

PERVERSE SCHOEBERS AND MCKAY CORRESPONDENCE

§1 PERVERSE SCHOEBERS

SETUP: X - A (LOCALLY) COMPACT) TOPOLOGICAL SPACE WITH A STRATIFICATION $S = \{S_i\}_{i \in \mathbb{Z}}$ ^{TOPOLOGICAL}
 $\mathcal{S}h(X)$ - SHEAVES OF $\mathbb{R}$ -MODULES ON X ^{$\mathbb{R}$ - AN ALG. CLOSED FIELD}
 $D^b(X)$ - THE DERIVED CATEGORY OF COMPLEXES IN $\mathcal{S}h(X)$ WITH BOUNDED COHOMOLOGY
 $\mathcal{S}h_{lc}(X)$ - LOCAL SYSTEMS (LOCALLY CONSTANT, LOCALLY FIN.DIM. SHEAVES) ON X
 $\cong$ FIN.DIM. REPS OF $k\pi_1(X)$ WHEN X IS CONNECTED.
 $\mathcal{S}h_S(X)$ - S-CONSTRUCTIBLE SHEAVES $\Leftrightarrow F \in \mathcal{S}h(X)$ ST. $\forall S_i \in S \quad F|_{S_i}$ IS A LOCAL SYSTEM.
 $\mathcal{S}h_c(X)$ - CONSTRUCTIBLE SHEAVES $\Leftrightarrow$ S-CONSTRUCTIBLE WRT SOME S
 $D_{lc}^b(X) \subseteq D_S^b(X) \subseteq D_c^b(X) \subseteq D^b(X)$: COMPLEXES WITH COHOMOLOGY
 IN $\mathcal{S}h_{lc}(X) \subseteq \mathcal{S}h_S(X) \subseteq \mathcal{S}h_c(X)$.

RIEMANN-HILBERT CORRESPONDENCE:

X - SMOOTH ALGEBRAIC VAR. / $\mathbb{R} = \mathbb{C}$

WE HAVE AN EQUIVALENCE OF CATEGORIES:

$$\begin{array}{ccc} \text{CONN}(X) & \xrightleftharpoons[\text{V=deid}]{\text{Ker}(\nabla)} & \mathcal{S}h_{lc}(X^{an}) \\ \text{VECT. BUNDLE} & & \text{LOCAL SYSTEM} \\ \text{FLAT CONN. \nabla} & & \\ \text{OX-COHERENT DX-MODULE} & & \end{array}$$



$$\begin{array}{ccc} \text{REGULAR, HOLONOMIC} & & \text{CONSTRUCTIBLE} \\ \text{DX-MODULES} & & \text{SHEAVES OF VECT. STACS} \\ D_{\text{REG, HOLON}}^b(D_X\text{-MOD}) & \xrightleftharpoons{\sim} & D_c^b(X^{an}) \cong \mathcal{S}h_c(X^{an}) \\ \cup & & \cup \\ D_X\text{-MOD}_{\text{REG, HOLON}} & \xrightleftharpoons{\sim} & \text{Per}(X^{an}) \\ \text{HEART OF STD. t-STRUCT.} & & \text{HEART OF PERVERSE t-STRUCT.} \\ & \nwarrow \nearrow & \\ & \text{PRESERVED BY VERDIER DUALITY} & \end{array}$$

PERVERSE SHEAVES:

DEFINITION: LET $p: S \rightarrow \mathbb{Z}$ BE A PERVERSITY FUNCTION. $F \in D_c^b(X)$ IS A P-PERVERSE SHEAF IF FOR ANY STRATUM $S_k \in S$ WE HAVE:

$$\begin{array}{l} \bullet H^i i_{S_k}^* F = 0 \quad \forall n > p(S_k) \rightsquigarrow D_p^{>0} \\ \bullet H^i i_{S_k}^! F = 0 \quad \forall n < p(S_k) \rightsquigarrow D_p^{<0} \end{array} \left. \vphantom{\begin{array}{l} \bullet H^i i_{S_k}^* F = 0 \\ \bullet H^i i_{S_k}^! F = 0 \end{array}} \right\} \text{P-PERVERSE t-STRUCTURE}$$

NOW LET X = THE ANALYTIC SPACE OF AN ALGEBRAIC VARIETY / $\mathbb{C}$

$d = \dim_{\mathbb{C}} X$ $S = \{S_i\}_{i=0}^d$ COMPLEX STRATIFICATION

"PERVERSE SHEAVES" $\Leftrightarrow$ P-PERVERSE FOR $p(S_k) = -\dim_{\mathbb{C}} S_k = -k$ "THE AUTODUAL PERVERSITY"

$F \in \text{Per}(X) \Rightarrow i_{S_k}^* F = i_{S_k}^! F$ IS CONCENTRATED IN DEGREE $-d \Rightarrow i_{S_k}^* F$ IS A (SHIFTED) LOCAL SYSTEM.

THUS PERVERSE SHEAVES ARE (CERTAIN) DERIVED EXTENSIONS OF LOCAL SYSTEMS ON THE OPEN STRATUM.

EXAMPLE: A DISC WITH A MARKED POINT

X : A DISC Δ HOMEOM TO $\mathbb{C}^1$, STRATIFIED AS $S = (0, U = \Delta \setminus \{0\})$ ^{A POINT}

(BEILINSON, '84) $\text{Per}_S(X) \simeq \{V_0 \xrightarrow{u} V_1 \text{ IN } \underline{\text{Vect}}_{\mathbb{C}} \mid \text{Id}_{V_0} - \partial u \text{ AND } u^* - \text{Id}_{V_1} \text{ ARE INVERTIBLE}\}$ ^{THE CAT. OF C-VECTORS}

HERE $(V_1, u^* - \text{Id}_{V_1})$ IS THE DATA OF THE CORRESPONDING LOCAL SYSTEM ON U . ^{FIBER MONODROMY}

CATEGORIFICATION: PERVERSE SCHOEBERS

WANT: A PERVERSE SHEAF OF TRIANG. CATEGORIES

HAVE: A PERVERSE SCHOEBER ON (X, S) IS A CATEGORIFICATION OF A QUIVER-THEORETIC DESCRIPTION OF $\text{Per}_S(X)$.

EXAMPLE: A PERVERSE SCHOEBER ON (Δ, pt) IS $\text{Core}(F \rightarrow \text{Id}_B)$ $\text{Core}(\text{Id}_A \rightarrow R)$ ARE AUTOEQUIV. ^{TRIAN. CAT. EXACT FUNCT. TRIAN. CAT.}

SPHERICAL FUNCTORS!

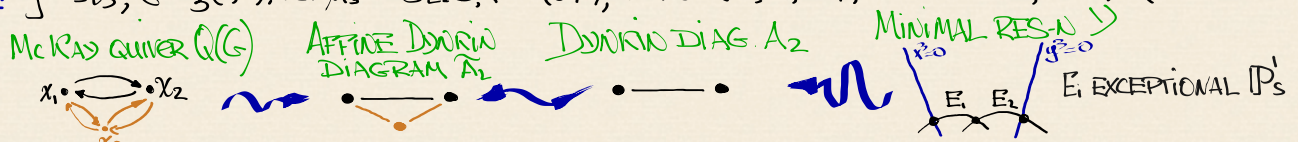
PERVERSE SCHOEBERS ARE (CERTAIN) EXTENSIONS OF CATEGORICAL GROUP ACTIONS. ^{TRIAN. LOCAL SYSTEMS OF TRIAN. CATS}

§2. THE MCKAY CORRESPONDENCE

- $G \subseteq \mathrm{SL}_2(\mathbb{C})$, FINITE SUBGROUP
- $X := \mathbb{C}^2/G$, SINGULAR AT THE ORIGIN
- $Y \xrightarrow{\pi} X$, THE MINIMAL RESOLUTION
- $Q(G) :=$ THE **MCKAY QUIVER** OF G : VERTICES - IRRED REPS OF G
ARROWS $P \rightarrow P' = \dim \mathrm{Hom}(P \otimes V_G, P')$

(MCKAY, '80) FORGETTING ARROW DIRECTION AND EDGE MULTIPLICITY IN $Q(G)$ YIELDS THE AFFINE DYNKIN DIAGRAM FOR A SEMISIMPLE LIE ALGEBRA $\mathfrak{g}$ OF ADE TYPE, WHILE THE EXCEPTIONAL DIVISORS ON Y FORM REGULAR DYNKIN DIAG. OF $\mathfrak{g}$.

EXAMPLE $\mathfrak{g} = \mathrm{SL}_3$, $G = \frac{1}{3}(1,2)$, i.e. $\mu_3 \subset \mathrm{SL}_2(\mathbb{C})$, $\xi \mapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi^2 \end{pmatrix}$, $\mathrm{IRR} G = \{x_0, x_1, x_2\}$, CHARACTERS, $\chi_i(\xi) = \xi^i$.



CATEGORICAL ACTION OF $\mathrm{Br}(\mathfrak{g})$ ON Y

$E_i \in \mathrm{Br}(Y)$ ARE $\mathbb{Z}$ -RATIONAL CURVES $\Rightarrow \mathcal{O}_{E_i} \in D_{\mathrm{gm}}^1(Y)$ ARE SEIDEL-THOMAS SPHERICAL OBJECTS
SPHERICAL TWISTS $T_i := \mathrm{cone}(E_i \otimes \mathrm{RHom}(E_i, -) \xrightarrow{\mathrm{id}} \mathrm{id}_Y)$ ARE AUTOEQUIVALENCES OF $D_{\mathrm{gm}}^b(Y)$.

$E_i \cap E_j = \emptyset \Rightarrow T_i$ AND T_j COMMUTE.

$E_i \cap E_j = \mathrm{pt} \Rightarrow T_i$ AND T_j BRAID

$T_i T_j = T_j T_i$
 $T_i T_j T_i = T_j T_i T_j$ } THE RELATIONS ON GENERATORS OF $\mathrm{Br}(\mathfrak{g})$

THUS WE HAVE $\mathrm{Br}(\mathfrak{g}) \curvearrowright D_{\mathrm{gm}}^b(Y)$ VIA $G_i \mapsto T_i$.

(SEIDEL-THOMAS, '01) IN THE A_n CASE, THIS ACTION IS FAITHFUL.

LET $\mathfrak{h} \subset \mathfrak{g}$ BE A CARTAN SUBALGEBRA AND FIX A CHOICE $\Delta_+ \subset \Delta_+^*$ OF POSITIVE/SIMPLE ROOTS.

$\Delta_+ =$ VERTICES OF DYNKIN DIAGRAM, SO $\mathrm{IRR} G|_{\mathfrak{h}} \xrightarrow{\sim} \Delta_+ \xrightarrow{\sim} \mathrm{Exc} Y$.

LET S BE THE STRATIFICATION OF $\mathfrak{h}$ BY $\{s^\perp\}_{s \in \Delta_+}$, AND ALSO THE INDUCED STRAT. OF $\mathfrak{h}/W$.

$\mathrm{Br}(\mathfrak{g}) = \pi_1(\mathfrak{h}/W)_0$, THE OPEN STRATUM.

THE Weyl GROUP
GEN'D BY REF'L IN $s^\perp$

THE ACTION $\mathrm{Br}(\mathfrak{g}) \curvearrowright D_{\mathrm{gm}}^b(Y) \Leftrightarrow$ A LOCAL SYSTEM OF TRIANG. CATS ON $(\mathfrak{h}/W)_0$ WITH FIBRE $D_{\mathrm{gm}}^b(Y)$.

MAIN CONJECTURE

THIS LOCAL SYSTEM EXTENDS TO A PERVERSE SCHOBER ON THE QUOTIENT STACK $[\mathfrak{h}/W]$.

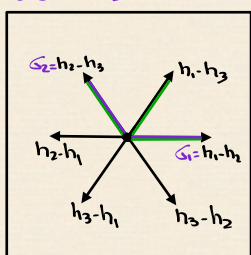
NO GOOD DESCRIPTION OF PERVERSE SHEAVES/SCHOBERS ON $[\mathfrak{h}/W]$ $\rightsquigarrow$ WORK-IN-PROGRESS W/A. BONDAL
THERE IS A GOOD DESCRIPTION FOR $\mathfrak{h}$ ITSELF. (KAPRANOV-SCHUCHTMAN, '16)

IDEA: CONSTRUCT A NATURAL SCHOBER ON $\mathfrak{h}$ WHICH WOULD DESCEND TO $[\mathfrak{h}/W]$.

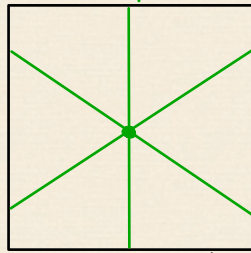
JOINT WORK WITH RINA ANCO AND CHRIS SEAMAN

EXAMPLE (CONTINUED)

$G = \mathrm{SL}_3$, $\mathfrak{h} = \{(h_1, h_2, h_3) \in \mathbb{C}^3 \mid \sum h_i = 0\}$, $\mathfrak{h}^* = \{d_1 h_1 + d_2 h_2 + d_3 h_3 \mid \sum d_i = 0\}$, $\Delta = \{\pm(h_1 - h_2), \pm(h_2 - h_3), \pm(h_1 - h_3)\}$



$\mathfrak{h}^*$ AND ROOT SYSTEM Δ
IR-SKELETON



STRATIFICATION S OF $\mathfrak{h}$ BY Δ_+
IR-SKELETON

A PERVERSE SCHOBER ON $\mathfrak{h}$ IS:

- FOR EACH CELL $s \in S_{\mathrm{IR}}$ A TRIANG. CAT. $\mathcal{C}_s$.
- FOR EACH $s \leq t$ IN S_{IR} AN EXACT FUNCTOR $\chi_{st}: \mathcal{C}_s \rightarrow \mathcal{C}_t$ WITH RIGHT ADJOINT $\bar{\chi}_{st}: \mathcal{C}_t \rightarrow \mathcal{C}_s$ SUCH THAT:
 - EACH $\bar{\chi}_{st}$ IS FULLY FAITHFUL
 - $\forall$ COLLINEAR $s, t, u \in S_{\mathrm{IR}}$ $\chi_{st} \chi_{tu} \simeq \chi_{su}$ (where $\chi_{st} = \chi_{st} \circ \chi_{s0}$, $0 = \text{origin}$)
 - $\forall$ d -DIM $s \leq t$ SEPARATED BY $(d-1)$ -DIM CELL u , $\bar{\chi}_{st}$ IS AN EQUIVALENCE OF CATEGORIES

GIT CONSTRUCTION OF Y

(KING, '94) GIT QUOTIENT CONSTRUCTION OF MODULI OF QUIVER REPS AS $H_T // \mathrm{PGL}_T$

NOTION OF θ -STABILITY FOR $\Theta := \{\theta: \mathrm{IRR} G \rightarrow \mathbb{Z} \mid \theta(\text{WEIGHT VECTOR}) = 0\}$

REGULAR REP. OF G

FOR EACH $\theta \in \Theta$ LET M_θ BE THE MODULI SPACE OF θ -SEMISTABLE REPS OF $Q(G)$ OF WT VECTOR V_{REG} .

(BKR, '01) $\forall \theta \in \Theta$ GENERIC $M_\theta \simeq Y$ ($G \subseteq \mathrm{SL}_2(\mathbb{C}) \Rightarrow M_\theta$ IS A CREPANT RES-N).

$\theta = 0 \Leftrightarrow \text{ALL}$

FOR NON-GENERIC θ , M_θ IS A PARTIAL RES-N OF $\mathbb{C}^2/G$ WITH SEVERAL E_i CONTRACTED.

KEY IDEA:

AS $\mathrm{IRR} G|_{\mathfrak{h}} \xrightarrow{\sim} \Delta_+$, WE CAN IDENTIFY $\mathfrak{h} \simeq \Theta_{\mathfrak{h}}$. THE NAT. STRATIFICATION OF $\Theta_{\mathfrak{h}}$ THEN REFINES S_{IR} .

$\forall$ CELL $s \in S_{\mathrm{IR}}$ WE WANT $\mathcal{C}_s$ TO BE $D_{\mathrm{gm}}^b(\text{STACKS } M_\theta)$ FOR $\theta \in s$.

LET $H := H_{\mathrm{ACC}} = \prod_{\theta \in \Theta} \mathrm{Hom}(V_{\theta}, V_{\theta})$ BE THE SPACE IN THE GIT CONSTRUCTION OF M_θ .

"STACKS M_θ " IS $\tilde{M}_\theta := \text{QUOTIENT STACK } [H_{\theta=0} // \mathrm{PGL}_{\mathrm{ACC}}]$. M_θ IS THE COARSE MODULI SPACE OF $\tilde{M}_\theta$.

HALPERN-LEISTNER'S CONSTRUCTION

(HALPERN-LEISTNER '12) "DERIVED KIRLOV SURJECTIVITY"

H -AND QUASI-FROS VARIETIES WITH AN ACTION OF REDUCTIVE GROUP P .

FOR ANY $\theta, \theta' \in \Theta$ S.T. $H_{\theta} \supseteq H_{\theta'}$ CONSTRUCTS A FAMILY OF FULLY FAITHFUL EMBEDDINGS

$$\mathcal{E}_w: D_{\text{Der}}^b(\tilde{M}_\theta) \hookrightarrow D_{\text{Der}}^b(\tilde{M}_{\theta'}) \quad w \in \mathbb{Z}^{\# \text{KN-STRATA OF } H_{\theta} \setminus H_{\theta'}} \quad (+)$$

(HALPERN-LEISTNER-SAM, '16) WHEN $H \simeq \tilde{A}^n$ IS A QUASI-SYMMETRIC LINEAR REP,
(SPENRO-VAN DEN BERGH, '19) (+) PACKAGE UP INTO A PERVERSE SCHOBER ON Θ .

WE AIM TO DO THE SAME, HOWEVER IN OUR CASE:

- H_{QCG} IS NOT SMOOTH
- ACTION OF $\text{PGL}_n(\mathbb{C})$ ON H_{QCG} IS NOT QUASI-SYMMETRIC
- THE INCLUSIONS OF KN-STRATA INTO H_{QCG} ARE NOT ALWAYS REGULAR IMMERSIONS
- RESTRICTION OF THE RELAT. COTANGENT CX OF A KN-STRATUM TO THE FIXED LOCUS DOES NOT ALWAYS HAVE STRICTLY POSITIVE WEIGHTS

THIS RESULTS IN SOME TECHNICAL COMPLICATIONS.