Categorification and $\mathbb{C}_q[Gr(k,n)]$

Xiuping Su
University of Bath

<u>Goal</u>: to introduce an invariant $\kappa(M,N)$ and discuss its applications, including its role in categorifying the quantum cluster structure in $\mathbb{C}_q[Gr(k,n)]$.

Plan.

I. $CM(A)$: the Grassmannian cluster category

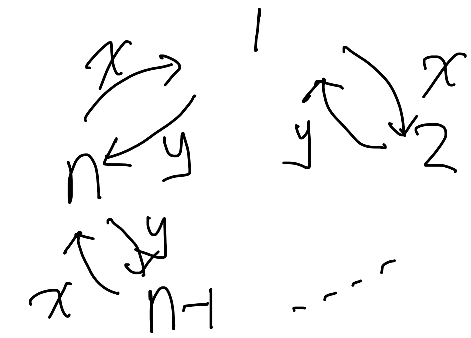
II. $K(M,N)$: the invariant

III. Main results: quantum seed data (B, L) and $\mathbb{C}_q[Gr(k,n)]$

Reference: arXiv 1904.07849
(jt. w. B T Jensen and A King)

I. $CM(A)$: $k < n \in \mathbb{N}$

1. Q = double cyclic quiver



2. A = complete path algebra $\widehat{\mathbb{C}Q}$ modulo relations: $xy = yx$, $x^k = y^{n-k}$.

Remark: • $R = \dfrac{\mathbb{C}[[x, y]]}{\langle x^k - y^{n-k} \rangle}$ & $G = C_n$: the cyclic group of order n

• $A = R * G$, twisted group ring

• $Z = \mathbb{C}[[xy]]$: the centre of A.

3. $CM(A)$ = Cohen – Macaulay A–modules

= A-modules that are free over Z

= G-equivariant CM-modules of R

<u>Remark</u>: · $\operatorname{rank}_Z e_i M = \operatorname{rank}_Z e_{i+1} M$, $\forall i$

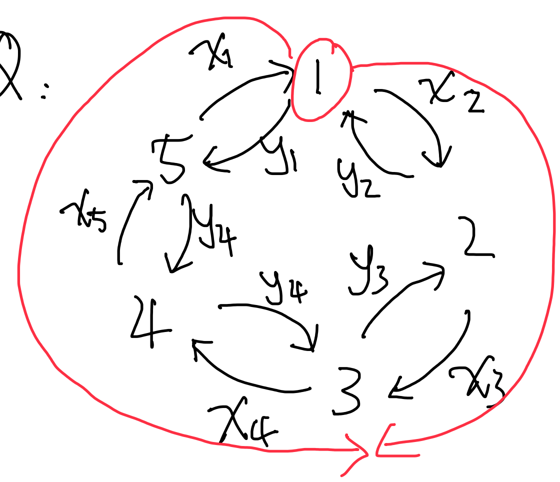
· $\operatorname{rank} M := \operatorname{rank}_Z e_i M$ well-defined

(e_i = the trivial path at vertex i)

4. <u>Examples</u>:

$k=2$, $n=5$

Q:

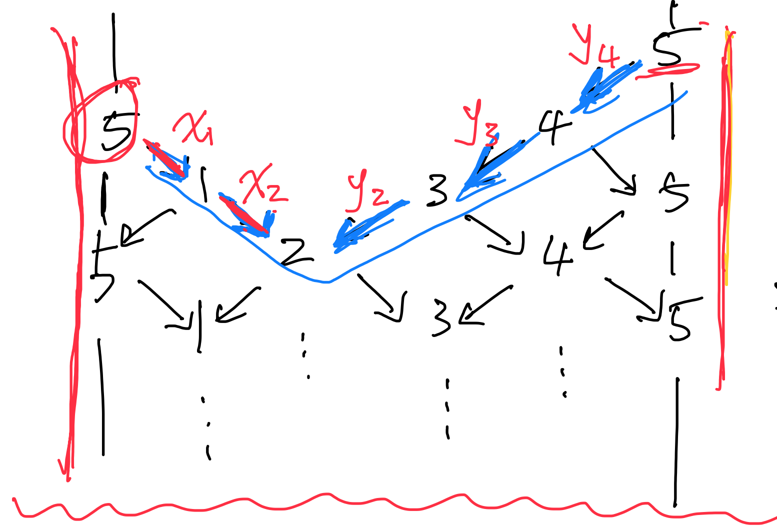


relations:

$$\begin{cases} x_i y_i = y_{i+1} x_{i+1} \\ x_{i+1} x_i = y_{i-3} y_{i-2} y_{i-1} \end{cases}$$

① $P_5 = Ae_5$

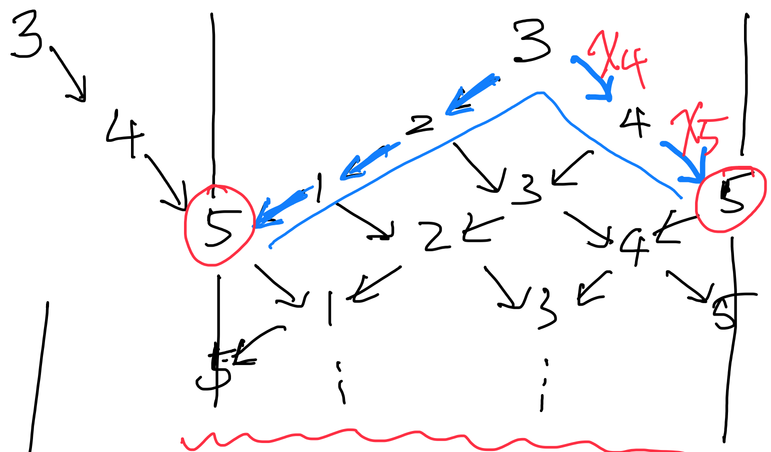
- indec. proj.
- CM(A)
- rank $=1$



$= M_{\{1,2\}}$

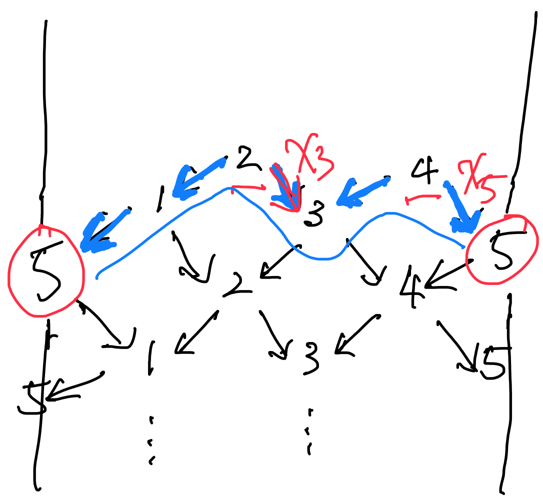
contour of P_5

② $P_3 = Ae_3$



$= M_{\{4,5\}}$

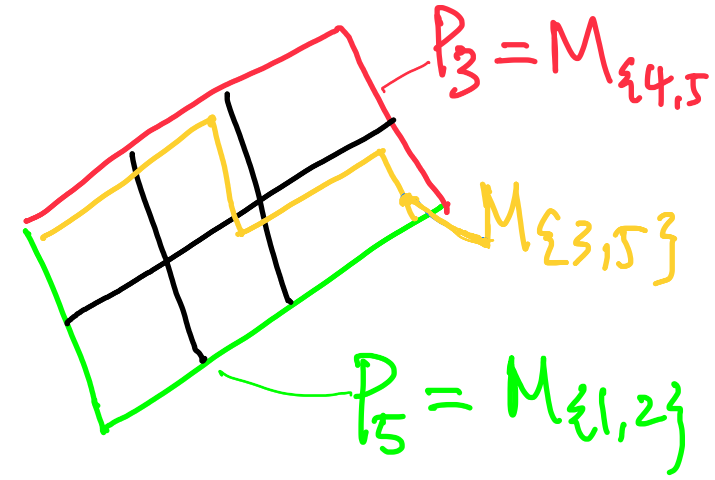
③



$= M_{\{3,5\}}$

$$\{\text{rank } 1 \text{ modules}\}$$
$$M \quad \mapsto \quad \text{indices of d-r arrows}$$
$$\updownarrow \ 1\text{-}1$$
$$\{I \subseteq \{1,\cdots,n\} : |I| = k\}$$

By identifying the "top 5":

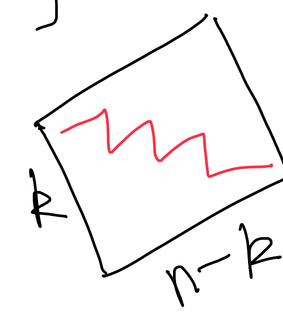


$P_5 \leq M_{\{3,5\}} \leq P_3$

(In general, $P_n \leq M_I \leq P_{n-k}$)

rank 1 modules $\overset{1-1}{\longleftrightarrow}$ k-sets in $\{1, \cdots, n\}$
(down-right steps)

$\overset{1-1}{\longleftrightarrow}$ Young diagrams in



($\overset{1-1}{\longleftrightarrow}$ certain partitions)

Remark: • $k=2$: rank of any indec. CM A-module $= 1$.

• $\exists$ cluster tilting objects $(\forall k, n)$

• $\forall M$, $\exists$ cluster char. $\psi_M \in \mathbb{C}[Gr(k,n)]$.

Remark (continued):

- In particular, $\Psi_{M_I} = \Delta_I$, the minor associated to I.
- Via the cluster character Ψ, $\mathrm{CM}(A)$ provides an additive categorification of the Grassmannian cluster algebra.

II. $K(M,N)$

1. $E: \mathrm{mod}\,A \to \mathrm{mod}\,Z$ (restriction functor)

$$M \longrightarrow M_n = e_n M = \mathrm{Hom}_A(Ae_n, M)$$
$$= e_n A \otimes_A M$$

- left adjoint P: $\mathrm{mod}\,Z \to \mathrm{mod}\,A$, $W \longmapsto Ae_n \otimes_Z W$
- right adjoint J: $\mathrm{mod}\,Z \to \mathrm{mod}\,A$, $W \longmapsto \mathrm{Hom}_Z(e_nA, W)$
- $\mathrm{Hom}_A(M, JN_n) \cong \mathrm{Hom}_Z(M_n, N_n) \cong \mathrm{Hom}_A(PM_n, N)$

2. For $M, N \in CM(A)$, define

$$\varphi: \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_Z(M_n, N_n), \quad f \mapsto f_n = Ef$$

and $K(M,N) = \mathrm{cok}\,\varphi$.

Lemma (1) $\exists$ embeddings and so s.e. seq's

$0 \to PM_n \xrightarrow{\alpha} M \to \Pi(M) \to 0$, $\quad 0 \to N \xrightarrow{\beta} JN_n \to \omega(N) \to 0$ fin. dim.

(2) $\exists$ commutative diagram

$$\begin{tikzcd} & \mathrm{Hom}_A(PM_n, N) \arrow[d, "\cong"] \\ \mathrm{Hom}_A(M,N) \arrow[ur, "\alpha^*(N)"] \arrow[r, "\varphi"] \arrow[dr, "\beta_*(M)"'] & \mathrm{Hom}_{\mathbb{Z}}(M_n, N_n) \\ & \mathrm{Hom}_A(M, JN_n) \arrow[u, "\cong"'] \arrow[r] & \mathrm{Hom}_A(M, \omega(N)) \end{tikzcd}$$

fin. dim.

$\left\{ \begin{array}{l} \alpha, \beta \ \exists : \text{unit-counit} \\ \varepsilon\alpha = Id, \ \varepsilon\beta = Id \end{array} \right.$

$\Rightarrow \operatorname{cok} \alpha^*(N) \cong K(M,N) \cong \operatorname{cok} \beta_*(M) \leq \mathrm{Hom}_A(M, \omega(N))$

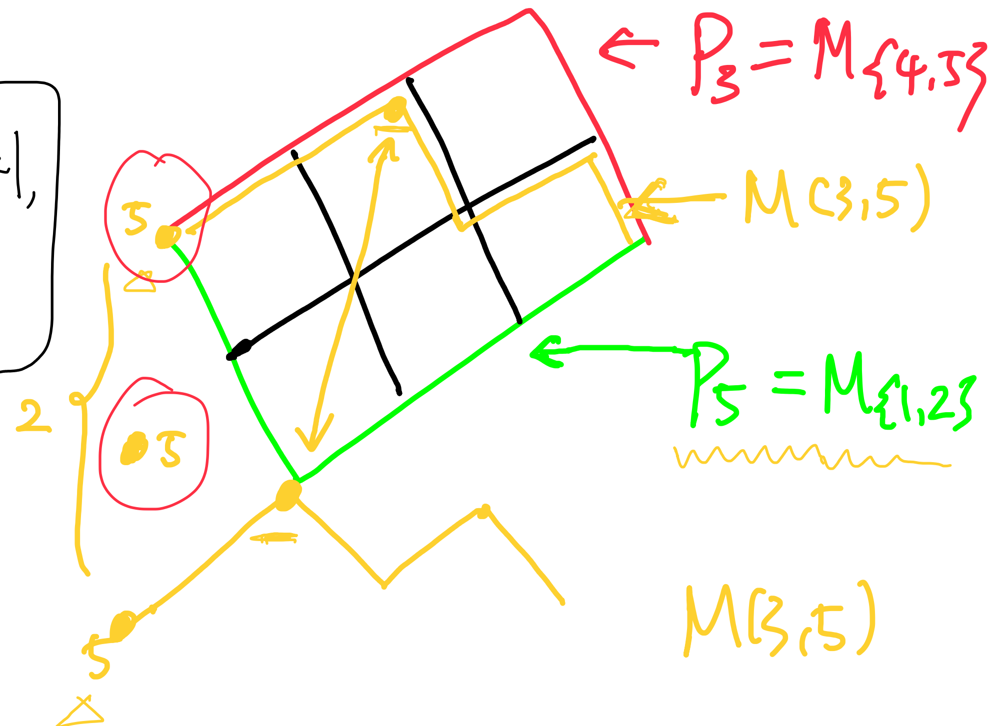
In particular, $K(M,N)$ is finite dimensional.

3. Define $k(M,N) = \dim K(M,N)$, $\forall M, N \in CM(A)$

4 Examples

<u>Note</u>: when $\operatorname{rank} M = \operatorname{rank} N = 1$,

$$\operatorname{Hom}_{\mathbb{Z}}(M_n, N_n) \cong \mathbb{Z}$$



① $K(P_5, M_I) = 0$

② $K(M_I, P_3) = 0$

③ $K(M_I, P_5) = 2$

$$\varphi : \operatorname{Hom}_A(M,N) \to \operatorname{Hom}_{\mathbb{Z}}(M_n, N_n)$$

$$\cong \mathbb{Z} \qquad \text{rank } 1$$

Remark

- Let S = maximal collection of non-crossing k-sets $\subseteq \{1, \cdots, n\}$
- $T_S = \bigoplus_{I \in S} M_I$ is a cluster tilting object.
- $K(T_S, M_J) = (K(M_I, M_J))_{I \in S}$ is the exponent vector of a flow polynomial defined on the network chart determined by S.
- $K(T_S, M_J)$, $\forall J$: the lattice point of the Newton-Okounkov body constructed by the Rietsch-Williams.

III. Main results: (B, L) and $\mathbb{C}_q[Gr(k,n)]$

1. $T = \bigoplus_{i=1}^{N} T_i$, a cluster tilting object, $= \bigoplus_{i=1}^{N-n} T_i \oplus_A A$

$B = (b_{ij})_{N\times(N-n)}$: $b_{ij} = \dim \mathrm{Irr}(T_j, T_i) - \dim \mathrm{Irr}(T_i, T_j)$

$L = (\lambda_{ij})_{N\times N}$: $\lambda_{ij} = K(T_j, T_i) - K(T_i, T_j)$

Theorem

① (B, L) is compatible, i.e. $B^T L = (\underbrace{I}_{N-n}, \underbrace{0}_{n})$

② $\underline{\mu_k(B(T), L(T))} = (B(\underline{\underline{\mu_k(T)}}), L(\mu_k(T)))$

↳ a new cluster tilting obj.

Berenstein-Zelevinsky's mutation

Theorem: $T = T_S = \bigoplus_{I \in S} M_I$. $L = (\lambda_{IJ})$ coincide with Leclerc-Zelevinsky's quasi-commutation rules for the quantum minors Δ_I $(I \in S)$. That is

$$\lambda_{IJ} = K(M_I, M_J) - K(M_J, M_I) = c(I, J),$$

where $\Delta_I \Delta_J = q^{c(I,J)} \Delta_J \Delta_I$.

2. (B, L) is compatible and so determines a quantum cluster algebra $\mathcal{C}_q(B, L)$ over $\mathbb{C}[q]$

Theorem: $T = T_S$ and $(B, L) = (B(T), L(T))$

$$\mathcal{C}_q(B, L) \otimes_{\mathbb{C}[q]} \mathbb{C}(q) \cong \mathbb{C}_q[\mathrm{Gr}(k, n)] \otimes_{\mathbb{C}[q]} \mathbb{C}(q).$$

Sketch of Proof:

$$\begin{tikzcd} \mathbb{C}_q[Gr(k,n)] \arrow[r, hook, "\eta_1"] \arrow[d, dashed, "\exists\,\eta"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}_q(Gr(k,n)) \arrow[d] \\ \mathbb{C}_q(B,L) \arrow[ur, hook, "\eta_2"'] & \text{skew field of fractions} \end{tikzcd}$$

① η exists because $\operatorname{Im}\eta_2$ contains all the q-minors

② $\mathbb{C}_q[Gr(k,n)]$, $\mathbb{C}_q(B,L)$ are flat deformations of their classical counterparts. (Kelly-Lenagen-Rigal; Geiss-Leclerc-Schröer). ∎

Remark: Grabowski-Launois proved a version of this result, using delicate homogenisation of GLS-result. Our approach is completely different from theirs.