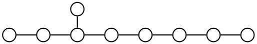

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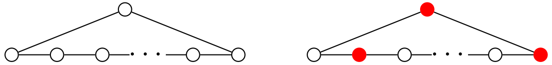

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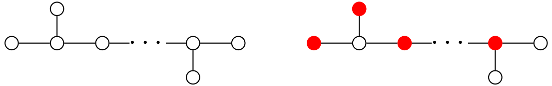

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Tits Cone Intersections

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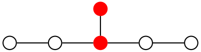
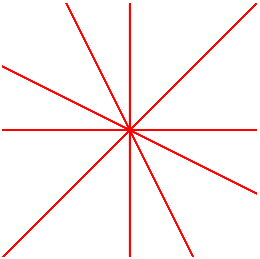
A similar story as to before, intersecting now inside the Tits Cone (instead of the root system) gives an *infinite* hyperplane arrangement, written Level($\mathcal{K}$).

This lives in $\mathbb{R}^{|\mathcal{K}|-1}$.

Finite Inside Infinite





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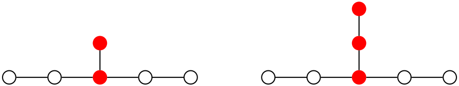

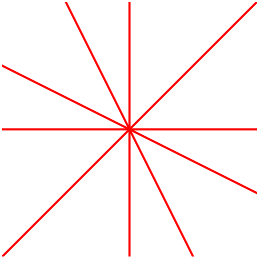

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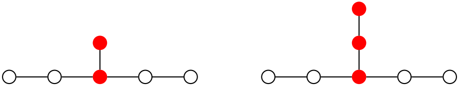

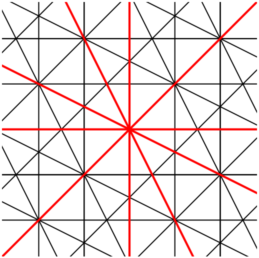

May as well develop the infinite theory; finite theory comes for free.

Labels and Wall Crossing

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The key is what we actually do is the following:

- Every chamber is labelled by a pair $(w, \mathcal{I})$, where w is an element in some group, and $\mathcal{I}$ is a subset of nodes.
- If $(x, \mathcal{I})$ and $(y, \mathcal{J})$ label adjacent chambers, it is possible to describe one from the other combinatorially, via a *wall crossing rule*.

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The rule is a bit technical, but it allows us to start anywhere, and iterate. The rule is also important for geometric applications.

The Wall Crossing Rule

Number of wall crossings = number of red nodes in subset.

To cross one of these walls, choose red node. Temporarily delete *all other* red nodes, apply Dynkin involution, then put back in the deleted vertices.

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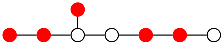

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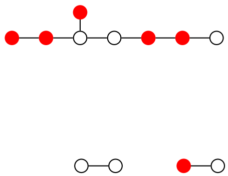

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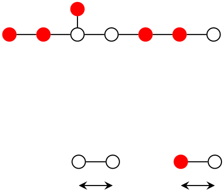

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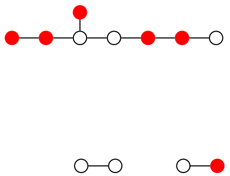

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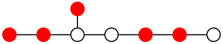
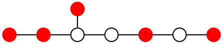
Example





Theorem (Iyama–W)

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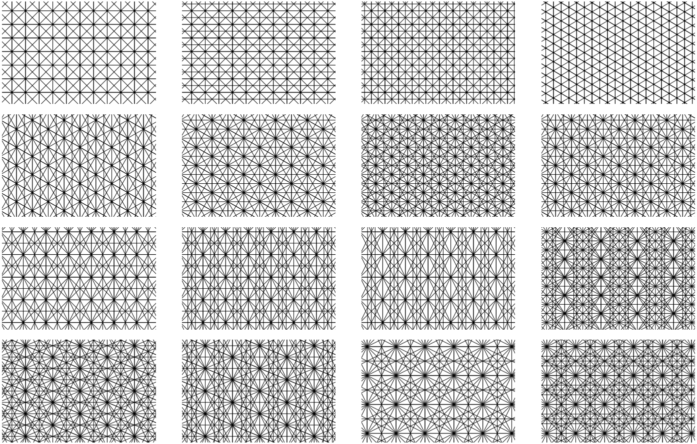
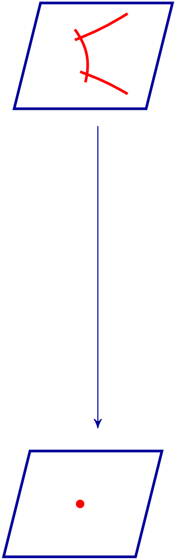


Application 1: Dimension Two



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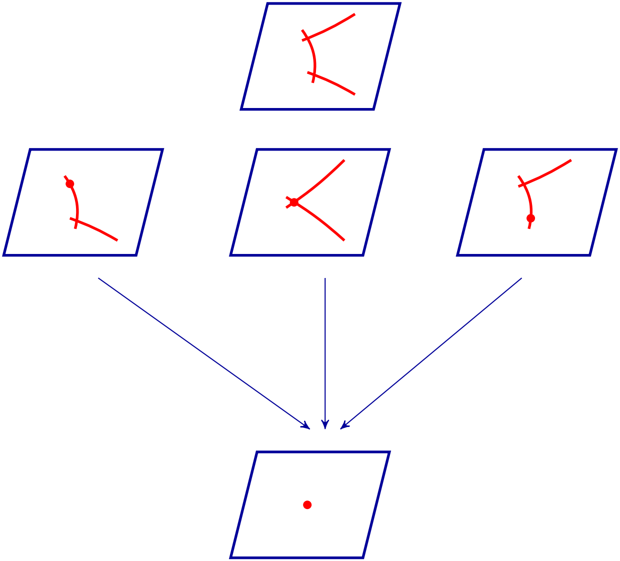

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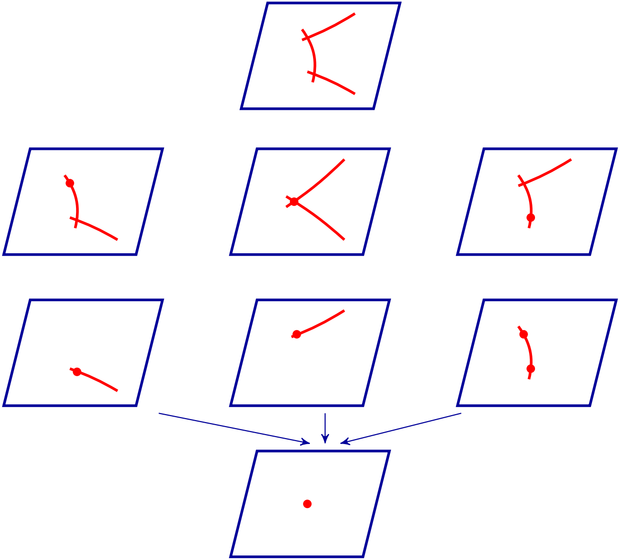

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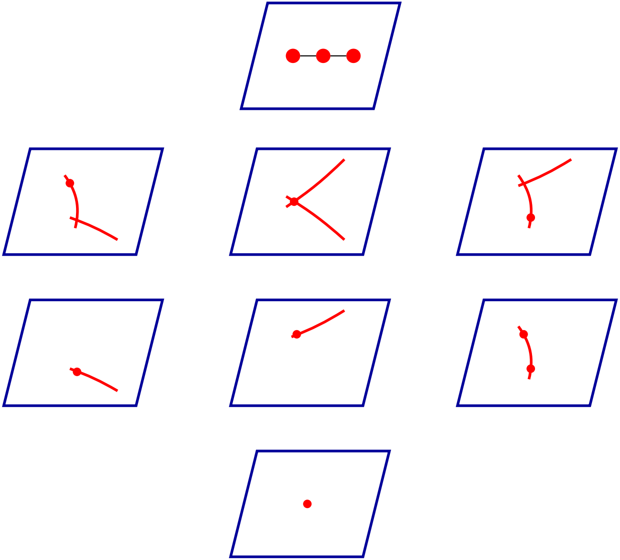

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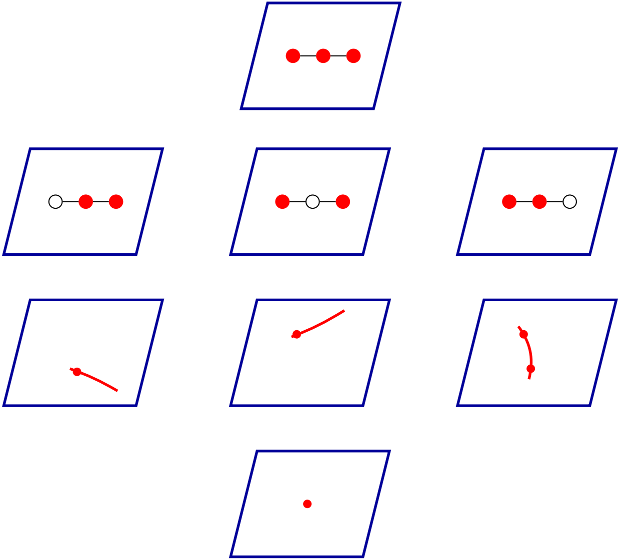

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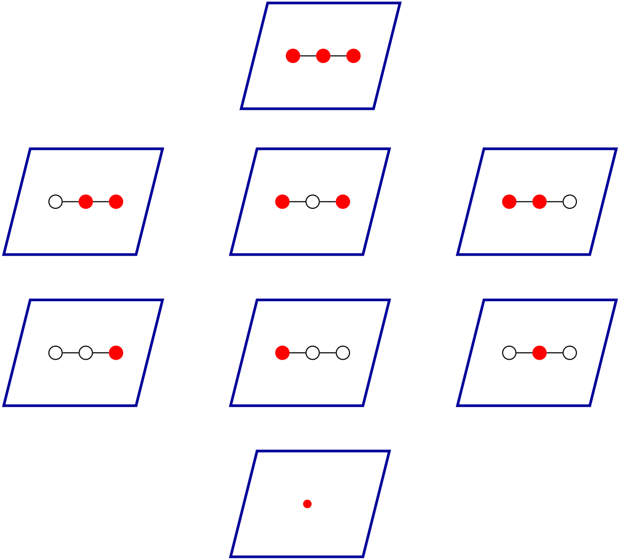

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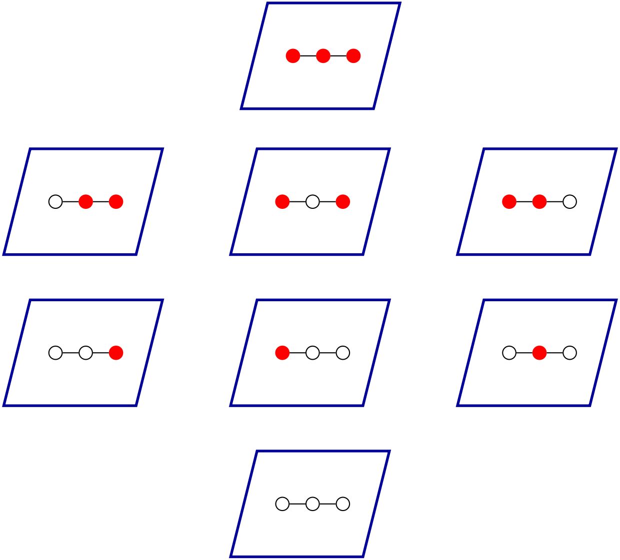

So: crepant partial resolutions $\longleftrightarrow X_{\mathcal{J}}$ for some $\mathcal{J} \subseteq \Delta$.

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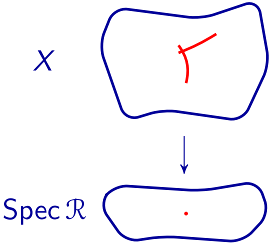
Theorem (Iyama–W)

Conjecture ($\Leftarrow$) always holds. Further, ($\Rightarrow$) holds for all types except possibly D_n with $n \geq 8$.

Application 2: Dimension Three

Consider now three-dimensional multi-curve flops, which are pictorially:



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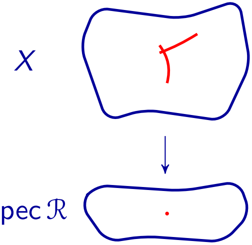
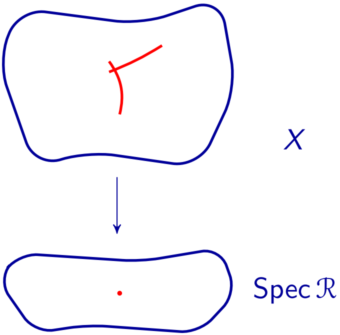


where X is smooth. Smoothness is not important, all statements later hold more generally, with tweaks. We are interested in:

- Classification.
- Invariants, curve counting.
- Derived categories and stability conditions.
- Symmetries: derived autoequivalences.
- Noncommutative resolutions.

Slicing Gives Combinatorics



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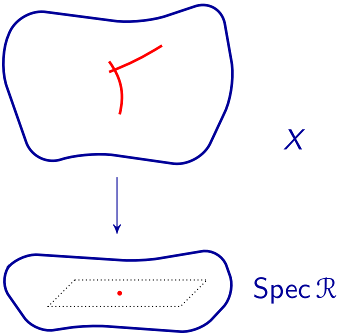

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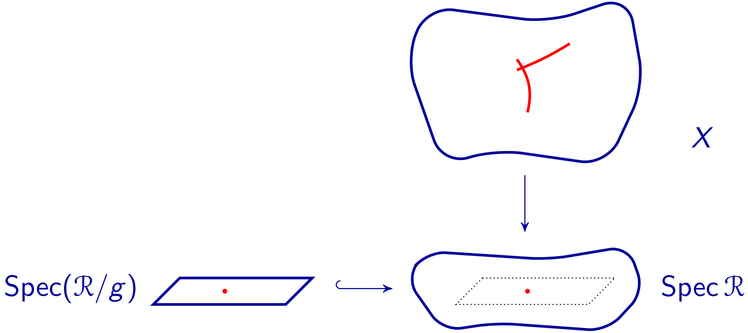

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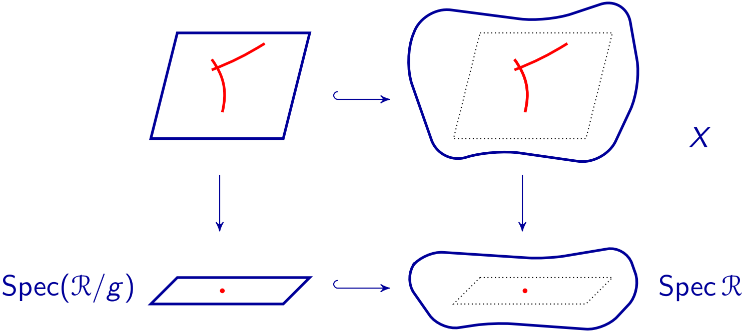

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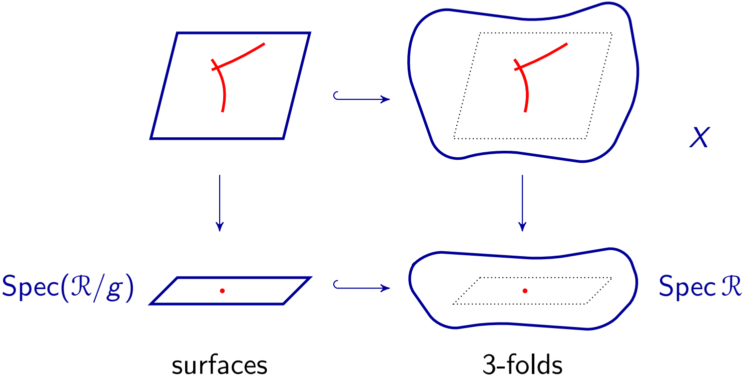

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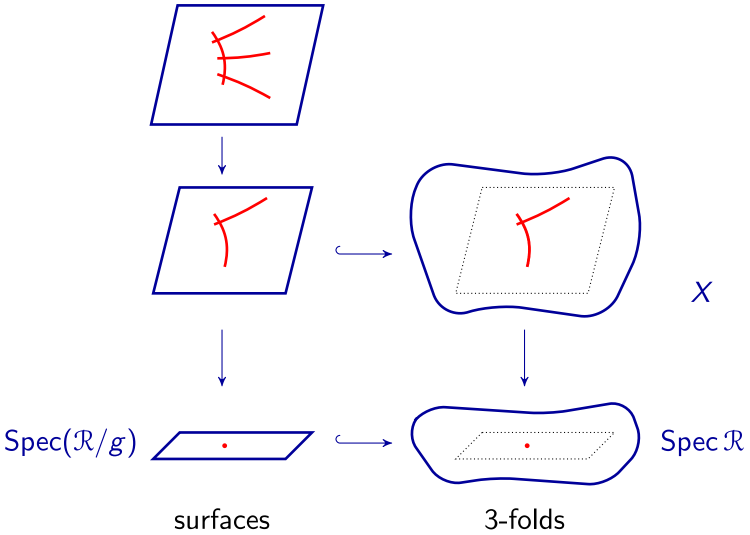

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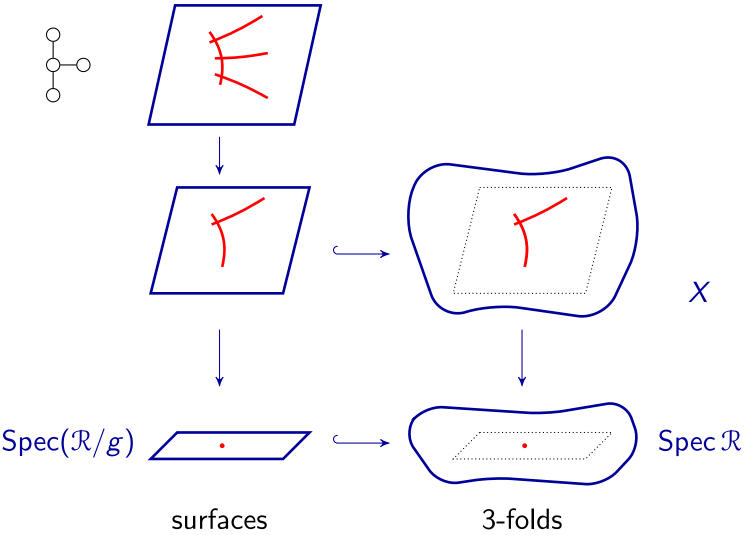

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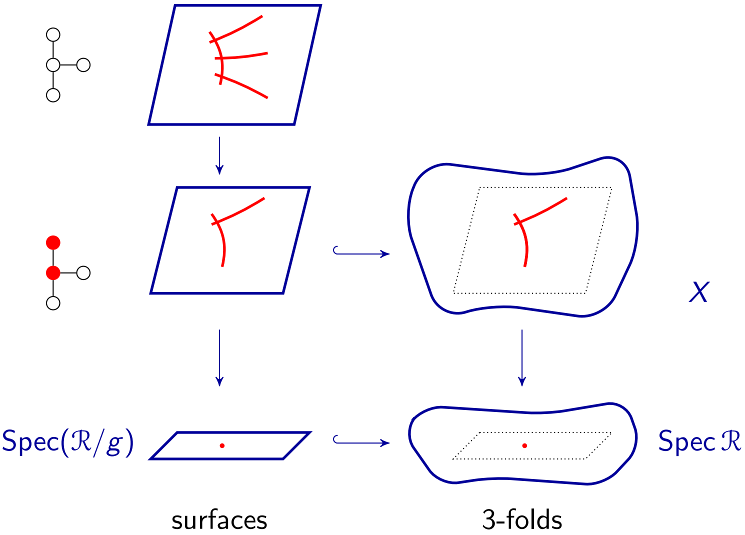

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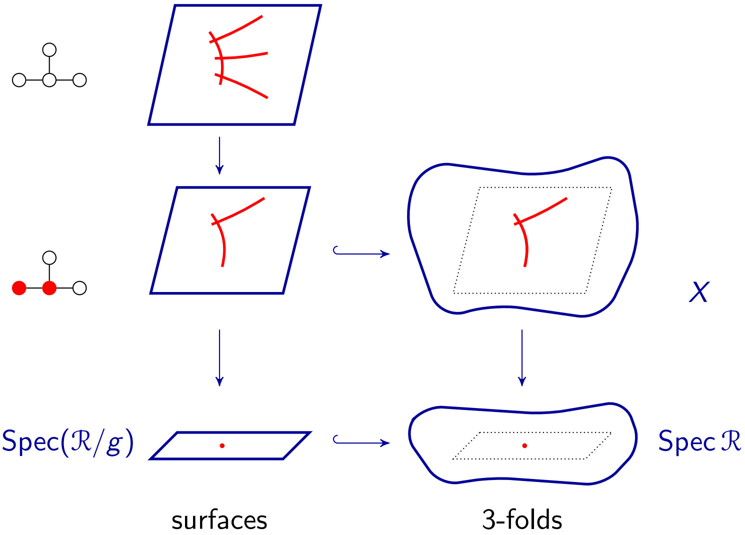
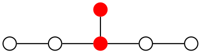


Upshot

For every 3-fold flop $X \to \operatorname{Spec} \mathcal{R}$, obtain a pair $(\Delta, \mathcal{J})$, namely a shaded ADE Dynkin diagram.

As before, from this we can always just add in the extended vertex:



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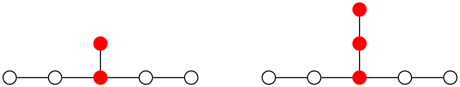

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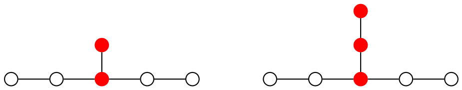

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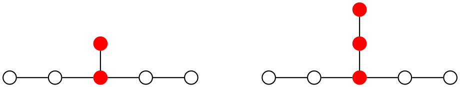

Start of talk: the left one gives us a finite hyperplane arrangement $\mathcal{H}$, the right hand one gives us an infinite arrangement $\mathcal{H}_{\mathsf{aff}}$.

Question

What do these combinatorics have to do with $X \to \operatorname{Spec} \mathcal{R}$?

Answer

...many things! I'll focus here on noncommutative resolution aspects, then move to autoequivalences and stability conditions.

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For finite story, interested in those finitely generated $\mathcal{R}$-modules M such that

- M is Cohen–Macaulay, namely $\mathrm{Ext}^i_{\mathcal{R}}(M, \mathcal{R}) = 0$ for all $i > 0$.
- M is rigid, namely $\mathrm{Ext}^1_{\mathcal{R}}(M, M) = 0$.
- M is maximal with respect to the above property.

In the lingo, 'maximal rigid objects in the category $\mathrm{CM}\mathcal{R}$'.

Remarkably, below it will turn out that there are only finitely many such maximal rigid objects. Thus, for the infinite arrangement story, we need (infinitely!) more.

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Consider those finitely generated $\mathcal{R}$-modules M such that:

- M is reflexive, namely there is an isomorphism

$$M \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{R}}(\mathrm{Hom}_{\mathcal{R}}(M, \mathcal{R}), \mathcal{R})$$

- M is modifying, namely $\mathrm{End}_{\mathcal{R}}(M)$ satisfies

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In the lingo, 'maximal modifying modules'. These are the building blocks of *noncommutative resolutions*.

Main Theorem (Iyama–W)

Suppose that $X \to \operatorname{Spec} \mathcal{R}$ is a flopping contraction. Associate $\mathcal{J} \subseteq \Delta$ by slicing, which gives a finite arrangement $\mathcal{H}$ and an infinite arrangement $\mathcal{H}_{\mathsf{aff}}$.

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1. Maximal rigid objects in $\mathrm{CM}\mathcal{R}$ are in bijection with chambers of the finite hyperplane arrangement $\mathcal{H}$.
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...in particular, we get a *complete* classification of noncommutative resolutions in this setting!

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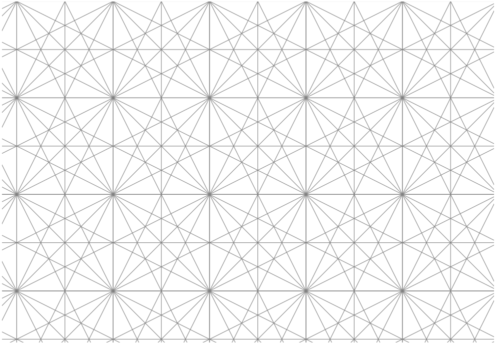

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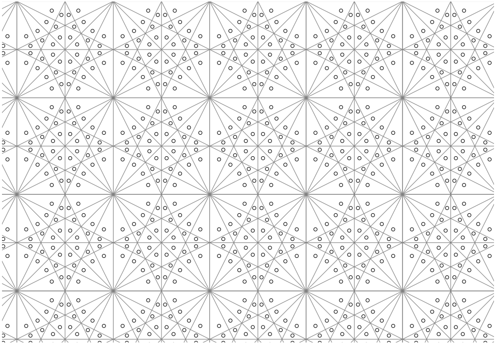

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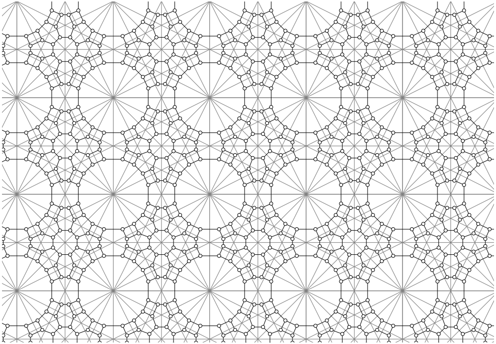


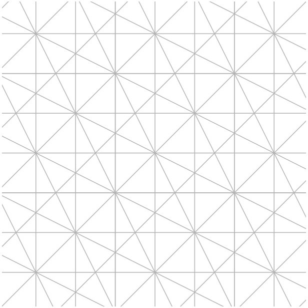
The dots are those $M \in \operatorname{ref} \mathcal{R}$ which give NCCRs. The edges connecting dots are the *mutations* of these; the above is really a picture of the exchange graph.

To have such highly regular structure is very unusual.

Now categorify...

The *mutation functors* lift the above combinatorial statements.

Consider the following groupoid:



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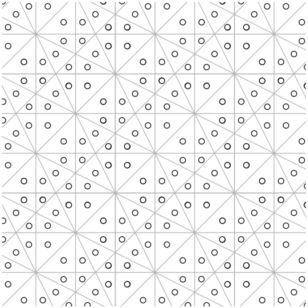

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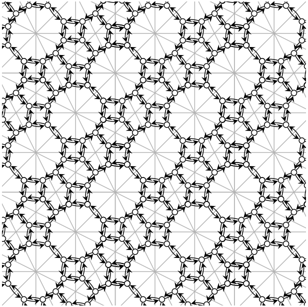

with relations give by identifying shortest paths. This is called the *Deligne groupoid*.

There is another way to build a groupoid. By last theorem:

- Each chamber has associated M, thus $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \operatorname{End}_{\mathcal{R}}(M))$.
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Theorem (Iyama–W)

There exists a functor from the Deligne groupoid to the groupoid described above.

Corollary (Iyama–W)

$\pi_1(\mathbb{C}^n \backslash (\mathcal{H}_{\mathrm{aff}})_{\mathbb{C}})$ acts on $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$.

And categorify again...

Consider the following two subcategories of $\mathrm{D^b}(\operatorname{coh} X)$.

$$\mathcal{C} = \{\mathcal{F} \in \mathrm{D^b}(\operatorname{coh} X) \mid \mathbf{R}f_*\mathcal{F} = 0\}$$
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Theorem (Hirano–W)

Given flopping contraction $X \to \operatorname{Spec}\mathcal{R}$, associate finite $\mathcal{H}$ and infinite $\mathcal{H}_{\mathsf{aff}}$ by slicing. Then the forgetful maps

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Autoequivalences of the last slide are the deck transformations.