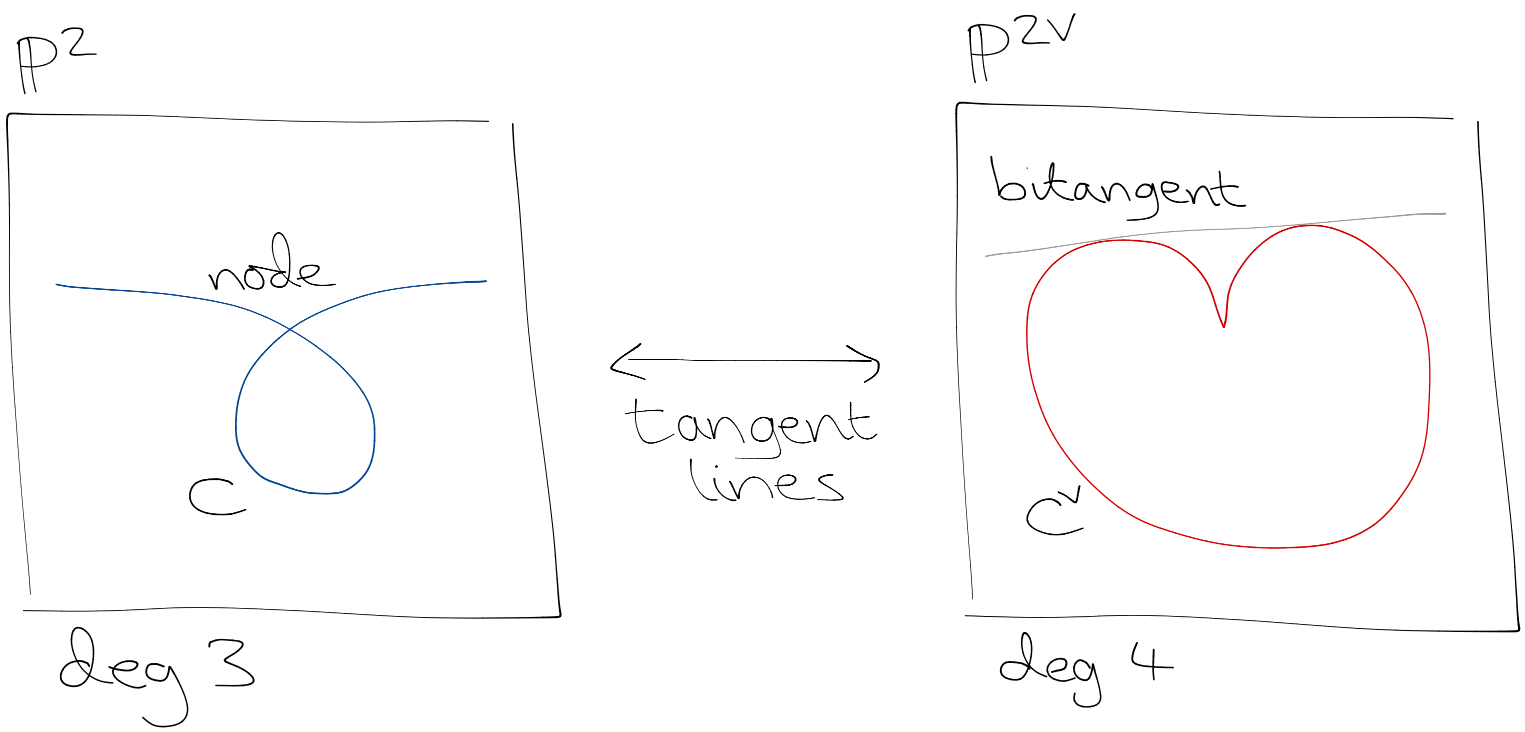
Mutation of

Windows on the Pfaffian-Grassmannian correspondence

Projective duality

Ex

[M. Monks]



Def $\quad C^\vee = \{ L \in \mathbb{P}^{2\vee} \mid L \cap C \text{ singular} \}$

Pfaffian $\mathrm{Pf}_r = \{\mathrm{rk}\, M \le \underset{\text{even}}{r}\} \subset \mathrm{Mat}^{\text{skew}}_{n\times n}$

$\cong \Lambda^2 V^\vee =: W$

2-forms

Write: $\mathbb{P}\mathrm{f}_r = \mathbb{P}(\mathrm{Pf}_r) \subset \mathbb{P}W$

Ex $\mathbb{P}\mathrm{f}_2 = \mathbb{P}\{f \wedge g\} = \mathrm{Gr}_2(V^\vee)$

$\boxed{\mathbb{P}\mathrm{f}_r \subset \mathbb{P}W} \xleftrightarrow{\text{duality}} \boxed{\mathbb{P}W^\vee \supset \mathbb{P}\mathrm{f}_s}$ $\quad r+s(+1)=n$

Ex $(r,n)=(2,7)$

$\boxed{\mathrm{Gr}_2(7) \subset \mathbb{P}W} \longleftrightarrow \boxed{\mathbb{P}W^\vee \supset \mathbb{P}\mathrm{f}_4}$

Cut by linear $\Pi \subset W$, codim 7, generic:

$$\underbrace{\boxed{\mathrm{Gr}_2(7) \cap \mathbb{P}\Pi}}_{X_1} \qquad \underbrace{\boxed{\mathbb{P}\Pi^\perp \cap \mathbb{P}\mathrm{f}_4}}_{X_2}$$

3-folds

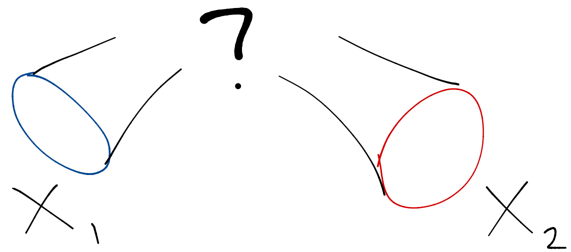
smooth

Calabi-Yau

$X_1 = \boxed{Gr_2(7) \cap \mathbb{P}\Pi}$ $\quad X_2 = \boxed{\mathbb{P}\Pi^{\perp} \cap \mathbb{P}f_4}$

Conj (Rødland ~00) X_i have same mirror,

correspond to LRLs, same moduli:



Rem X_i have $h^{1,1} = 1 \Rightarrow X_i$ not birational

Thm (Bor-Cal 06) $\Phi : \boxed{D(X_1)} \xrightarrow{\sim} \boxed{D(X_2)}$

Pf Fourier-Mukai techniques

Hori-Tong 06 X_i from gauged linear σ-model

Add-Don-Seg 14 Reprove equiv (different Φ)

Hosono-Takagi, Segal-Thomas, Rennemo-Segal...

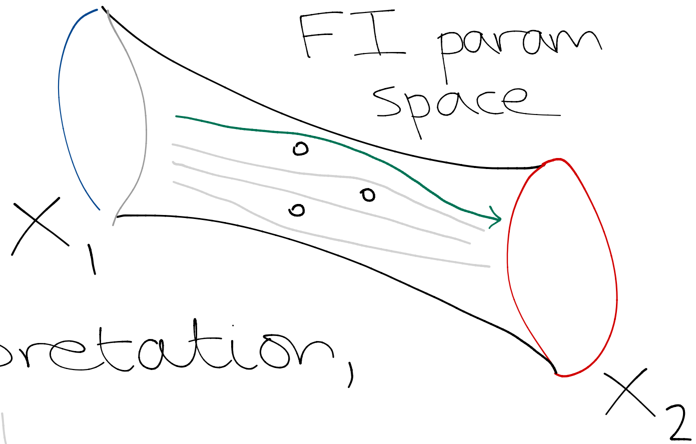
$X_1 = \boxed{\mathbb{G}r_2(7) \cap \mathbb{P}\Pi}$ $X_2 = \boxed{\mathbb{P}\Pi^\perp \cap \mathbb{P}f_4}$

Thm (Bor-Cal 06) $\Phi : \boxed{D(X_1)} \xrightarrow{\sim} \boxed{D(X_2)}$

Pf Fourier-Mukai techniques

Hori-Tong 06 X_i from gauged linear σ-model

Hori, Eager-H-Knapp-Romo 16



Physics: Φ associated to $\boxed{\text{path}}$

This talk: explain math interpretation, and $\boxed{\text{other paths}}$

Matrix factorizations

variety M, $f : M \to \mathbb{C}$

$\mathbb{C}^* \curvearrowright M$, where -1 acts trivially, $\mathrm{wt}(f) = 2$

<u>Def</u> <u>bundle $\mathcal{E}$ on (M,f)</u> is $\mathbb{C}^*$-eqvt

bundle $\mathcal{E}$ on M with $d : \mathcal{E} \to \mathcal{E}$

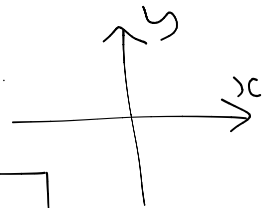
$\mathrm{wt}(d) = 1$, $d^2 = f$

<u>Note</u> $\mathcal{E} = \underset{\text{even/ odd wt}}{\mathcal{E}_0 \oplus \mathcal{E}_1}$ so write $\mathcal{E} = \left(\mathcal{E}_0 \underset{d}{\overset{d}{\rightleftarrows}} \mathcal{E}_1 \right)$

$D(M,f)$: derived category

<u>Ex</u> $M = \mathbb{C}^2$ coords x y $\quad f = xy$

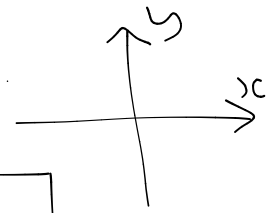
$\mathrm{wt}_{\mathbb{C}^*}$ $\;0$ $\;2$



$$\mathcal{E} = \left(\mathcal{O}_M \underset{y}{\overset{x}{\rightleftarrows}} \mathcal{O}_M[1] \right)$$
$$\cong \mathcal{O}_{\{y=0\}}, \mathcal{O}_{\{x=0\}}[1]$$

$$\boxed{\begin{array}{c}\text{compare} \\ \begin{pmatrix} & x \\ y & \end{pmatrix}\begin{pmatrix} & y \\ x & \end{pmatrix} = f\,\mathrm{id}\end{array}}$$

$\underline{\text{Ex}}$ $M = \mathbb{C}^2$ coords $x \quad y$ $\quad f = xy$
$\text{wt}_{\mathbb{C}^*}$ $\quad 0 \quad 2$



$$\mathcal{E} = \left(\mathcal{O}_M \underset{y}{\overset{x}{\rightleftarrows}} \mathcal{O}_M[1] \right)$$
$$\cong \mathcal{O}_{\{y=0\}}, \mathcal{O}_{\{x=0\}}[1]$$

$\boxed{\text{compare } \begin{pmatrix} & x \\ y & \end{pmatrix}\begin{pmatrix} & y \\ x & \end{pmatrix} = f \cdot \text{id}}$

$\underline{\text{Knorrer periodicity}}$ "$D(M,f)$ only sees Crit_f"

$M = \mathbb{C}^{2n}, f = \sum_1^n x_i y_i$ $\quad \boxed{D(M,f) \cong D(pt)}$

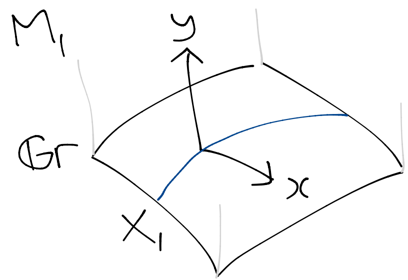
$\underline{\text{Ex}}$ $f = xy$ $\quad \mathcal{E}$ generates, $\text{Hom}(\mathcal{E},\mathcal{E}) = \mathbb{C}$

$\underline{\text{3-fold}}$ $X_1 = \boxed{Gr_2(7) \cap \mathbb{P}\Pi} = Z(x_1, \ldots, x_7)$

$\uparrow \in \Gamma_{Gr} \mathcal{O}(1)$

$$M_1 = \text{Tot}\begin{pmatrix} \mathcal{O}(-1)^{\oplus 7} \\ \downarrow \\ Gr \end{pmatrix}$$

fibre coords y_i
$f = \sum x_i y_i$

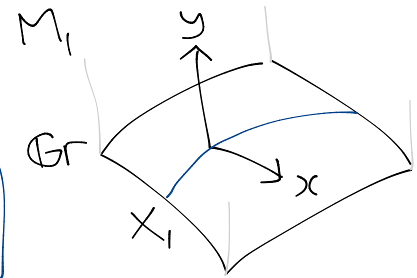


Knorrer (Shipman) $\Rightarrow$ $\boxed{D(M_1, f) \cong D(X_1)}$

3-fold $X_1 = \boxed{Gr_2(7) \cap \mathbb{P}\Pi}$

$M_1 = \mathrm{Tot}\begin{pmatrix} \mathcal{O}(-1)^{\oplus 7} \\ \downarrow \\ Gr \end{pmatrix}$ fibre coords y_i

$f = \sum x_i y_i$



Knörrer (Shipman) $\Rightarrow$ $\boxed{D(M_1, f) \cong D(X_1)}$

3-fold $X_2 = \boxed{\mathbb{P}\Pi^{\perp} \cap \mathbb{P}f_4} \subset \mathbb{P}^6$

GIT prob $M = [S^{\vee \oplus 7} \oplus \Lambda^2 S^{\oplus 7} / GL_2]$,

where S = std GL_2-rep

Quotients: M_1 and M_2

Add-D-Seg $\boxed{D(M_2, f) \cong D(X_2)}$

Hori-Tong GL_2-gauged linear σ-model from $(M, f) \rightsquigarrow X_i$

GIT prob $\mathcal{M} = [S^{\vee \oplus 7} \oplus \Lambda^2 S^{\oplus 7} / GL_2]$

GIT quotients $M_1 \hookrightarrow \mathcal{M} \hookleftarrow M_2$

Omitting f, have $\underline{D(X_1) \cong D(M_1)} \; \overset{?}{\cdots} \; \underline{D(M_2) \cong D(X_2)}$

Std technique: $$D(M_1) \xleftarrow{res_1} D(\mathcal{M}) \xrightarrow{res_2} D(M_2)$$
$$\cup$$

seek <u>window</u> $\mathcal{W}$ so $res_i|_{\mathcal{W}}$ equivs

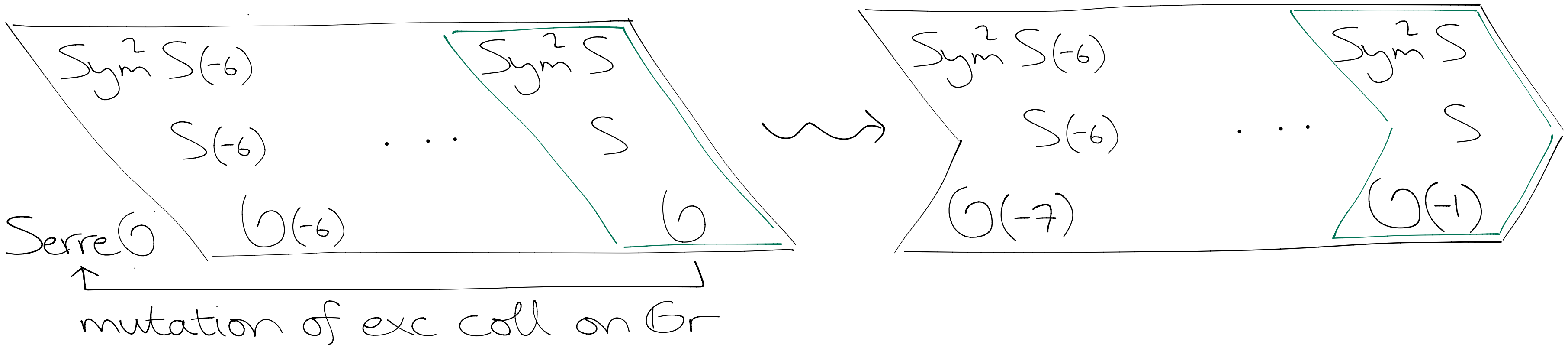
$$\boxed{\text{Notation } F_{\mathcal{W}} : D(M_1) \xrightarrow{\sim} D(M_2)}$$

$\mathcal{W}$ on $\mathcal{M}$ generated by sheaves $\mathrm{Sym}^k S(l) \in$

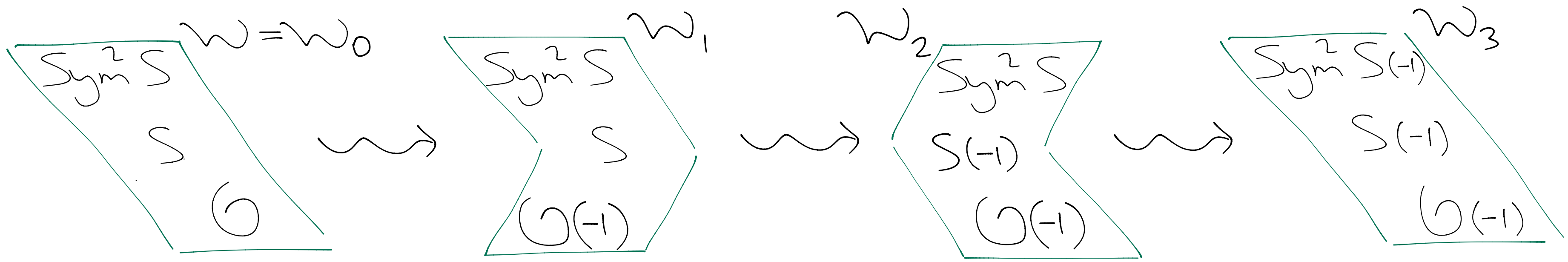
$$\begin{array}{ccc} \mathrm{Sym}^2 S(-6) & & \mathrm{Sym}^2 S \\ S(-6) & \cdots & S \\ \mathcal{O}(-6) & & \mathcal{O} \end{array}$$

<u>Note</u> Same GL_2-reps give exc coll on $Gr_2(7)$.

Get further new $\mathcal{W}$ by mutation:



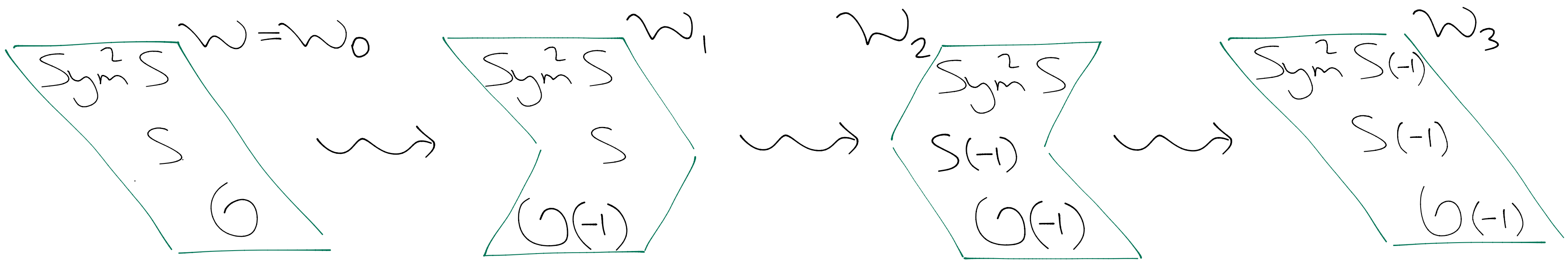
... determined by "blocks" as follows [thanks to Hori]



Check Each gives tilting bundle on M_1 and M_2

Thm Get $F_i : D(M_1) \xrightarrow{\sim} D(M_2)$ from each $\mathcal{W}_i$

$$\begin{array}{ccc} & D(M_1) & & D(M_2) \\ & \cong & & \cong \\ \text{also } \Phi_i : & D(X_1) & & D(X_2) \end{array}$$



<u>Check</u> Each gives tilting bundle on M_1 and M_2

$$\boxed{\begin{array}{ll} \underline{\text{Thm}} & \text{Get } F_i : D(M_1) \xrightarrow{\sim} D(M_2) \text{ from each } \mathcal{W}_i \\ & \qquad\qquad\quad\ \ \, \wr\wr \qquad\qquad\quad\ \wr\wr \\ & \text{also } \Phi_i : D(X_1) \qquad\quad\ \ D(X_2) \end{array}}$$

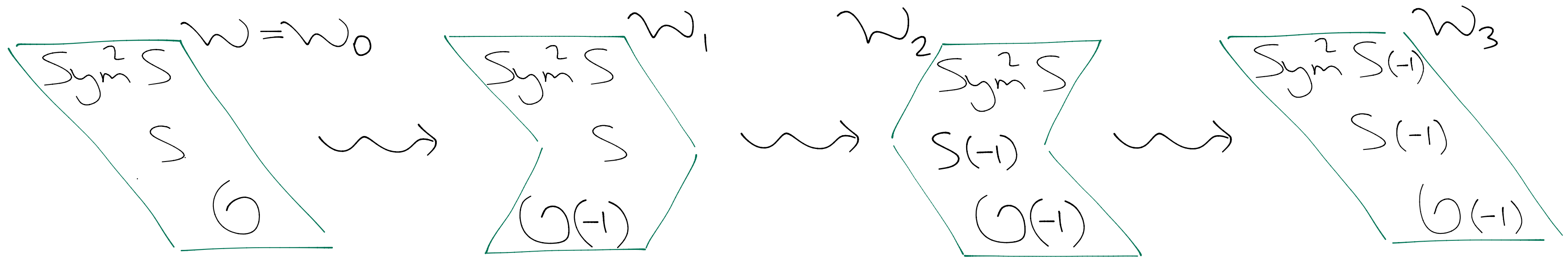
<u>Prop</u> $F_1^{-1} F_0 = \mathrm{Tw}_{\omega_{\mathrm{Gr}}}$ on $D(M_1)$

<u>Prop</u> $\Phi_1^{-1}\Phi_0 = \mathrm{Tw}_{\mathcal{O}}$ on $D(X_1)$ Calabi-Yau

sim. $\Phi_2^{-1}\Phi_1 = \mathrm{Tw}_S$

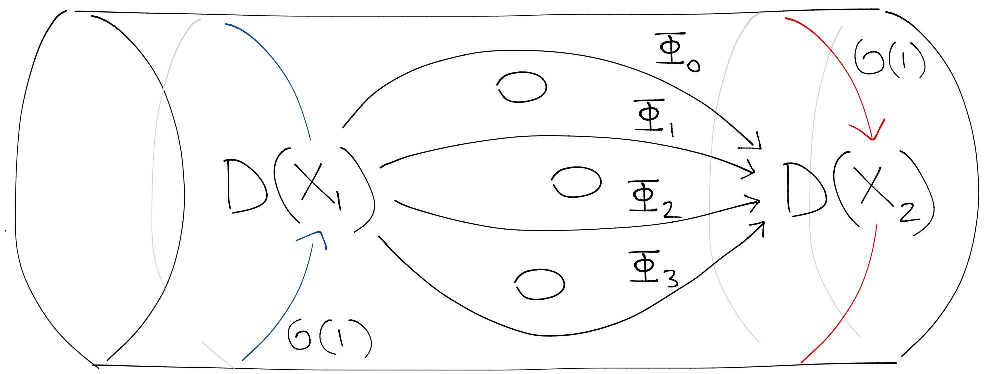
$\Phi_3^{-1}\Phi_2 = \mathrm{Tw}_{\mathrm{Sym}^2 S}$

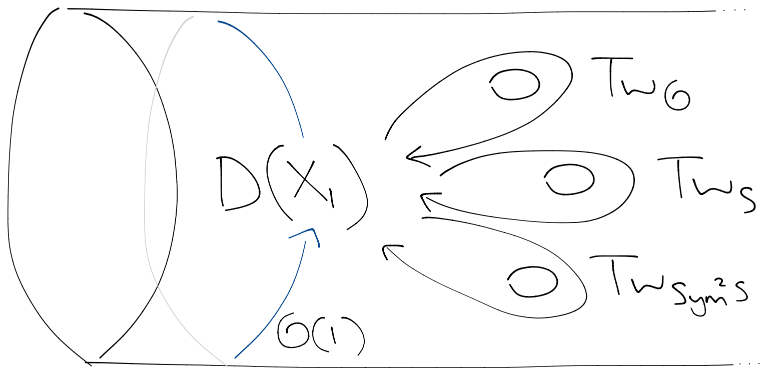
sheaves on $X_1 \subset \mathrm{Gr}_2(7)$



<u>3-folds</u> $X_1 = \boxed{\mathrm{Gr}_2(7) \cap \mathbb{P}\Pi}$ $X_2 = \boxed{\mathbb{P}\Pi^\perp \cap \mathbb{P}f_4}$

<u>Thm</u> $\pi_1(\text{cylinder} - 3\text{pt})$ acts on $D(X_1)$ and $D(X_2)$:





<u>Rem</u> agrees with FI param space picture, <u>Hori et al</u>.

THANKS!