The McKay correspondence, mutation and related topics

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Grassmannian categories of infinite Rank

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[ACFGS]

Idea: Categorify Grassmannian cluster algebras of infinite Rank

Fomin-Zelevinsky: Cluster algebra $\mathcal{A}$ subalgebra of $\mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}]$

generaters : cluster variables ⇝ clusters

$\mathcal{A}$ generated by mutation of clusters.

1 Grassmannian cluster algebras

$Gr(k,n)$ Grassmannian of k-subspaces of $\mathbb{C}^n$

Scott 06 : $\mathbb{C}[Gr(k,n)]$ is a cluster algebra

$$\mathbb{C}[\,p_I \mid I \subset \underset{\substack{\| \\ [n]}}{\{1,\dots,n\}},\ |I|=k\,] \big/ \mathcal{I}_p$$

where
$$\mathcal{I}_p = \Big\langle \sum_{r=0}^{k} (-1)^r \, p_{J' \cup \{j_r\}} \, p_{J \setminus \{j_r\}} \;\Big|\; J, J' \subset [n],\ |J| = k+1,\ |J'| = k-1,\ J = \{j_0, \dots, j_k\} \Big\rangle$$

Def:

- $I, J \subseteq \mathbb{Z}$ k-subsets are crossing if $\exists$ $i_1, i_2 \in I \setminus J$ & $j_1, j_2 \in J \setminus I$ s.t. $i_1 < j_1 < i_2 < j_2$ or $j_1 < i_1 < j_2 < i_2$

- p_I & p_J are compatible if I & J are non-crossing.

Scott 06: Maximal sets of compatible Plücker coordinates are (examples of) clusters.

$k=2$: Plücker coordinates $\overset{1:1}{\longleftrightarrow}$ cluster variables

Plücker relations $\longleftrightarrow$ exchange formulas

Jensen-King-Su 16: Grassmannian cluster categories

$\mathbb{C}[x,y] \curvearrowleft \mu_n = \{ \xi \in \mathbb{C} \mid \xi^n = 1 \}$

$x \mapsto \xi x \qquad y \mapsto \xi^{-1} y$

$\mathbb{C}[Gr(k,n)]$

ADE

$$R_n = \mathbb{C}[x,y] \big/ (x^k - y^{n-k})$$

$MCM_{\mu_n}(R_n)$ = μ_n-equivariant maximal Cohen-Macaulay R_n-modules

$MCM_{\mu_n}(R_n)$ is a Frobenius category:

- {Rank 1 modules} $\overset{1-1}{\longleftrightarrow}$ { Plücker coordinates }
  $M_I \longleftarrow\!\shortmid\; p_I$
- $Ext^1(M_I, M_J) = 0 \iff p_I$ & p_J are compatible
- cluster-tilting subcategories $\overset{1-1}{\longleftrightarrow}$ maximal sets of compatible Plücker coordinates
- Define a cluster character (using Geiß-Leclerc-Schröer)

2 Infinite Rank

Idea: $n \longrightarrow \infty$ in $Gr(k,n)$

Grabowski – Gratz 14: $\mathcal{A}_k = \mathbb{C}[p_I \mid I \subset \mathbb{Z},\ |I| = k] / \mathcal{I}_p$

$\mathcal{A}_k$ can be endowed with structure of an infinite rank cluster algebra in uncountably many ways.

Gratz 15: $\mathcal{A}_k$ colimit of cluster algebras of finite rank in the category of rooted cluster algebras.

Groechening 14: Construction of $\mathcal{A}_k$ as a coordinate Ring of infinite Rank Grassmannian

$\rightsquigarrow$ $k=2$: $\mathcal{A}_k$ homogeneous coordinate ring of 'infinite' Grassmannian: 2-dim subspaces of a profinite-dimensional vector space

3 Grassmannian categories of infinite rank

$Gr(k,\infty)$

$\mathbb{C}[x,y] \circlearrowleft \mathbb{G}_m = \mathbb{C}^* \ni \xi$

$x \mapsto \xi x \qquad y \mapsto \xi^{-1} y$

$\mathbb{C}[x,y]/(x^k - y^{n-k})$

$R := \mathbb{C}[x,y]/(x^k)$ (think of $n \to \infty$ in $x^k - y^{n-k}$)

$MCM_{\mathbb{G}_m} R$ $\quad$ $\mathbb{G}_m$-equivariant maximal Cohen-Macaulay modules

$\rightsquigarrow \quad \mathrm{mod}_{\mathbb{G}_m} R \simeq \mathrm{gr}\, R$ $\quad$ since $\mathrm{Hom}(\mathbb{G}_m, \mathbb{C}^*) \simeq \mathbb{Z}$

$\mathrm{mod}_{\mathbb{G}_m} R$ = f.g. $\mathbb{G}_m$-equivariant R-modules; $\mathrm{gr}\, R$ = f.g. $\mathbb{Z}$-graded R-modules, $|x| = 1$, $|y| = -1$

$\rightsquigarrow \quad MCM_{\mathbb{G}_m} R \simeq MCM_{\mathbb{Z}} R$ $\quad$ $\mathbb{Z}$-graded maximal Cohen-Macaulay R-modules

$\boxed{\mathrm{MCM}_{\mathbb{Z}} R =: \text{Grassmannian category of infinite rank}}$

Buchweitz 86 : $\underline{\mathrm{MCM}}_{\mathbb{Z}} R \simeq D_{sg}(\mathrm{gr}\, R)$

$k=2$

Holm - Jørgensen 12 : $D^f_{dg}(\mathbb{C}[y])$, $|y| = -1$

has cluster combinatorics of type A.

Rk: $D^f_{dg}(\mathbb{C}[y]) \simeq \underline{\mathcal{C}}$, $\mathcal{C} = \langle$ generically free rank 1 $\mathrm{MCM}_{\mathbb{Z}} R$ modules, $R = \mathbb{C}[x,y]/\langle x^2 \rangle$

Yildirim - Paquette 20 : Completion of discrete cluster categories of infinite type by Igusa - Todorov (15)

$\rightsquigarrow$ If $k=2$, 1 accumulation point

$$\simeq \mathrm{MCM}_{\mathbb{Z}}\, \mathbb{C}[x,y]/(x^2)$$

$\mathcal{F} = \mathbb{C}[x, y^{\pm}] / (x^k)$ total Ring of fractions

<u>Def</u>: $M \in MCM_{\mathbb{Z}} R$ is generically free of Rank n if $M \otimes_R \mathcal{F}$ is a graded free $\mathcal{F}$-module of Rank n.

<u>Proposition</u>: (1) If $M \in MCM_{\mathbb{Z}} R$ generically free then $M = \Omega(N)$ for some $N \in \operatorname{gr} R$ with $\dim N < \infty$

(2) $M \in MCM_{\mathbb{Z}} R$ is generically free of Rank 1 $\iff$ M is a graded ideal of R & $y^n \in M$ for some $n \geq 0$

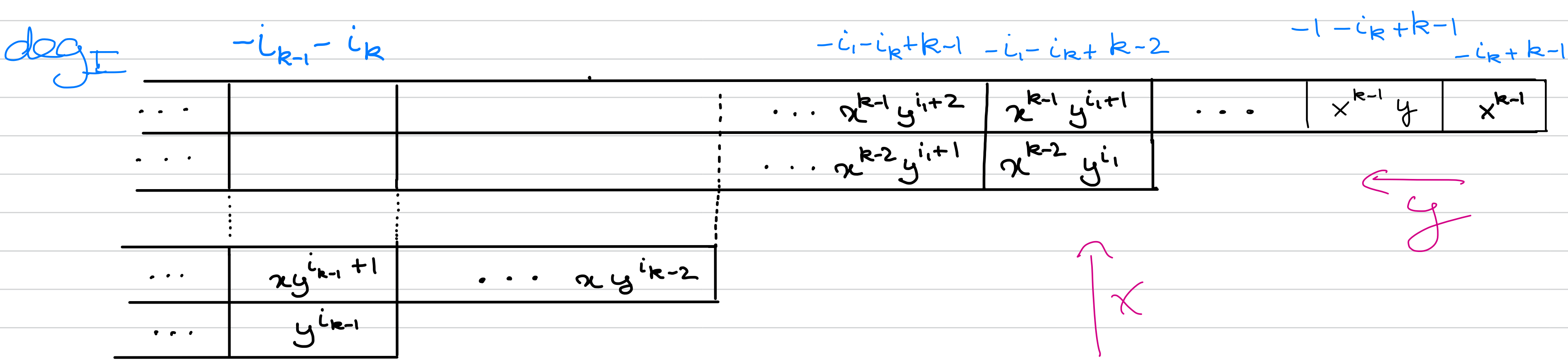
(3) Every homogeneous ideal I of R can be generated by monomials.

Thm [ACFGS 20]: $Gr(k,\infty)$

$I \in MCM_{\mathbb{Z}} R$ is generically free of rank 1

$$\iff \quad I = (x^{k-1}, x^{k-2}y^{i_1}, x^{k-3}y^{i_2}, \dots, xy^{i_{k-2}}, y^{i_{k-1}})(i_k)$$

with $0 \le i_1 \le i_2 \le \dots \le i_{k-1}$ and $i_k \in \mathbb{Z}$



Def: $\ell_I := (-i_{k-1} - i_k, -i_{k-2} - i_k + 1, \dots, -i_k + k-1)$

strictly non-decreasing

Corollary:

$$\left\{ \begin{array}{c} \text{generically free Rank 1} \\ \text{modules in } MCM_{\mathbb{Z}} R \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Plücker cordinates} \\ \text{in } \mathcal{A}_k \end{array} \right\}$$

$$\mathcal{A}_k = \mathbb{C}[p_I \mid I \subset \mathbb{Z}, |I| = k] / \mathcal{I}_p$$

$$I \longmapsto p_{\underline{\ell}_I}$$

$$I(\underline{\ell}) \longleftarrow\!\shortmid \ \underline{\ell} = (\ell_1, \dots, \ell_k)$$

$$I(\underline{\ell}) = (x^{k-1}, x^{k-2}y^{i_1}, x^{k-3}y^{i_2}, \dots, xy^{i_{k-2}}, y^{i_{k-1}})(i_k) \quad \text{with } i_k = k-1-\ell_k$$

Structure preserving

Rigidity in $MCM_{\mathbb{Z}} R \overset{1-1}{\longleftrightarrow}$ compatability of Plücker cordinates

<u>Thm [ACFGS 20]</u>: $I, J \in MCM_{\mathbb{Z}} R$, generically free of rank 1. Then

$$\mathrm{Ext}^1(I, J) = 0 \iff \rho_{\ell_I} \text{ and } \rho_{\ell_J} \text{ are compatible}$$

$$\iff \mathrm{Ext}^1(J, I) = 0$$

<u>Corollary</u>: $I \in MCM_{\mathbb{Z}} R$ generically free of rank 1 $\Rightarrow \mathrm{Ext}^1(I, I) = 0$

<u>Idea of Proof:</u> Use matrix factorisation

$$R^k \xrightarrow{M} R^k \xrightarrow{N} R^k \longrightarrow I \longrightarrow 0$$

$\rightsquigarrow$ graded projective presentation of I

apply graded $\mathrm{Hom}(-, J)$ $\quad$ ($\mathrm{Hom}(R(m), J) = J(-m)$)

$$\rightsquigarrow \quad J \xrightarrow{N^T} J(1) \xrightarrow{M^T} J(k)$$

$$\rightsquigarrow \mathrm{Ext}^1 = \left(\ker M^T \big/ \mathrm{im}\, N^T \right)_0 = \ker(M^T)_0 \big/ \mathrm{im}(N^T)_0$$

$$\Rightarrow \quad \dim \mathrm{Ext}^1(I, J) = \dim \ker(M^T)_0 - \dim \mathrm{im}(N^T)_0$$

$$\dim(\ker(M^T)_0) = \dim J(1)_0 - \dim \mathrm{im}(M^T)_0$$

$$\dim(\mathrm{im}(N^T)_0) = \dim J_0 - \dim \ker(N^T)_0$$

$$\rightsquigarrow \quad \dim J_0 - \dim J(1)_0 = |\underline{\ell}_I \cap \underline{\ell}_J|$$

$$\dim \operatorname{im}(M^T)_0 = k - \beta(\underline{\ell}_I, \underline{\ell}_J)$$

$$\dim \ker(N^T)_0 = \alpha(\underline{\ell}_I, \underline{\ell}_J)$$

new combinatorial tool

⇝ staircase paths

<u>Thm [ACFGS 20]:</u>

$$\rightsquigarrow \quad \dim \operatorname{Ext}^1(I, J) = \alpha(\underline{\ell}_I, \underline{\ell}_J) + \beta(\underline{\ell}_I, \underline{\ell}_J) - k - |\underline{\ell}_I \cap \underline{\ell}_J|$$
$$= \dim \operatorname{Ext}^1(J, I)$$

Example of calculation of dim $\mathrm{Ext}^1(I, J)$:

$k = 3$ $\qquad l_I = (-2, 0, 2) = (l_1, l_2, l_3) = l$

$\qquad\qquad l_J = (-1, 2, 3) = (m_1, m_2, m_3) = m$

$$\begin{array}{lll} l_1 < m_1 & l_1 < m_2 & l_1 < m_3 \\ l_2 > m_1 & l_2 < m_2 & l_2 < m_3 \\ l_3 > m_1 & l_3 = m_2 & l_3 < m_3 \end{array}$$

$B(l, m)$

$A(l, m)$

staircase paths.

$\alpha(l, m) = \#$ diagonals strictly above $A(l, m) \quad = 3$

$\beta(l, m) = \#$ diagonals strictly below $B(l, m) \quad = 2$

$|l \cap m| = 1$, $k = 3$ $\Rightarrow \dim \mathrm{Ext}^1(I, J) = 3 + 2 - 3 - 1 = 1$

$\alpha(l, m) = \beta(m, l)$ & $\beta(l, m) = \alpha(m, l)$

$\Rightarrow \dim \mathrm{Ext}^1(J, I) = 1$

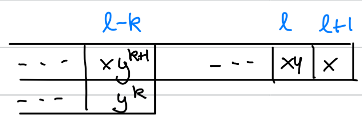
$$\begin{array}{lll} m_1 > l_1 & m_1 < l_2 \, , & m_1 < l_3 \\ m_2 > l_1 & m_2 > l_2 \, , & m_2 \leq l_3 \\ m_3 > l_1 & m_3 > l_2 \, , & m_3 > l_3 \end{array}$$

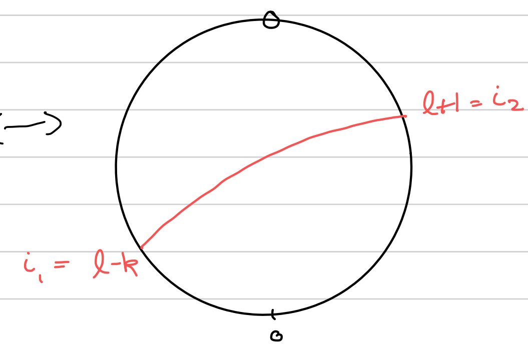
$B(m, l)$

$A(m, l)$

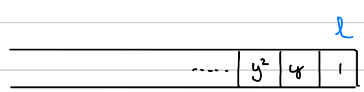
$\boxed{k=2}$ $\qquad R = \mathbb{C}[x,y]/(x^2)$

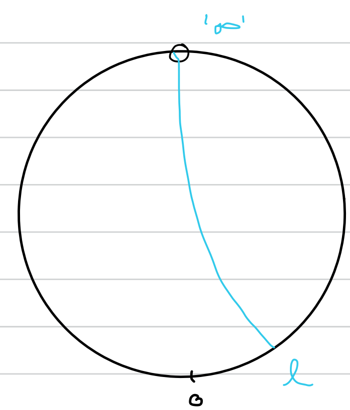
<u>Proposition:</u> The indecomposable $MCM_{\mathbb{Z}}$ R modules correspond to

• $(x,y^k)(-\ell)$ 

$\longleftrightarrow$ 

generically free rank 1 modules.

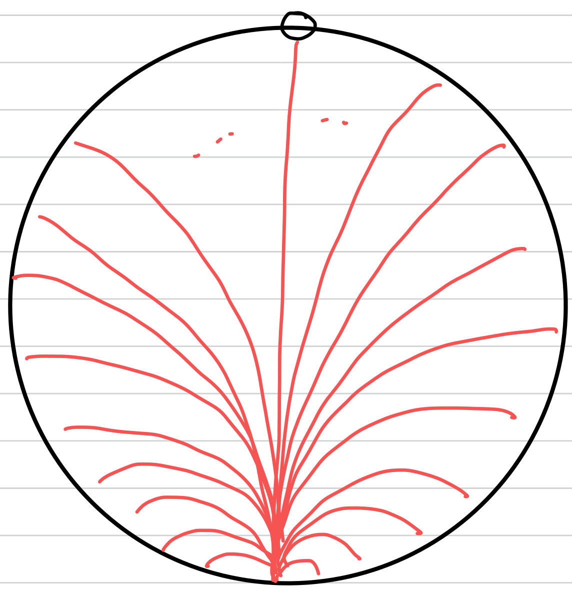
• $\mathbb{C}[y](-\ell)$ 

$\longleftrightarrow$ 

"(ℓ,∞)"

We can describe Hom-spaces $\rightsquigarrow$

Thm [ACFGS]: $\mathcal{MCM}_{\mathbb{Z}} R$ has cluster-tilting subcategories and they are of the form



Thank you!