COMPLEX REFLECTION GROUPS, THEIR MCKAY QUIVERS, AND THE MCKAY CORRESPONDENCE

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Big goal: establish a McKay correspondence for finite complex reflection groups:
rep. theory $\leftrightarrow$ geometry $\leftrightarrow$ algebra

<u>Today</u>: combinatorial description of McKay quivers of $G(r,p,n)$
McKay correspondence?

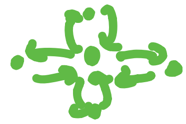
I McKay quivers

Let $G \subset GL(V)$ finite, V cx. vector space
$K = \mathbb{C}$ $\qquad \dim_{\mathbb{C}} V = n$
(V = defining rep of G
$G \hookrightarrow GL(V)$)

and let $\operatorname{Irrep}(G) = \{\text{irreps of } G\} = \{V_1, \ldots, V_d\}$
$MK(G) :=$ McKay quiver of G, defined by

- vertices: $V_1, \dots, V_d$ $\qquad G \hookrightarrow GL(V)$
- arrows: $V_i \xrightarrow{m_{ij}} V_j$ if V_j appears with multipl. m_{ij} in $V \otimes V_i$, i.e.

$$m_{ij} = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}G}(V \otimes V_i, V_j)).$$

ex: $G \subset SL(V)$, $\dim V = 2$, then
$MK(G)$ = extended Dynkin diagram of type ADE with arrows in both directions. ex. $G = D_4$ 

McKay correspondence (à la AUSLANDER)
$MK(G)$ = AR-quiver of $MCM(R)$

R = invariant ring of group action $G \circlearrowright S = \mathrm{Sym}_{\mathbb{C}} V \cong \mathbb{C}[x_1, x_2]$
$= S^G = \{f \in S : g(f) = f \ \forall g \in G\}$

And:
$\{\text{irreps of } G\} \xleftrightarrow{1-1} \{\text{indec. MCM-modules over } R\} \leftrightarrow \{R\text{-direct summands of } S\}$

via: $S * G \cong \mathrm{End}_R(S)$

" $s \otimes g$
skew grp ring
$\mathrm{Proj}(S*G) \leftrightarrow \mathrm{Irrep}(G)$

S rep. generator over R
$\mathrm{End}_R S$ is NCR of R

Rmk: $S * G \cong \mathrm{End}_R S$ whenever G does NOT contain complex reflections.

II Complex reflection groups

Questions: (1) Can we describe $MK(G)$ for $G \subset GL(V)$ cx. reflection group?

(2) algebra: $\{\text{reps of } G\} \overset{1-1}{\longleftrightarrow} \{?\}$

$S * G \cong ?$

In particular: $\dim V = 2$: do we get a nice description?

First some definitions:

$G \subset GL(V)$ finite, $\dim_{\mathbb{C}} V = n$

$G \curvearrowright V$ and also $G \curvearrowright \mathrm{Sym}_{\mathbb{C}} V \cong \mathbb{C}[x_1, \ldots, x_n] =: S$

G is a complex reflection group if it is generated by reflections g.

($g \in G$ reflection $:\Leftrightarrow$ g conjugate to $\begin{pmatrix} \xi & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$ with $\xi \neq 1$, ξ root of unity)

$\Leftrightarrow$ g fixes a hyperplane pointwise

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g is true reflection if $\zeta = -1$ ($\Leftrightarrow$ ord $g = 2$)

G is true reflection group if it is generated by true reflections.

Finite

Complex reflection groups were classified by [SHEPHARD-TODD '1954]:

Family $G(r,p,n)$ + 34 exceptional cases

- cyclic groups $\mathbb{Z}/m\mathbb{Z} \cong G(m,1,1)$
- symmetric groups $S_n \cong G(1,1,n)$
- $G(r,p,n)$ with $p \mid r$ and $(r,p,n) \neq (2,2,2)$

Abstractly: $G(r,1,n) = \mu_r \wr S_n$ WREATH prod.

$p > 1$ $\quad G(r,p,n)$ = normal subgp of index p of $G(r,1,n)$.

$$1 \to \boxed{G(r,p,n)} \to G(r,1,n) \to \mu_p \to 1$$

e.g: $G(2,1,n) = B_n$ Coxeter group $\mu_2^n \rtimes S_n$

Concretely: $G_n(r,p,n) \subseteq GL(V)$, $\dim V = n$

$\{P \cdot D$: P is $n \times n$ permutation mat. D diag. mat of roots of unity and $(\det D)^{1/p} = 1\}$

III McKay quivers of $G(r,p,n)$

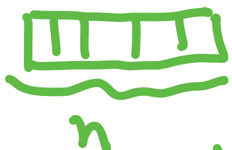
1. $S_n \cong G(1,1,n)$

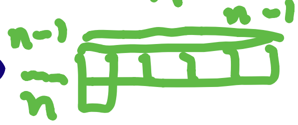
Folclore result:

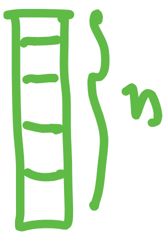
reps of $S_n \xleftrightarrow{1-1}$ Partitions of n $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_i \geq \lambda_{i+1}$ $\forall i$ and $\sum \lambda_i = n$

$\xleftrightarrow{1-1}$ Young diagrams of size n

Easy to see:

- $\lambda = (\underline{n}) \longleftrightarrow$  $\longleftrightarrow V_{triv}$

- $\lambda = (n-1, 1) \longleftrightarrow$  $\longleftrightarrow V_{std}$

- $\lambda = (\underbrace{1, \dots, 1}_{n}) \longleftrightarrow$  $\longleftrightarrow V_{det}$

Use Frobenius reciprocity and

$\mathrm{Res}: S_n \to S_{n-1}$

$\mathrm{Ind}: S_{n-1} \to S_n$

(Fulton-Harris!

$\mathrm{Res}\, V_\lambda = \bigoplus_{\tau} V_\tau$

to describe $V \otimes V_\lambda$:

τ obt. from λ by deleting one block

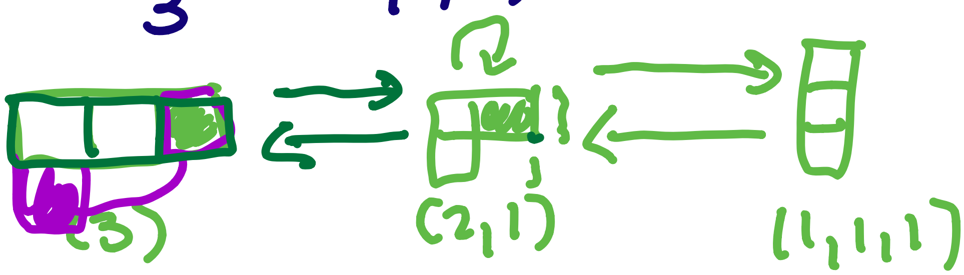
<u>Thm</u> [Buchweitz-F-Ingalls-Lewis]

Let λ, τ be partitions of $n \overset{1-1}{\longleftrightarrow} V_\lambda, V_\tau$ irreps

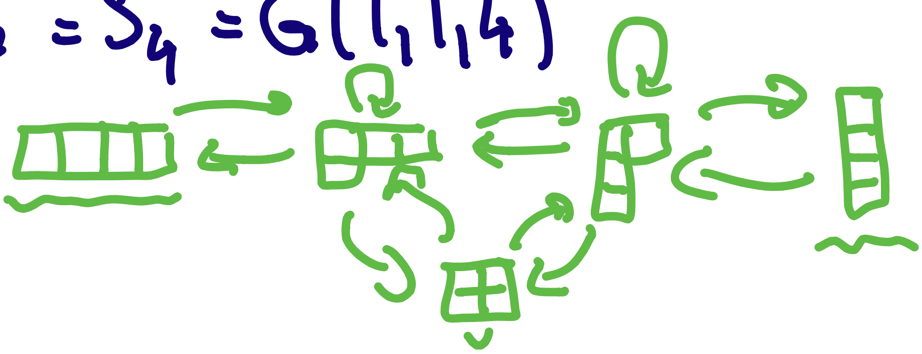
<u>If $\lambda \neq \tau$</u>: then there is an arrow from λ to τ in $MR(G(1,1,n))$
$\iff \lambda$ can be formed from τ by moving a single block

<u>$\lambda = \tau$</u>: # loops = (# parts of λ) − 1

<u>ex</u>: $G = S_3 = G(1,1,3)$



• $G = S_4 = G(1,1,4)$



→ always have loops in $MK(G(1,1,n))$:

2. $G(r,1,n)$ $r \geq 2$

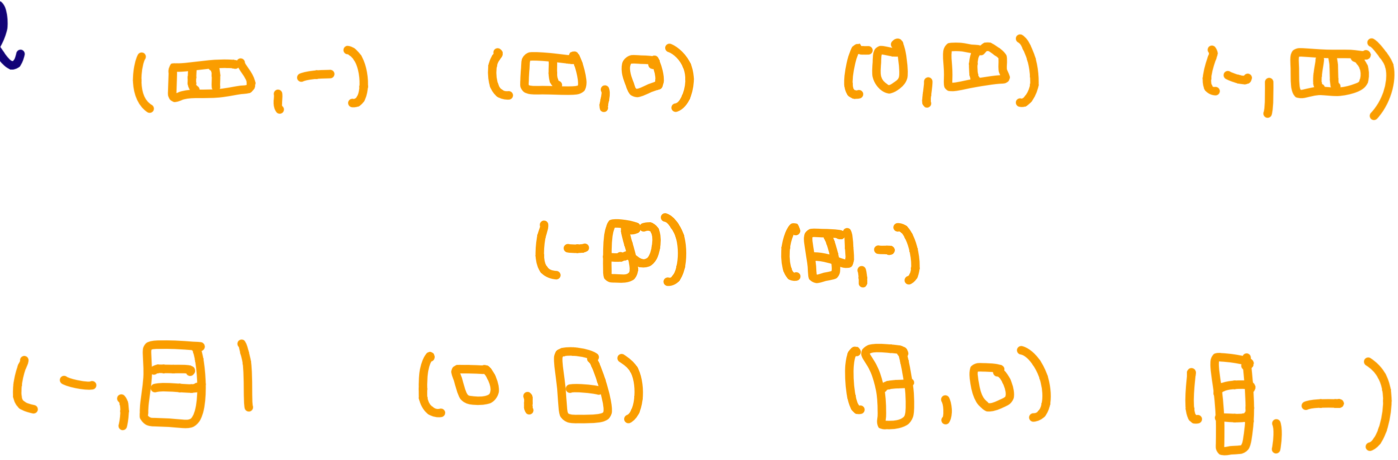
[ARIKI-KOIKE 1994]: describe generators + relations and irreps in terms of Young diagrams (partitions λ of n):

irreps of $G(r,1,n)$ V_λ $\longleftrightarrow$ r-tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ with $\lambda^{(i)}$ partition of $0 \leq n_i \leq n$ and $\sum_{i=1}^{r} n_i = n$

λ = vector of Young diagrams

e.g: $G(2,1,3)$

$r = 2$



10 irreps

For McK(G) same strategy:

$\boxed{V_{std} = (\underbrace{\square\square\square}_{n-1}, \underbrace{\square}_{1}, -)}$ and use Restriction and induction to decompose $V_{std} \otimes V_\lambda$

$V_{triv} = (\underbrace{\square\square\square}_{n}, -, -\ldots)$

[AK]: branching rule $\mathrm{Res}^{G(r,1,n)}_{G(r,1,n-1)}$

PROBLEM: This does not quite work

$\mathrm{Ind}^{G(r,1,n)}_{G(r,1,n-1)}$ will multiply dimension by $n \cdot r$, but $\otimes V_{std}$ only multiplies by n

$\rightsquigarrow$ restrict to larger subgroups: $G(r,1,n-1) \times \mu_r$

<u>Thm</u> [BFIL] McK$(G(r,1,n))$:

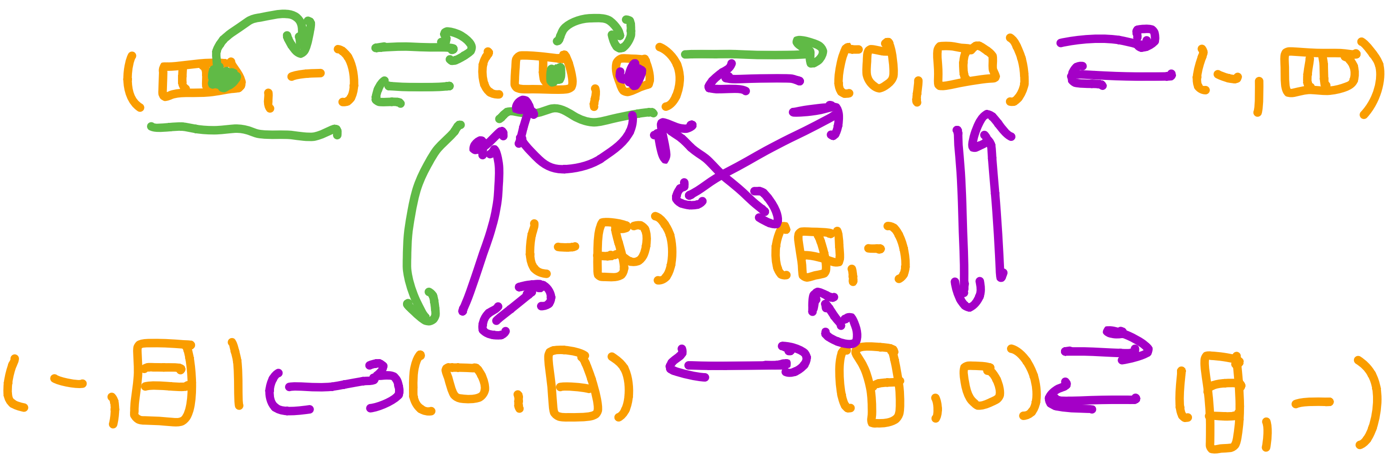
Vertices: $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ as above

Arrows: $\underset{(\alpha^{(1)}, \ldots, \alpha^{(r)})}{\alpha}$ to $\beta = (\beta^{(1)}, \ldots, \beta^{(r)})$

$\Leftrightarrow$ the r-tuple of β is obtained from

r-tuple of α by deleting a cell in position i of α and then adding it to position $(i+1)$ in β modulo r.

eg: $G(2,1,3)$



Cor: $MK(G(r,1,n))$ has arrows in both directions $\Leftrightarrow r=2$.
There are no loops in $MK(G(r,1,n))$ if $r \geq 2$.

3. $G(r,p,n)$

Same idea, use Clifford theory for Irr-Res.

$\boxed{\text{For simplicity: let } r=p}$ ← can do it in general

$G(r,p,n) \triangleleft G(r,1,n)$ index p
$\quad \| \qquad\qquad \|$
$\quad H \qquad\qquad G$

Irreps of H [see STEMBRIDGE 1989]:

$G/H \cong \langle \delta_1 \rangle$, where $\delta_1 = (-, \underbrace{\boxed{}\boxed{}\boxed{}}_{n}, -, -)$

If $\underset{(\lambda^{(1)},\dots,\lambda^{(r)})}{\lambda} \in \mathrm{Irrep}(G)$, then $\lambda^{\delta_1} := \lambda \otimes \delta_1$
$= (\lambda^{(r)}, \lambda^{(1)}, \dots, \lambda^{(r-1)})$

λ^{δ_1} is SHIFT of λ

and $\lambda^{\delta_i} = \lambda \otimes \underbrace{\delta_1 \otimes \dots \otimes \delta_1}_{i\text{-times}}$

Let $[\lambda]$ be the (G/H)-orbit of λ.

Say that for $\mu \in \mathrm{Irrep}(G)$ we have
$\lambda \simeq \mu :\Leftrightarrow \mu = \lambda^{\delta_i}$ for some $i \in \{0, \dots, p-1\}$

$\Rightarrow [\lambda] = \{\mu \in \mathrm{Irrep}(G) : \mu \simeq \lambda\}$

Let $(G/H)_\lambda := \{\delta \in G/H : \lambda = \delta \otimes \lambda\}$
stabilizer of λ

$|(G/H)_\lambda| =: u(\lambda)$

Then [Stembridge]

$\{\text{Irreps of } H = G(r,p,n)\} \overset{1-1}{\longleftrightarrow} ([\lambda], \delta)$, $\lambda \in \mathrm{Irrep}(G)$, $\delta \in (G/H)_\lambda$

ex: $\lambda = ([\square], 0, [\square])$
$G(3,1,3)$
$H = G(3,3,3)$ $\quad u(\lambda) = 3$

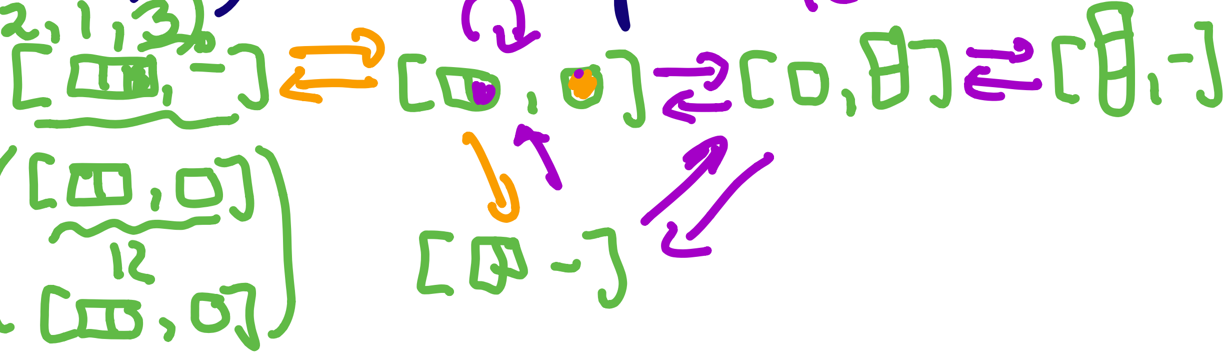
$\to$ Can describe $\mathrm{Res}\text{-}\mathrm{Ind}_H^G$
+ be careful with symmetries of λ!

<u>Thm</u> [BFIL] Let $([\alpha], \delta)$, $([\beta], \delta')$ be irreps of $H = G(r,p,n)$.
If $(G/H)_\alpha$ is trivial, then there is an arrow from $([\alpha], \delta)$ to $([\beta], \delta')$ in $\mathrm{MK}(H)$
$\iff$ $[\beta]$ can be obtained from $[\alpha]$ by moving a single cell cyclically to the right.

If $|u(\alpha)| > 1$: define fundamental domain
($r \neq p$: $\frac{r}{p}$-d-tuples in α)

ex: $G(2,2,3)$ $\quad r = 2$ $\quad \frac{r}{p} = 1$

$G(2,1,3)$

Irreps: 

$MR(G(2,2,3)) = MR(G(1,1,4))$

Indeed: $G(2,2,3) \cong G(1,1,4)$

IV McKay correspondence:

$G \circlearrowright S = \mathbb{C}[x_1, \dots, x_n]$, set $A := S * G$

Thm [BFI 2020] Let $G \subset GL(V)$ be true reflection group, then

$\{\text{nontrivial irreps of } G\} \xleftrightarrow{1:1} \{\text{indec. } R/\Delta\text{-direct summands of } S/z\}$

discriminant of G

$$\begin{array}{ccc} & G & \\ & \circlearrowright & \\ \pi: & \mathbb{C}^n & \longrightarrow \mathbb{C}^n/G \\ & \cup & \\ & V(z) & \longmapsto \pi(V(z)) = V(\Delta) \end{array}$$

SMOOTH

hyperplane arrang.

More precisely: $A/AeA \cong \mathrm{End}_{R/\Delta}(S/z)$

$e = \frac{1}{|G|}\sum_{g \in G} g$

NCR of R/Δ

In $\dim(V) = 2$: $MR(G)/\mathrm{triv}$ = AR-quiver of $MCM(R/\Delta)$

ADE curve

ex: $G(1,1,3) = S_3$

$\Delta = A_2$-curve

What about higher order reflections?

[BFI]: If G has at least one generating reflection of order ≥ 3 (i.e. NOT true refl. grp)

then $\boxed{A/AeA \not\cong \mathrm{End}_{R/\Delta}(\ldots)}$

(look at codim 1 structure of A/AeA)

→ But what about $\mathrm{End}_{R/\Delta}(S/z)$?
Still NCR? (Not clear in general(!))

Look at $n=2$ case first, for $G(r,p,n)$:

[BANNAI] dim 2 discriminants are all ADE curves

For $G(r,1,2)$: A_2-curve sing. $= \Delta$

[MAY 2020] For $G(r,1,2)$
$\mathrm{End}_{R/\Delta}(S/z)$ is an NCR of R/Δ)
is rep. generator, can describe isotypical components

(+ $G(1,p,2)$ nearly finished).

Further questions:

- What is S/z for $G(1,p,n)$, $n \geq 3$?
- How does $\mathrm{End}_{R/\Delta}(S/z)$ relate to $\underbrace{A/AeA}_{\text{algebra structure?}}$?