Combinatorial Reid's Recipe for Dimer Models

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There are currently three incarnations of Reid's Recipe, a generalisation of the McKay correspondence in the threefold case.

Classical Reid's recipe (Reid 1997, Craw 2005)

Given an affine Gorenstein toric simplicial singularity $\mathbb{C}^3/G$, with $G \subset SL(3,\mathbb{C})$, there is a combinatorial algorithm that decorates the fan Σ of a distinguished crepant resolution with irreducible representations of the group G.

The resolution is in fact G-Hilb, and the construction also encodes a minimal set of relations in its Picard group. Understanding these relations leads to constructing a $\mathbb{Z}$-basis of its cohomology ring. Our aim is to generalise this construction.

This studies the equivalence of categories appearing in works of Nakamura and proven by Bridgeland–King–Reid, in the case where G is abelian:

$$\Psi\colon D^b\big(G\text{-}\operatorname{coh}(\mathbb{C}^3)\big) \longrightarrow D^b\big(\operatorname{coh}(G\text{-Hilb})\big)$$

Key question (Cautis–Logvinenko 2009):

What can be said of the images of simple G-sheaves $\mathcal{O}_0 \otimes \rho$ on $\mathbb{C}^3$, for ρ a nontrivial irreducible rep?

Answer (Logvinenko 2010, Cautis–Craw–Logvinenko 2012):

They are pure sheaves which can be computed explicitly from Reid's recipe, i.e the image sheaf depends on what ρ marks in Σ.

Any generalisation of the recipe from the simplicial case to an arbitrary Gorenstein affine variety would have to:

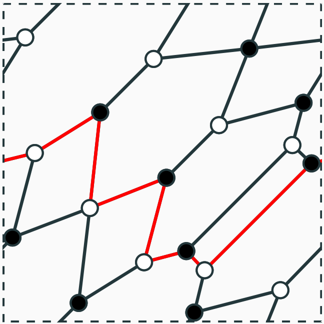
- Find an appropriate equivalent to the derived correspondence.

→ Blocklandt–Craw–Quintero-Vélez (2014) used consistent dimer models as key instruments in writing this dictionary

- Find a combinatorial way to deduce the markings.

→ Craw–H.–Tapia Amador (2020) deduce the markings on points and line segments of the fan Σ.

	0-dimensional	1-dimensional	2-dimensional
Dimer model Γ	Nodes ($\mathfrak{n}$)	Edges ($\mathfrak{e}$)	Tiles ($\mathfrak{t}$)
Quiver Q	Vertices (i)	Arrows (a)	
Triangulation Σ	Lattice points (ρ)	Line segments (τ)	Triangles (σ)



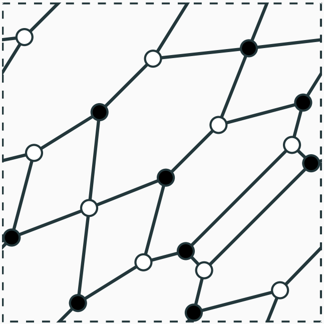
(a) Consider this consistent dimer model on the 2-torus $\mathbb{T}$ (i.e. there is no homologically trivial zig-zag path).

Finally, consider the Jacobian algebra of the quiver:

$$A := \mathbb{C}Q/\langle p_a^+ - p_a^- \mid a \in Q_1\rangle$$

is the quotient of the path algebra $\mathbb{C}Q$ of the quiver Q by the ideal of relations $\langle p_a^+ - p_a^- \mid a \in Q_1\rangle$.

The running example: Longhex



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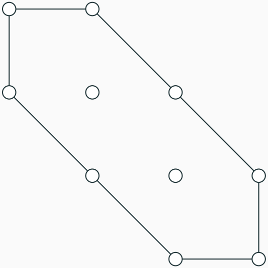

(b) This is its characteristic polygon. Consider the Gorenstein toric threefold X whose ray generators are the six vertices of this hexagon.

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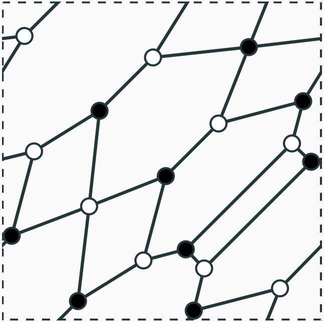

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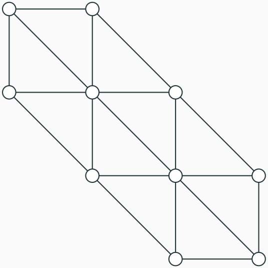

(b) We consider the distinguished crepant resolution $Y \to X$ associated to this triangulation.

Finally, consider the Jacobian algebra of the quiver:

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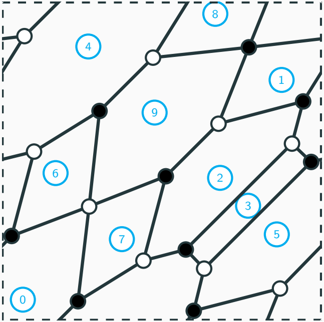
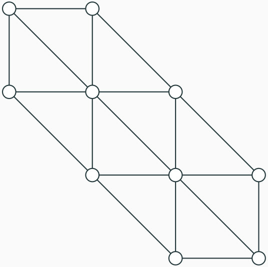


(a) Construct the dual quiver: each tile is dual to a vertex $i \in Q_0$



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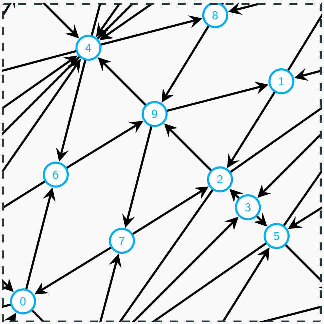
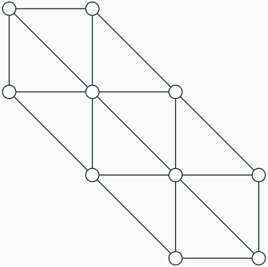


(a) Construct the dual quiver: each tile is dual to a vertex $i \in Q_0$ and each arrow $a \in Q_1$ is perpendicular to an edge.



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Finally, consider the Jacobian algebra of the quiver:

$$A := \mathbb{C}Q / \langle p_a^+ - p_a^- \mid a \in Q_1 \rangle$$

where $p_a^{\pm}$ are paths in Q with tail at $\mathrm{h}(a)$ and head at $\mathrm{t}(a)$, that are clockwise (p_a^+) when the dual dimer node and anticlockwise (p_a^-) when it is black.

The distinguished resolution Y is a moduli space of representations of the quiver Q, so it comes with a tautological vector bundle

$$T = \bigoplus_{i \in Q_0} L_i,$$

where $L_0 \simeq \mathcal{O}_Y$. Ishii–Ueda showed that

1. the natural map $A \to \mathrm{End}(T) = \bigoplus_{i,j} \mathcal{H}om(L_i, L_j)$ is an isomorphism; and
2. there is an equivalence of categories

$$\Psi(-) := T^\vee \otimes_A (-) \colon D^b(\text{mod-}A) \longrightarrow D^b(\text{coh-}Y).$$

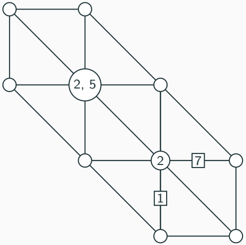
Bocklandt–Craw–Quintero-Vélez (2014) show:

Theorem (Geometric Reid's recipe)

For $i \in Q_0$, let $S_i := \mathbb{C}e_i$ be the vertex simple A-module. If i is not the zero vertex then exactly one of the following happens:

$$\Psi(S_i) = \begin{cases} L_i^{-1}|_{D_i} & \text{for some divisor } D_i; \\ L_i^{-1}|_{C_\tau} & \text{for some } (-1,-1) \text{ curve } C_\tau; \\ F|_{Z_i}[1] & \text{for some sheaf } F \text{ and divisor } Z_i \end{cases}$$

In the case of our running example, this suggests the following markings for the dimer version of the recipe:

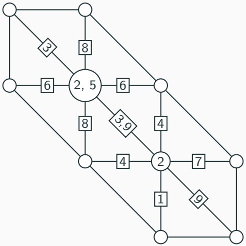


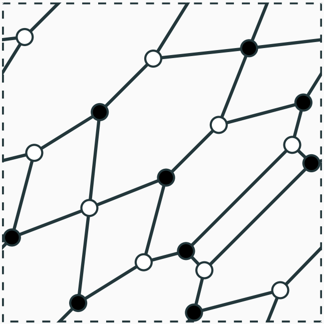
Theorem (Combinatorial Reid's recipe, Craw – H. – Tapia Amador (2020))

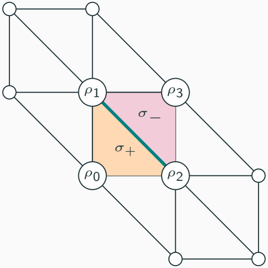
We introduce a marking for every interior lattice point and segment of Σ, such that:

1. *The recipe for points and vertices marking unique line segments coincides with the geometric recipe.*
2. *The recipe agrees with Reid's original recipe for marking cones in the toric fan of G-Hilb in the special case when Q is the McKay quiver of a finite abelian subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$.*

Here is what we obtain for Longhex:

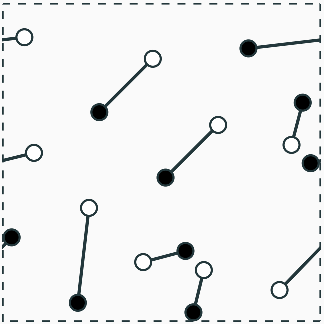


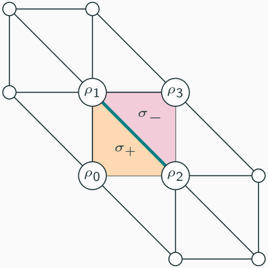




Suppose we want to mark the green line segment between ρ_1 and ρ_2. This belongs to two triangles, $\sigma_+ = \langle \rho_0, \rho_1, \rho_2 \rangle$ and $\sigma_- = \langle \rho_1, \rho_2, \rho_3 \rangle$. Each ρ_i corresponds to a perfect matching Π_i.

Key tool: the fundamental hexagon H_{σ_+}, formed by dimer tiles in $\mathbb{R}^2$ bounded by the connected component in $\Pi_0 \cup \Pi_1 \cup \Pi_2$ which is non-isolated.





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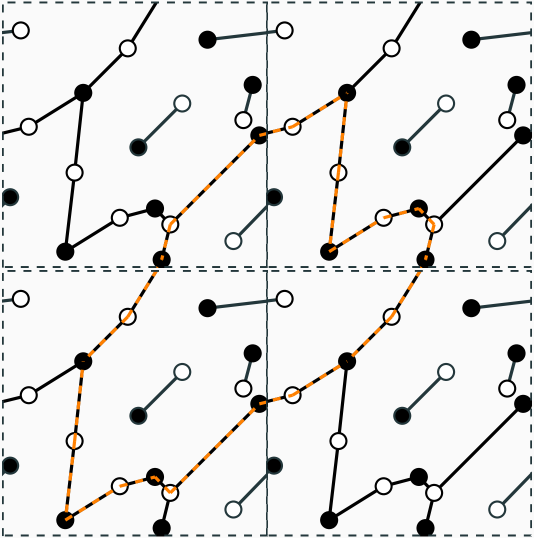
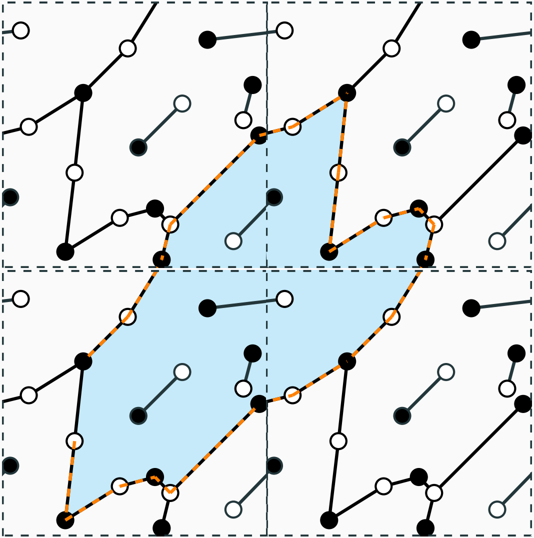
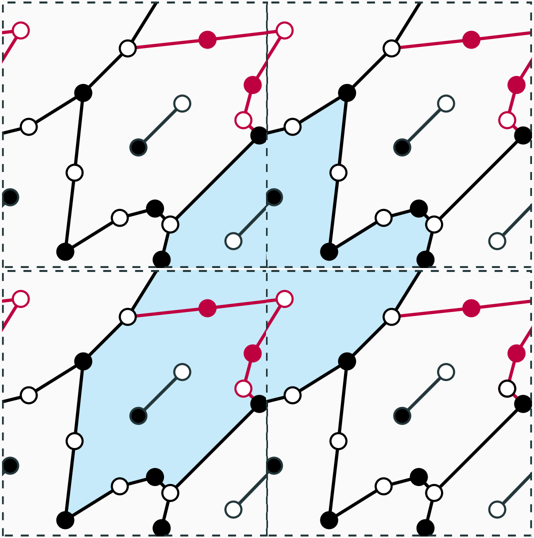
Key tool: the fundamental hexagon H_{σ_+} , formed by dimer tiles in $\mathbb{R}^2$ (or $\mathbb{T}$) bounded by the connected component in $\Pi_0 \cup \Pi_1 \cup \Pi_2$ which is non-isolated.



The edges of $\bigcup_{0 \leq i \leq 2} \Pi_i$ shown in the universal cover of $\mathbb{T}$.

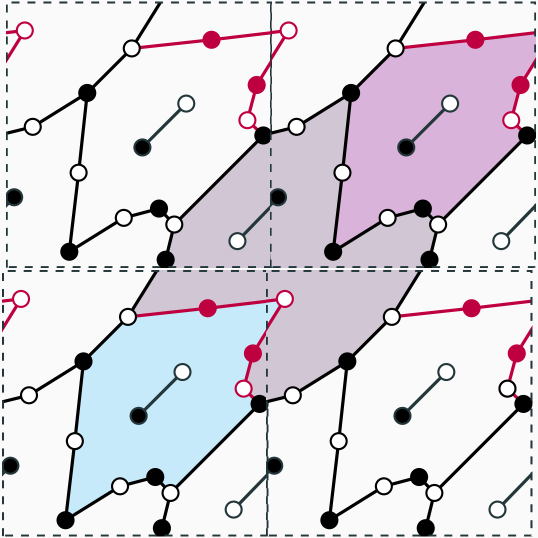


The interior of $Hex(\sigma_+)$.

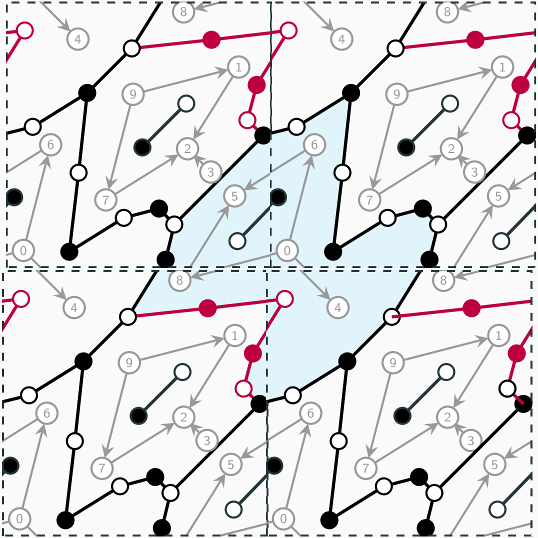


$Hex(\sigma_+)$ and the edges in Π_3

The markings



$Hex(\sigma_+)$ and $Hex(\sigma_-)$



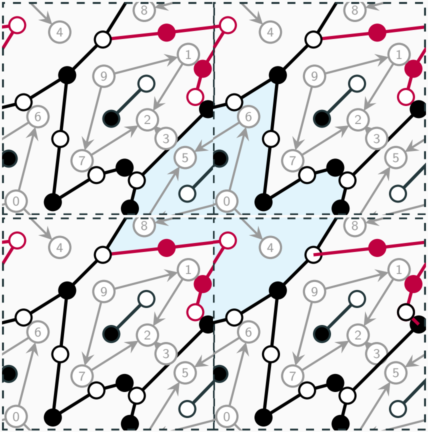
The quivers that don't contain arrows dual to the boundaries of $Hex(\sigma_+) \cup Hex(\sigma_+)$.

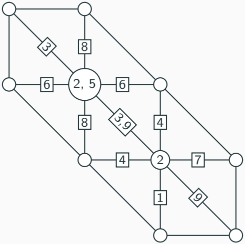
Remark

1. The closures of connected components of $\mathbb{T} \setminus \bigcup_{0 \leq i \leq 3} \Pi_i$ are called jigsaw pieces (Nakamura).
2. There is a unique jigsaw piece J_0 containing the 0-tile, which is common to all $Hex(\sigma)$.
3. In full generality, comparing $Hex(\sigma_+)$ and $Hex(\sigma_-)$ can lead to many jigsaw pieces (arranged differently in $\mathbb{R}^2$), however the quivers obtained by restricting to the outside of J_0 all have the same source vertices.
4. One can associate an A-module M_σ to every $Hex(\sigma)$, $\sigma \in \Sigma(3)$. This is the key object to proving all our results and choosing the marking on lattice points.

Markings

1. A vertex $i \in Q_0$ marks a single line segment $\tau \in \Sigma$ if i is one of the common source vertices of the quivers outside J_0.
2. A vertex $i \in Q_0$ marks an interior lattice point ρ if S_i lies in the socle of the $A-module$ M_σ for every cone $\sigma \in \Sigma(3)$ satisfying $\rho \subset \sigma$.





Remark (New to the dimer case)

1. Interior lattice points can be marked with the same vertex $i \in Q_0$ (e.g. 2).
2. Interior line segments can be marked with more than one vertex (e.g. 3 and 9).
3. The marking of an interior line segment is not determined by the hyperplane containing it (e.g. 3 and 9).
4. The marking of an interior lattice point is not determined by the geometry of the toric surface D_ρ.
5. The Euler number of an irreducible component of the exceptional divisor is not bounded by 6 from above.

Thank you!