Stability and McKay

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MPIM

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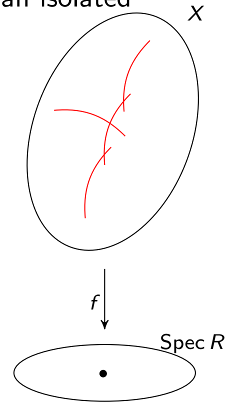
Geometric Setting

Classical Setting: $f : X \to \operatorname{Spec} R$ is the minimal resolution of a Kleinian (Du Val) singularity.

3-fold setting: $f : X \to \operatorname{Spec} R$ is a minimal model of an isolated compound Du Val (cDV) singularity:



- each $\operatorname{Spec} R$ has more than one, but finitely many, minimal models;
- taking a generic slice, f yields a partial resolution of a Kleinian singularity.

In both cases, work of Bridgeland–King–Reid and Van den Bergh shows how to construct a tilting bundle $\mathcal{V} = \mathcal{O}_X \oplus \mathcal{V}'$ on X so that, if $\Lambda := \operatorname{End}_X(\mathcal{V})$, then

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda).$$

Finite-Dimensional Algebras

Set Λ_{con} to be the stable endomorphism algebra

$$\underline{\text{End}}_X(\mathcal{V}) := \text{End}_X(\mathcal{V})/[\mathcal{O}_X]$$

where $[\mathcal{O}_X]$ consists of morphisms factoring through some $\mathcal{O}_X^{\oplus n}$.

- This is a finite dimensional algebra.
- In the surfaces case, Λ_{con} is the preprojective algebra of the corresponding ADE Dynkin diagram.

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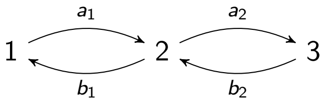
If Spec R is the A_3 surface singularity, Λ_{con} is given by the quiver and relations:

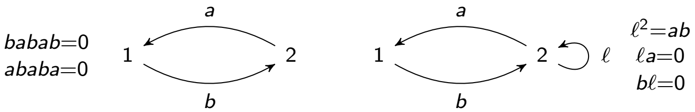


$$a_1 b_1 = 0$$
$$b_2 a_2 = 0$$
$$b_1 a_1 + a_2 b_2 = 0$$

Finite-Dimensional Algebras ctd.

In the 3-fold setting, Λ_{con} is called the *contraction algebra of f*.

There is a cA_2 singularity $R := \frac{\mathbb{C}[[u,v,x,y]]}{(uv-xy(x+y^2))}$ with 6 minimal models and where each of the contraction algebras is isomorphic to one of:



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$bab ab=0$
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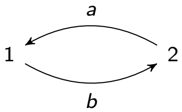

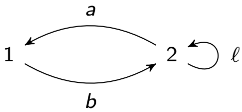

$\ell^2=ab$
$\ell a=0$
$b\ell=0$

In both the surface and the 3-fold setting, we have functors

$$\mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathsf{con}}) \xrightarrow{\mathrm{res}} \mathrm{D^b}(\mathrm{mod}\,\Lambda) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{coh}\,X).$$

What is Bridgeland Stability?

Definition

Let $\mathcal{T}$ be a triangulated category with $K_0(\mathcal{T}) \cong \mathbb{Z}^n$. A stability condition on $\mathcal{T}$ is a pair $(Z, \mathcal{A})$ where:

- *$\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{T}$;*
- *$Z\colon K_0(\mathcal{T}) \to \mathbb{C}$ is a group homomorphism which we call the central charge;*

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1. *If $E \in \mathcal{A}$ then $Z(E) \in \mathbb{H}$, where $\mathbb{H} = \mathbb{R}_{>0}e^{i\pi\phi}, \phi \in (0,1]$;*
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We write Stab$(\mathcal{T})$ *for the set of all (locally-finite) stability conditions on $\mathcal{T}$.*

What is Bridgeland Stability?

If $\mathcal{A} \subset \mathcal{T}$ has finite length, the locally-finite and HN properties are automatically satisfied, and so

$$\mathrm{Stab}(\mathcal{A}) = \{(Z, \mathcal{B}) \in \mathrm{Stab}(\mathcal{T}) \mid \mathcal{B} = \mathcal{A}\} \cong \mathbb{H}^n.$$

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Theorem (Bridgeland)

There is a topology on $\mathsf{Stab}(\mathcal{T})$ *such that the forgetful map*

$$\begin{aligned} p\colon\ \mathsf{Stab}(\mathcal{T}) &\to \mathsf{Hom}_{\mathbb{Z}}(K_0(\mathcal{T}), \mathbb{C}) \cong \mathbb{C}^n \\ (Z, \mathcal{A}) &\mapsto Z \end{aligned}$$

is a local homeomorphism. In particular, $\mathsf{Stab}(\mathcal{T})$ *is a complex manifold.*

One of the key difficulties in describing this space is understanding all the hearts of $\mathcal{T}$.

Stability for Minimal Resolutions

For both the surface and 3-fold settings we do not study stability of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, but instead on the full subcategory

$$\mathcal{C} := \{\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \mid \mathrm{R}f_* \mathcal{F} = 0\}.$$

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Suppose that $f\colon X \to \operatorname{Spec} R$ is the minimal resolution of a Kleinian singularity and $\mathfrak{h}$ is the corresponding ADE root system. The complexified complement of $\mathfrak{h}$ is $\mathfrak{h}_{\mathrm{reg}}$.

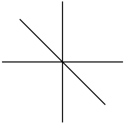
Theorem (Bridgeland, Brav–Thomas)

In this surfaces case, there is a connected component of $\operatorname{Stab}(\mathcal{C})$ *which is the universal cover of* $\mathfrak{h}_{\mathrm{reg}}$.

Stability for Minimal Models

In the 3-fold setting, since a minimal model $f\colon X \to \operatorname{Spec} R$ cuts to a partial resolution of a Kleinian singularity, it has an associated real hyperplane arrangement $\mathcal{H}$, given by (an intersection arrangement of) the corresponding ADE root system.

e.g. for the cA_2 example given before the associated hyperplane arrangement is the A_2 root system.



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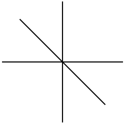

Theorem (Hirano–Wemyss)

In the 3-fold setting, there is a connected component of $\operatorname{Stab}(\mathcal{C})$ *given by the universal cover of* $\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}}$.

To show the stability manifold is contractible, they have to use a result known as Deligne's $K(\pi, 1)$ theorem for ADE root systems.

Silting/Tilting Theory

Suppose A is a finite dimensional algebra over $\mathbb{C}$.

Definition

A complex $T \in \mathrm{K}^{\mathrm{b}}(\operatorname{proj} A)$ *is called tilting (silting) if:*

1. $\operatorname{Hom}_A(T, T[n]) = 0$ *for all* $n \neq 0$ *(*$n > 0$*);*
2. T *generates* $\mathrm{K}^{\mathrm{b}}(\operatorname{proj} A)$ *as a triangulated category.*

e.g. $A, A[n]$ are tilting complexes over A.

Theorem (Rickard)

Two finite dimensional algebras A *and* B *are derived equivalent if and only if there exists a tilting complex* $T \in \mathrm{K}^{\mathrm{b}}(\operatorname{proj} A)$ *such that* $\operatorname{End}_A(T) \cong B$*.*

Connection with Stability

Theorem (Koenig–Yang Correspondence)

There is a bijection between:

- *silting complexes in* $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\, A)$;
- *hearts of bounded t-structures on* $\mathrm{D}^{\mathrm{b}}(A)$ *with finite length.*

So each silting complex in $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\, A)$ gives a piece of $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(A))$ isomorphic to $\mathbb{H}^n$, and mutation controls how these pieces fit together.

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If A is silting-discrete (a finiteness condition on the number of silting complexes), then Pauksztello–Saorin–Zvonareva show:

1. the heart of any bounded t-structure on $\mathrm{D}^{\mathrm{b}}(A)$ has finite length;
2. $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(A))$ is contractible.

Back to Contraction Algebras

For now, assume we are in the 3-fold setting:

Theorem (A.)

Contraction algebras are silting-discrete.

With a bit of work, this essentially comes down to the fact each cDV singularity has finitely many minimal models.

- So we know $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathsf{con}}))$ is contractible, and that all hearts can be described using silting complexes.
- But moreover, contraction algebras are symmetric!
- One consequence is that every silting complex is in fact a tilting complex.
- So all hearts of $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathsf{con}})$ can be described using tilting complexes, or equivalently, standard derived equivalences.

Flops

For a given isolated cDV singularity, all its minimal models are connected by *flops*.

- Loosely, we can think of these as choosing a curve in the exceptional locus, cutting it out, and gluing it back in differently.
- If we label the curves $C_1, \ldots, C_n$, and denote the minimal model obtained by flopping curve C_i by $f_i \colon X_i \to \operatorname{Spec} R$, there are associated derived equivalences

$$G_i \colon \mathrm{D^b}(\mathrm{coh}\, X) \to \mathrm{D^b}(\mathrm{coh}\, X_i)$$

  called flop functors.
- Understanding compositions of these flop functors, and in particular the autoequivalences obtained by composing flop functors is key to obtaining the Hirano-Wemyss result.

Flops and Contraction Algebras

Suppose $f\colon X \to \operatorname{Spec} R$ has contraction algebra Λ_{con} and the $f_i\colon X_i \to \operatorname{Spec} R$ has contraction algebra $\nu_i \Lambda_{\text{con}}$.

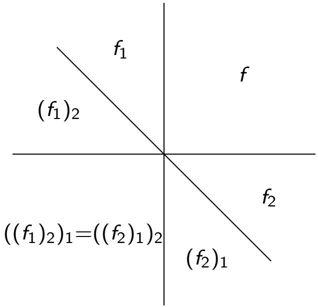
Theorem (A.)

There is a standard derived equivalence $F_i\colon \mathrm{D}^{\mathrm{b}}(\Lambda_{\text{con}}) \to \mathrm{D}^{\mathrm{b}}(\nu_i \Lambda_{\text{con}})$ *making the following diagram commute:*

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}(\Lambda_{\text{con}}) & \xrightarrow{\ \text{res}\ } & \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \\ \downarrow F_i & & \downarrow G_i \\ \mathrm{D}^{\mathrm{b}}(\nu_i \Lambda_{\text{con}}) & \xrightarrow{\ \text{res}\ } & \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X_i) \end{array}$$

Derived Equivalences

If $f : X \to \operatorname{Spec} R$ is a minimal model with associated hyperplane arrangement $\mathcal{H}$, Wemyss shows the chambers are in bijection with the minimal models of $\operatorname{Spec} R$, and wall crossing corresponds to flopping.



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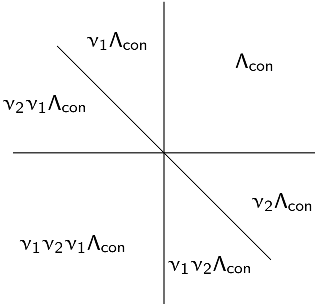

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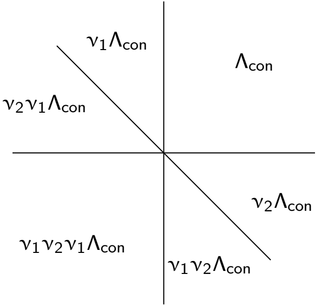

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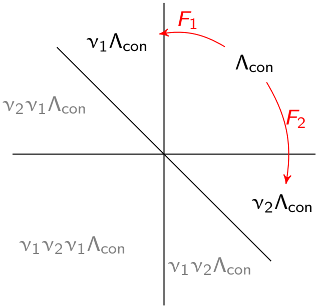

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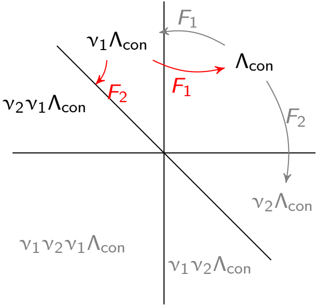

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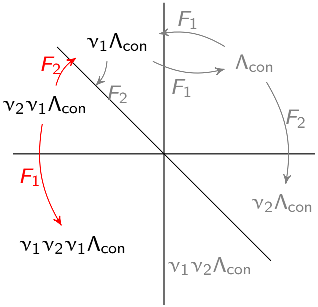

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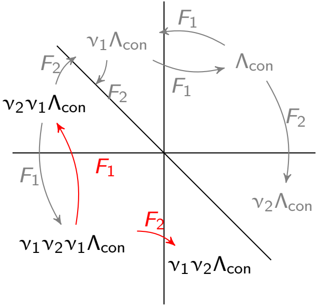

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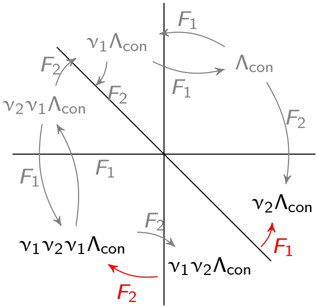

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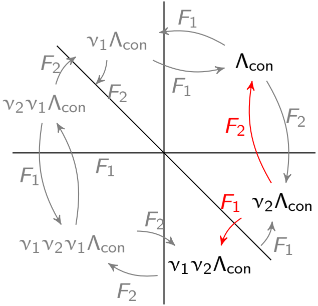

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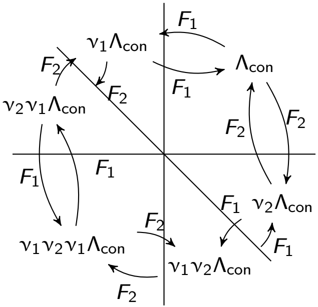

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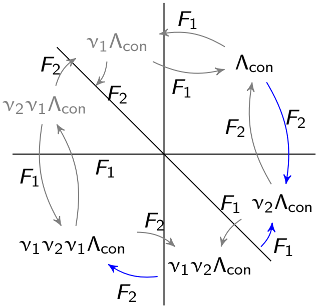

$F_2 \circ F_1^{-1} \circ F_2 : \ \mathrm{D}^{\mathrm{b}}(\Lambda_{con}) \to \mathrm{D}^{\mathrm{b}}(\nu_1\nu_2\nu_1\Lambda_{con})$

Results

Theorem (A.)

1. *Since Λ_{con} is silting-discrete, any standard equivalance from $\mathrm{D}^{\mathrm{b}}(\Lambda_{\text{con}})$, up to algebra isomorphism, is obtained as a path in this picture starting at $C_{\Lambda_{\text{con}}}$.*
2. *Or equivalently, any heart of $\mathrm{D}^{\mathrm{b}}(\Lambda_{\text{con}})$ is obtained as a path ending at $C_{\Lambda_{\text{con}}}$.*
3. *The F_i satisfy the Deligne relations coming from $\mathcal{H}$ e.g. the braid relation $F_1 \circ F_2 \circ F_1 \cong F_2 \circ F_1 \circ F_2$.*
4. *As a consequence, there is a group homomorphism*

$$\phi\colon \pi_1(\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}}) \to \mathsf{Auteq}(\mathrm{D}^{\mathrm{b}}(\Lambda_{\text{con}}))$$
$$\alpha\colon C_{\Lambda_{\text{con}}} \to C_{\Lambda_{\text{con}}} \mapsto F_\alpha.$$

Corollary (A.–Wemyss)

If $\alpha\colon C_{\Lambda_{\text{con}}} \to C_{\Lambda_{\text{con}}}$, then the isomorphism that F_α induces on the Grothendieck group is the identity.

What does this mean for stability?

Consequence: Every point of $\mathrm{Stab}(\mathrm{D^b}(\Lambda_{\mathrm{con}}))$ can be described as $(Z, \alpha\colon C_A \to C_{\Lambda_{\mathrm{con}}})$ where α is a path in our picture, and Z is a compatible group homomorphism.

The heart of $\mathrm{D^b}(\Lambda_{\mathrm{con}})$ corresponding to α is $F_\alpha(\operatorname{mod} A)$ where recall,

$$F_\alpha\colon \mathrm{D^b}(A) \to \mathrm{D^b}(\Lambda_{\mathrm{con}}).$$

When is Z compatible?

- By definition, we need $Z(E) \in \mathbb{H}$ for all $E \in F_\alpha(\operatorname{mod} A)$;
- Or equivalently, $Z \circ F_\alpha(E) \in \mathbb{H}$ for all $E \in \operatorname{mod} A$;

Group Actions

In any setting, the group Auteq($\mathcal{T}$) acts on Stab($\mathcal{T}$) via

$$\Phi \cdot (Z, \mathcal{A}) = (Z \circ \Phi^{-1}, \Phi(\mathcal{A})).$$

Restricting to $\mathrm{im}(\phi) \leq \mathsf{Auteq}(\mathrm{D^b}(\Lambda_{\mathsf{con}}))$, if $\beta\colon C_{\Lambda_{\mathsf{con}}} \to C_{\Lambda_{\mathsf{con}}}$ then

$$F_\beta \cdot (Z, \alpha) = (Z, \beta \circ \alpha).$$

Since the action does not effect the central charge it is clear the forgetful map factors as

$$\begin{array}{ccc} \mathsf{Stab}(\mathrm{D^b}(\Lambda_{\mathsf{con}})) & \xrightarrow{\quad p \quad} & \mathbb{C}^n \\ {\scriptstyle p_1} \searrow & & \nearrow {\scriptstyle p_2} \\ & \mathsf{Stab}(\mathrm{D^b}(\Lambda_{\mathsf{con}}))/\mathrm{im}(\phi) & \end{array}$$

Stability Conditions on Contraction Algebras

$$\begin{array}{ccc} \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})) & \xrightarrow{\quad p \quad} & \mathbb{C}^n \\ & {\scriptstyle p_1} \searrow \quad \nearrow {\scriptstyle p_2} & \\ & \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}))/\mathrm{im}(\phi) & \end{array}$$

It remains to show that:

- p_2 gives an isomorphism onto $\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}}$:
  - show that $\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}}$ can be written as a disjoint union of $\mathbb{H}^n$'s;
  - observe that the action identifies (Z, α) and (Z, β), where α and β start in the same chamber.
- p_1 is a regular covering map:
  - show the action is of $\mathrm{im}(\phi)$ on $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}))$ is free and properly discontinuous;
  - the commutative diagram linking F_i with the flop functors is crucial.

Theorem (A.–Wemyss)

$\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}))$ *is the universal cover of* $\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}}$.

What about the surface story?

We can summarise the 3-fold story as saying that studying stability for a minimal model, and for its contraction algebra, both give the same answer. Since things are so similar in the surfaces story, should we expect the same answer there?

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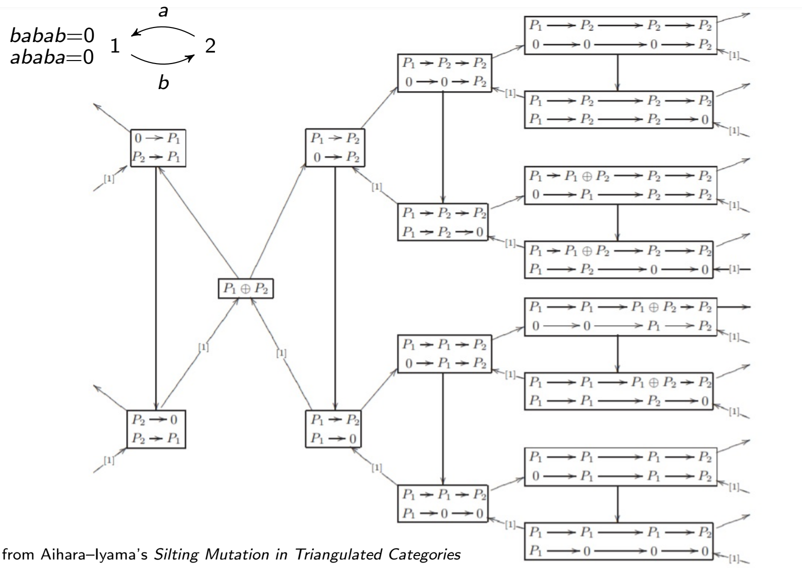
Two obstacles:

- Preprojective algebras of ADE Dynkin type are not symmetric, only self-injective. As a consequence, there might be some silting complexes which are not tilting.
- Aihara–Mizuno show preprojective algebras of ADE Dynkin type are tilting-discrete, but not silting-discrete so we can't use the results of Paukzstello–Saorin–Zvonareva.

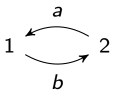
Silting Quiver of Contraction Algebra with A_2 root system

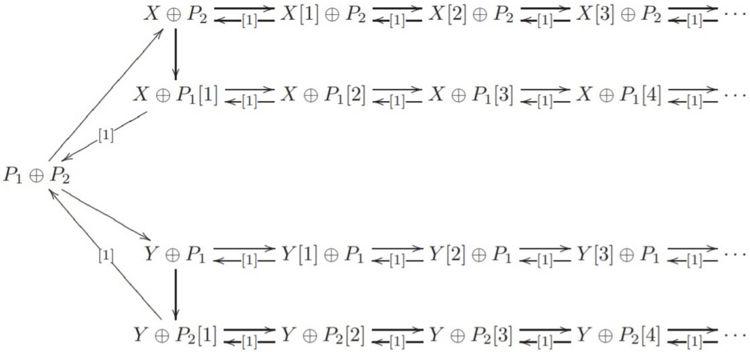


Picture from Aihara–Iyama's *Silting Mutation in Triangulated Categories*

Silting Quiver for A_2 Preprojective Algebra

$ab=0$
$ba=0$





Picture from Aihara–Iyama's *Silting Mutation in Triangulated Categories*

Weakly symmetric algebras

The preprojective algebras of type D_{2n}, E_7 and E_8 have trivial Nakayama permutation i.e. they are weakly symmetric algebras.

Theorem

A tilting-discrete weakly symmetric algebra is in fact silting-discrete, and every silting complex is a tilting complex.

Now Mizuno's work completely describes the tilting theory of these algebras, including providing a commutative diagram linking their derived autoequivalences to the twist functors (surface analogue of flop functors) on the geometric side.

Theorem

If Λ is a weakly symmetric preprojective algebra of ADE Dynkin type, then $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(\Lambda))$ *is the universal cover of the corresponding Dynkin root system.*

Thank you!