Nil Hecke bimodule categories
(joint work with Timothy Logvinenko)

Rina Anno

Kansas State University

Nil Hecke algebras

Let $W = S_n$ (full generality for this slide: W is a Coxeter group).
The nil Hecke algebra $\mathcal{H}(W)$ is the complex algebra with generators $h_1, \ldots, h_{n-1}$ and relations

$$\begin{aligned} h_i h_j &= h_j h_i \qquad \text{for } |i-j| > 1; \\ h_i h_j h_i &= h_j h_i h_j \qquad \text{for } |i-j| = 1; \\ h_i^2 &= 0. \end{aligned}$$

Alternative description: let $s_i \in W$ permute i and $i+1$. Denote by $l(w)$ the length of each $w \in W$: the length of the minimal presentation $w = s_{i_1} \ldots s_{i_N}$. Then $l(w_1 w_2) \leq l(w_1) + l(w_2)$.

The monomial basis

$\mathcal{H}(W)$ has a basis h_w, $w \in W$, with $h_{w_1} h_{w_2} = h_{w_1 w_2}$ if $l(w_1 w_2) = l(w_1) + l(w_2)$ and $h_{w_1} h_{w_2} = 0$ otherwise. For $w = s_{i_1} \ldots s_{i_N}$,

$$h_w = h_{i_1} \ldots h_{i_N}.$$

Nil Hecke algebra objects: the naive case

Now let $\mathcal{A}$ be a small DG category. Suppose there are DG $\mathcal{A}$-$\mathcal{A}$-bimodules $H_1, \ldots, H_{n-1}$ such that

$$\begin{aligned} H_i \otimes_{\mathcal{A}} H_j &\simeq H_j \otimes_{\mathcal{A}} H_i && \text{for } |i-j| > 1; \\ H_i \otimes_{\mathcal{A}} H_j \otimes_{\mathcal{A}} H_i &\simeq H_j \otimes_{\mathcal{A}} H_i \otimes_{\mathcal{A}} H_j && \text{for } |i-j| = 1. \end{aligned}$$

Definition

Define the nil Hecke $\mathcal{A}$-$\mathcal{A}$-bimodule $\mathcal{H}$ as the direct sum

$$\bigoplus_{w \in W} H_w, \qquad \text{where } H_{s_{i_1} \ldots s_{i_N}} = H_{i_1} \otimes \ldots \otimes H_{i_N}.$$

Define the multiplication map $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{H}$ by concatenation+isomorphism $H_{w_1} \otimes H_{w_2} \to H_{w_1 w_2}$ when $l(w_1 w_2) = l(w_1) + l(w_2)$ and zero otherwise. We obtain an algebra object in the category of $\mathcal{A}$-$\mathcal{A}$-bimodules.

Nil Hecke algebra object: the homotopy case

In real life, we can't expect isomorphisms of DG bimodules. Suppose that we have homotopy equivalences instead:

$$H_i \otimes_{\mathcal{A}} H_j \sim H_j \otimes_{\mathcal{A}} H_i \qquad \text{for } |i-j| > 1;$$
$$H_i \otimes_{\mathcal{A}} H_j \otimes_{\mathcal{A}} H_i \sim H_j \otimes_{\mathcal{A}} H_i \otimes_{\mathcal{A}} H_j \qquad \text{for } |i-j| = 1.$$

Convention

From now on, let us drop tensor signs, i.e. let $H_i H_j$ denote $H_i \otimes_{\mathcal{A}} H_j$.

Let $n = 3$. Then there are 6 summands: $\mathcal{A}$, H_1, H_2, H_1H_2, H_2H_1, and one more. Let the last one be the cone of

$n=2$:

$\mathcal{A} \oplus H_1$

$\mathcal{A} \otimes \mathcal{A} \xrightarrow{\simeq} \mathcal{A}$

$\mathcal{A} \otimes H_1 \xrightarrow{\simeq} H_1$

$H_1 \otimes \mathcal{A} \xrightarrow{\simeq} H_1$

$H_1 \otimes H_1 \xrightarrow{0}$

$H_1H_2 \otimes H_1$

$H_2 \otimes H_1 H_2$

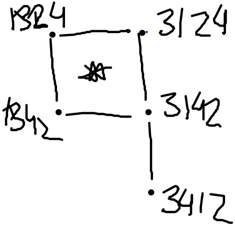
$$\left\{ H_2H_1H_2 \begin{array}{l} \xrightarrow{\sim} \\ \xrightarrow{\mathrm{Id}} \end{array} \begin{array}{c} H_1H_2H_1 \\ \oplus \\ H_2H_1H_2. \end{array} \right\}$$

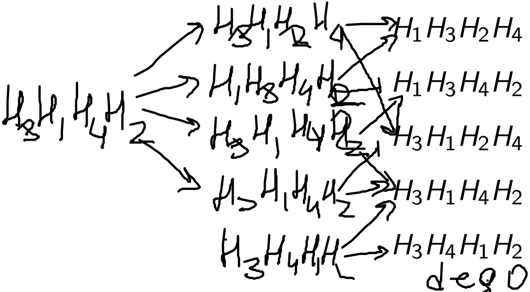
deg.0

The homotopy case (cont.)

Tweaking the bimodule

We define H_w as the convolution of a certain twisted complex in non-positive degrees. Its degree 0 term is the direct sum of $H_{i_1} \otimes \ldots \otimes H_{i_N}$ over all presentations $w = s_{i_1} \ldots s_{i_N}$ of length $l(w)$. Its convolution is homotopy equivalent to each of those monomials. For each $w_1 w_2 = w$, $l(w_1) + l(w_2) = l(w)$ there is a subcomplex isomorphic to $H_{w_1} \otimes_{\mathcal{A}} H_{w_2}$ which allows us to define strictly associative multiplication.





The block subalgebras

For each subset $I \subset \{1, \ldots, n-1\}$ of generators (or even for any subgroup of W) let $\mathcal{H}_I$ be the ~~direct sum~~ of the "monomials" in $\mathcal{H}$ generated by H_i, $i \in I$. This is an algebra object in $\mathcal{A}$-Mod-$\mathcal{A}$ as well.

$\mathcal{H}$, or each of its "block subalgebras" $\mathcal{H}_I$ can be viewed as a DG category with $\mathrm{Ob}\, \mathcal{H} = \mathrm{Ob}\, \mathcal{A}$. There are natural functors $\mathcal{H}_I \to \mathcal{H}_J$ when $I \subset J$. Each $\mathcal{H}_I$ can be viewed as an $\mathcal{H}_J$-$\mathcal{H}_K$-bimodule if $J, K \subseteq I$.

Restriction and induction

The functors $\mathcal{H}_J \to \mathcal{H}_I$ for $J \subset I$ induce adjoint pairs of functors $((-) \otimes_{\mathcal{H}_J} \mathcal{H}_I,\ (-) \otimes_{\mathcal{H}_I} \mathcal{H}_I)$ between the categories Mod-$\mathcal{H}_J$ and Mod-$\mathcal{H}_I$, where $\mathcal{H}_I$ is viewed as a $\mathcal{H}_J$-$\mathcal{H}_I$ and $\mathcal{H}_I$-$\mathcal{H}_J$-bimodule respectively. In particular, the monad algebra of this adjunction is $\mathcal{H}_I$ viewed as an algebra over $\mathcal{H}_J$.

Braid group action

Suppose that $(-)\overset{\mathbf{L}}{\otimes} H_i$ is an autoequivalence of $D(\mathcal{A})$. Then the restriction functor $\boxed{D(\mathcal{H}_{\{i\}}) \to D(\mathcal{A})}$ is spherical and $(-)\overset{\mathbf{L}}{\otimes} H_i[-1]$ is its inverse twist (cf. Segal'16).

$\mathcal{H}_{\{i\}} = \mathcal{A} \oplus H_i$ monad

$D(\mathcal{A}) \to D(\mathcal{H}_{\{i\}}) \to D(\mathcal{A})$

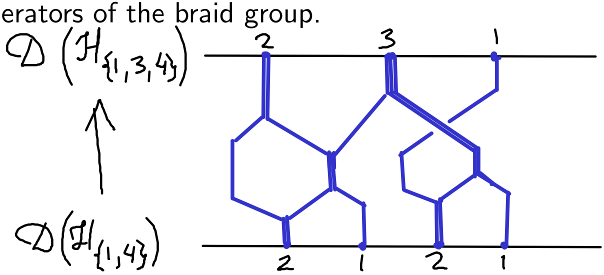
Suppose all $(-)\overset{\mathbf{L}}{\otimes} H_i$ are autoequivalences. Having a set of such H_i satisfying our previous conditions amounts to having a weak braid group action on $D(\mathcal{A})$.

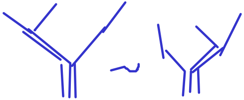
Generalized braids

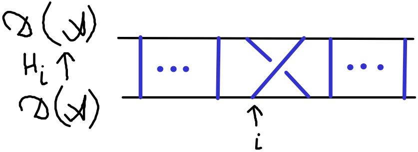
For any weak braid group Br_n action on a triangulated category (assuming DG enhancement) we can construct a network of other categories labeled by subsets of $\{1, \ldots, n-1\}$ and functors between them such that, in particular, the "top level" functors are spherical and their twists generate the braid group action.

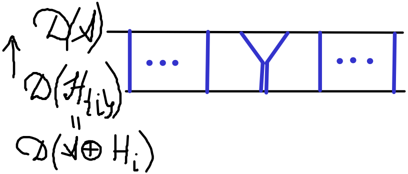
The diagram calculus

Our goal is to assign the triangulated categories $D(\mathcal{H}_I)$ to ordered tuples $i_1 + \ldots + i_k = n$ and represent certain functors between them as diagrams. In this representation, $\mathcal{A}$ corresponds to $(1, \ldots, 1)$ and H_i to the generators of the braid group.









Resolution of a Kleinian singularity

Consider the well-known braid group action on the resolution of a Kleinian singularity of type A_{n-1}, i.e. on the derived category of modules over $\mathbb{C}[x,y] \rtimes \mathbb{Z}_n$. Let $\rho_i \simeq \mathbb{C}$ be the ith irrep of $\mathbb{Z}_n$, and let $\mathbb{C}_0 \simeq \mathbb{C}$ be the representation of $\mathbb{C}[x,y]$ where x, y act by zero. Then the spherical objects that generate the braid group action are $\mathbb{C}_0 \otimes \rho_i$, and

$$H_i = \{\left(\mathbb{C}_0 \otimes \rho_i\right) \otimes_{\mathbb{C}} \left(\mathbb{C}_0 \otimes \rho_i\right)[-2] \to \mathbb{C}[x,y] \rtimes \mathbb{Z}_n\}.$$

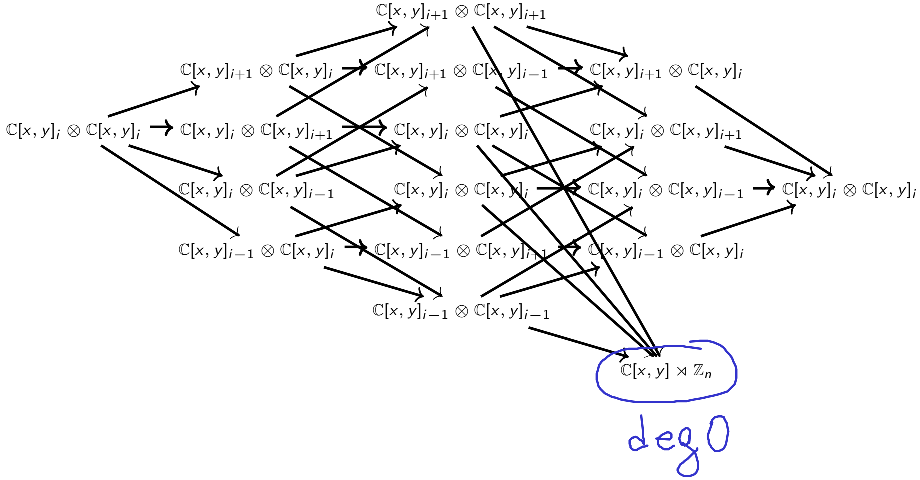
Here, the map is derived; this cone is represented by a complex of bimodules (see next slide).

Projective resolution for $\mathbb{C}_0 \otimes \rho_i$ (denote $\mathbb{C}[x,y]_i := \mathbb{C}[x,y] \otimes \rho_i$):

$$\begin{tikzcd}
 & \mathbb{C}[x,y]_{i+1} \arrow[dr, "\cdot y"] & \\
\mathbb{C}[x,y]_i \arrow[ur, "\cdot x"] \arrow[dr, "\cdot(-y)"'] & & \mathbb{C}[x,y]_i \\
 & \mathbb{C}[x,y]_{i-1} \arrow[ur, "\cdot x"'] &
\end{tikzcd}$$

deg 0

The bimodule H_i



Thank you!