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Crepant resolutions and moduli of G-constellations for abelian groups

July, 2020

§ Introduction

$G \subset SL(n, \mathbb{C})$: finite subgroup
$\pi : X \to \mathbb{C}^n/G$: crepant resolution (i.e. $K_X = \pi^* K_{\mathbb{C}^n/G}$)

Q Is X obtained as a moduli space of G-constellations on $\mathbb{C}^n$?

The G-Hilbert scheme gives a crepant resol.
when $n=2$ (Ito-Nakamura '96)
or $n=3$ (Bridgeland-King-Reid '02)

$\boxed{G\text{-Hilb} = \{ Z \subset \mathbb{C}^n : G\text{-invariant s.t. } H^0(\mathcal{O}_Z) \cong \mathbb{C}G \}}$

Conj. (Derived McKay Correspondence)

$$\boxed{\exists\, \Phi : D^b(\mathrm{coh}\, X) \xrightarrow{\sim} D^b(\mathrm{coh}^G \mathbb{C}^n)}$$

This conjecture is true when
- $X = G\text{-Hilb}$ and $n=2$ (Kapranov-Vasserot '98) or $n=3$ (BKR)
- $G \subset Sp(2n, \mathbb{C})$ and X is any crepant resolution (Bezrukavnikov-Kaledin '04)
- G : abelian and X is any projective crepant resol. (Kawamata '05)

Thm. (Craw-Ishii '04)

If $G \subset SL(3, \mathbb{C})$ is abelian and $X \to \mathbb{C}^3/G$ is any projective crepant resol, then $\exists\, \Phi$ s.t. $\Phi(\mathcal{O}_x)$ is a G-constellation for any $x \in X$, and X is isomorphic to the moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations for some stability condition θ.

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Today For any given $X \to \mathbb{C}^n/G$ with G: abelian, we compare X with a moduli space $\mathcal{M}$ of certain G-constellations using quotient constructions of X and $\mathcal{M}$.

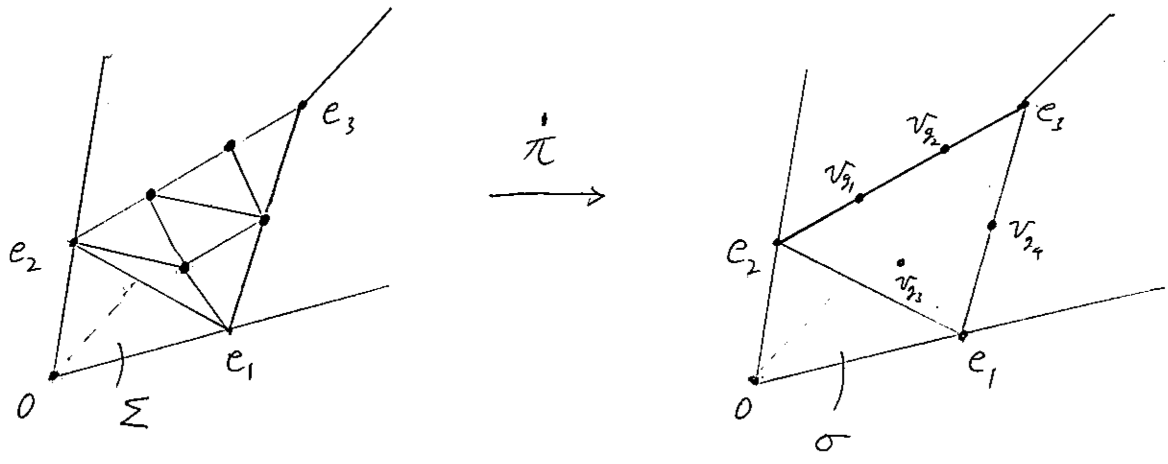
(Work in progress)

§ Crepant resolution of $\mathbb{C}^n/G$

$G \subset SL(n, \mathbb{C})$: finite abelian subgp (We may assume G is diagonal.)
$\pi : X \to \mathbb{C}^n/G$: not-necessarily-projective crepant resol.

Then π is obtained as a toric morphism $X(\Sigma) \longrightarrow X(\sigma)$

$$\left(\begin{array}{l} N := \mathbb{Z}^n \text{ with basis } e_1, \cdots, e_n \\ N' := N + \sum_{g \in G} \mathbb{Z} \cdot v_g \subset \mathbb{Q}^n \text{ where } v_g = \frac{1}{r}(a_1, \cdots, a_n) \text{ for } g = \begin{pmatrix} \zeta^{a_1} & & \\ & \ddots & \\ & & \zeta^{a_n} \end{pmatrix}, \quad \zeta = e^{\frac{2\pi i}{r}} \\ \sigma := \mathrm{cone}(e_1, \cdots, e_n) \subset N' \otimes_{\mathbb{Z}} \mathbb{R} . \end{array}\right.$$



$v_{g_1}, \cdots, v_{g_m}$: the lattice points on $\sigma \cap \{(a_1, \cdots, a_n) \mid \sum a_i = 1\}$ except $e_1, \cdots, e_n$

$\updownarrow$ 1:1

$E_1, \cdots, E_m$: the irreducible exceptional divisors of π

Quotient construction of X (due to Cox)

$T_m := (\mathbb{C}^*)^{\times m}$, $S = \mathbb{C}[\underbrace{x_1, \dots, x_n}_{\text{the coordinates of } \mathbb{C}^n}, y_1, \dots, y_m]$

$G \curvearrowright \mathbb{C}[x_1, \dots, x_n]$
$\chi_i \in G^\vee = \mathrm{Hom}(G, \mathbb{C}^*)$: the character defined by x_i (i.e. $g \cdot x_i = \chi_i(g) x_i$)

Define $a_{i,k} \in \mathbb{Z}_{\geq 0}$ $(1 \leq i \leq n, 1 \leq k \leq m)$ so that $\chi_i(g_k) = e^{\frac{2\pi i}{r} \cdot a_{i,k}}$ and $0 \leq a_{i,k} < r$.

$$T_m \curvearrowright S : \quad x_i \longmapsto t_1^{a_{i,1}} \cdots t_m^{a_{i,m}} x_i$$
$$(t_1, \dots, t_m) \qquad y_k \longmapsto t_k^{-r} y_k$$

Then: $\exists$ T_m-invariant open subset $U_\Sigma \subset \operatorname{Spec} S = \mathbb{C}^{n+m}$
s.t. $X(\Sigma) = U_\Sigma / T_m$.

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§ Construction of moduli of G-constellations (based on Craw-Mclagan-Thomas '07)

Def. A G-constellation (on $\mathbb{C}^n$) is a $\mathbb{C}[x_1, \dots, x_n]$-module M with a G-action s.t.
(1) $g \cdot (x \cdot m) = (g \cdot x_i)(g \cdot m)$ for $\forall g \in G$, $\forall i$, $\forall m \in M$
(2) $M \cong \mathbb{C}G := \bigoplus_{g \in G} \mathbb{C} \cdot g$ as a G-module

We identify a G-constellation with a representation of the McKay quiver Q_G with relations.

The quiver Q_G consists of:

$\begin{cases} \cdot \text{ vertices labelled by the irr. reps } \{\rho_0, \cdots, \rho_{r-1}\} \text{ of } G \\ \cdot \text{ arrows : } \{\alpha_{i,j}\}_{\substack{1\le i\le n \\ 0\le j\le r-1}} : \alpha_{i,j} : \rho_j \longrightarrow \rho_j \otimes \chi_i \end{cases}$

$R := \mathrm{Rep}(Q_G, \underbrace{(1, \cdots, 1)}_{\text{dimension vector}}, \underbrace{I}_{\text{relation}})$, $\quad I = \langle \alpha_1\alpha_2 - \beta_1\beta_2 \rangle$

$$\begin{tikzcd} & \circ \arrow[dr, "\alpha_2"] & \\ \circ \arrow[ur, "\alpha_1"] \arrow[dr, "\beta_1"'] & & \circ \\ & \circ \arrow[ur, "\beta_2"'] & \end{tikzcd}$$

R contains a distinguished irr. comp. $Z \subset R$ of dim $n+r-1$, and the moduli space $\mathcal{M}_\theta$ is defined as the GIT quotient $Z /\!\!/_{\chi_\theta} T_{r-1}$ where $T_{r-1} := \mathrm{Aut}_G(\mathbb{C}G) \cong (\mathbb{C}^*)^{r-1}$.

$$\begin{array}{ccl} \theta \in \Theta := \{ & \theta \in \mathrm{Hom}_{\mathbb{Z}}\left(\bigoplus_{\rho\in G^\vee} \mathbb{Z}\rho, \mathbb{Q}\right) & \mid \theta(\mathbb{C}G) = 0\} \\ \updownarrow & \cong & \\ \chi_\theta \in & \mathcal{X}(T_{r-1}) \otimes_{\mathbb{Z}} \mathbb{Q} & \end{array}$$

<u>Fact</u> (King '94)

$z \in Z$ is χ_θ-(semi)stable in the sense of GIT

$\Longleftrightarrow$ the corresponding rep. R_z is θ-(semi)stable

$\left(\overset{\text{def}}{\Longleftrightarrow} \text{ for any proper subrep. } R' \subset R_z, \text{ we have } \theta(R') \underset{(\geq)}{>} \theta(R_z) \right)$

<u>Note</u> $\mathcal{M}_\theta$ is always projective over $\mathbb{C}^n/G$.

Family of G-constellations on X

We use a "geometrically natural family" (gnat-family) introduced by Logvinenko. In our situation, such a family $\mathcal{F}$ is given as

$$\mathcal{F} = \bigoplus_{\rho \in G^\vee} L_\rho, \quad L_\rho = \mathcal{O}_X\left(-\sum_{k=1}^{m} b_{\rho,k} E_k\right) \quad (b_{\rho,k} \in \mathbb{Q})$$

satisfying

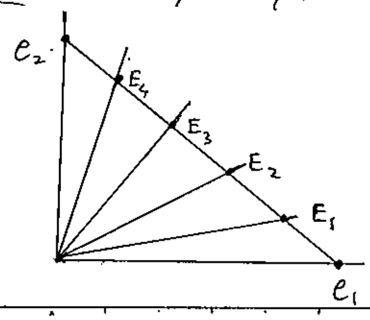
- $b_{\rho,k} + \underbrace{(v_{g_k})_i}_{\text{the } i\text{-th entry of } v_{g_k} \in \mathbb{Q}^n} - b_{\rho\otimes\chi_i,k} \in \mathbb{Z}_{\geq 0}$ for $\forall \rho, \forall k, \forall i$
- $L_{\rho_0} = \mathcal{O}_X$ (ρ_0: the trivial rep)

Note: $\{E_1, \dots, E_m\}$ is a $\mathbb{Q}$-basis of $\mathrm{Pic}(X)$

Facts (Logvinenko '08)

- There is a gnat-family $\mathcal{F}_{max}$, called the maximal shift family, whose coefficients $b_{\rho,k}$ achieve the maximal values among all the gnat-families for each (ρ, k).
- If $X = G\text{-Hilb}$, then the tautological bundle on X is isomorphic to $\mathcal{F}_{max}$.

Example (A_4 singularity)



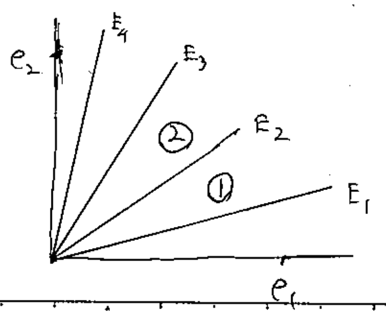
$|G| = 5$

	$\mathcal{F}_{max}$	$\mathcal{F}_1$	$\mathcal{F}_2$
ρ_0	0	0	0
ρ_1	$-\frac{1}{5}(4E_1+3E_2+2E_3+E_4)$	$-\frac{1}{5}(4E_1+3E_2+2E_3+E_4)$	$-\frac{1}{5}(4E_1+3E_2+2E_3+E_4)$
$\rho_1^{\otimes 2}$	$-\frac{1}{5}(3E_1+6E_2+4E_3+2E_4)$	$-\frac{1}{5}(3E_1+\underline{\underline{E_2}}+4E_3+2E_4)$	$-\frac{1}{5}(3E_1+E_2+4E_3+2E_4)$
$\rho_1^{\otimes 3}$	$-\frac{1}{5}(2E_1+4E_2+6E_3+3E_4)$	$-\frac{1}{5}(2E_1+4E_2+6E_3+3E_4)$	$-\frac{1}{5}(2E_1+4E_2+\underline{\underline{E_3}}+3E_4)$
$\rho_1^{\otimes 4}$	$-\frac{1}{5}(E_1+2E_2+3E_3+4E_4)$	$-\frac{1}{5}(E_1+2E_2+3E_3+4E_4)$	$-\frac{1}{5}(E_1+2E_2+3E_3+4E_4)$

$\rho_1 := \chi_1$

G-constellations for torus-fixed points are described by Laurent monomials!

	$\mathcal{F}_{max}$	$\mathcal{F}_1$	$\mathcal{F}_2$
①	$\begin{array}{l} y^3 \\ \mid \\ y^2 \\ \mid \\ y \\ \mid \\ 1 - x \end{array}$	$\begin{array}{ll} y^2 & \\ \mid & \\ y & \\ \mid & \\ 1 - & x \\ & \mid \\ & xy^{-1} \end{array}$	$\begin{array}{ll} y^2 & \\ \mid & \\ y & \\ \mid & \\ 1 - & x \\ & \mid \\ & xy^{-1} \end{array}$
②	$\begin{array}{l} y^2 \\ \mid \\ y \\ \mid \\ 1 - x - x^2 \end{array}$	$\begin{array}{l} x^{-1}y^2 - y^2 \\ \mid \\ y \\ \mid \\ 1 - x \end{array}$	$\begin{array}{l} x^{-1}y^2 \\ \\ y \\ \mid \\ 1 - x \end{array}$ x^2y^{-1}



§ Morphism from $\mathbb{C}^{n+m}$ to Z

Fix $X = X(\Sigma) \to \mathbb{C}^n/G$ and a family $\mathcal{F}$ on X.

Prop. There is a morphism $\varphi_{\mathcal{F}} : \mathbb{C}^{n+m} \to Z$ such that for any $\tilde{x} \in U_\Sigma \subset \mathbb{C}^{n+m}$, the G-constellation corresponding to $\varphi(\tilde{x})$ is isomorphic to $\mathcal{F}_x$ where $x \in X$ is the image of $\tilde{x}$ in $U_\Sigma / T_m = X$.

Moreover, there is a group homomorphism $\psi_{\mathcal{F}} : T_m \to T_{r-1}$ s.t. $\varphi_{\mathcal{F}}$ is equivariant with respect to $\psi_{\mathcal{F}}$.

Then we have:

$$
\begin{tikzcd}
T_m \arrow[r, "\psi_{\mathcal{F}}"] & T_{r-1} \\
\circlearrowright & \circlearrowright \\
\mathbb{C}^{n+m} \arrow[r, "\varphi_{\mathcal{F}}"] & Z \\
\cup & \cup \text{ open} \\
U_\Sigma \arrow[r] \arrow[d, "/T_m"] & V_{\Sigma,\mathcal{F}} := T_{r-1} \cdot \varphi_{\mathcal{F}}(U_\Sigma) \arrow[d] \\
X \arrow[r] \arrow[dr] & V_{\Sigma,\mathcal{F}} /\!/ T_{r-1} =: \mathcal{M}_{\Sigma,\mathcal{F}} \arrow[d] \\
 & \mathbb{C}^n/G
\end{tikzcd}
$$

may not be the orbit space

We take the normalizations:

$$
\begin{tikzcd}
 & T_{r-1} \\
 & \circlearrowright \\
U_\Sigma \arrow[r, "\overline{\varphi}_{\mathcal{F}}"'] \arrow[d] & \overline{V}_{\Sigma,\mathcal{F}} \arrow[r] \arrow[d, "/\!/ T_{r-1}"] & V_{\Sigma,\mathcal{F}} \arrow[d] \\
X \arrow[r, "\tilde{\varphi}_{\Sigma,\mathcal{F}}"'] & \overline{\mathcal{M}}_{\Sigma,\mathcal{F}} \arrow[r] & \mathcal{M}_{\Sigma,\mathcal{F}}
\end{tikzcd}
$$

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Thm. (—) Assume $\underbrace{\psi_{\mathcal{F}} : T_m \to T_{r-1} \text{ has finite kernel}}_{(*)}$.

Then, $\tilde{\varphi}_{\Sigma,\mathcal{F}}$ is an isomorphism

$\iff$ every T_{r-1}-orbit in $V_{\Sigma,\mathcal{F}}$ is closed

$\iff$ for every $x \in X$, $\mathcal{F}_x$ is indecomposable $\left(\begin{array}{l}\text{i.e. } \mathcal{F}_x \not\cong F_1 \oplus F_2 \\ \text{with subreps } F_i \neq 0\end{array}\right)$.

Moreover, for any $X = X(\Sigma)$, there is $\mathcal{F}$ such that $\tilde{\varphi}_{\Sigma,\mathcal{F}}$ is small i.e. no divisor is contracted.

$\boxed{\begin{array}{l}\text{Note} \\ m \le r-1\end{array}}$

Key points to show that $\tilde{\varphi}_{\Sigma,\mathcal{F}}$ is an iso. assuming closedness of orbits in $V_{\Sigma,\mathcal{F}}$.

(1) $T_{r-1} \cdot \varphi_{\mathcal{F}}(\{z_k = 0\}) \subset Z$ is a divisor

$\rightsquigarrow$ $\tilde{\varphi}_{\Sigma,\mathcal{F}}$ is small if all orbits in $V_{\Sigma,\mathcal{F}}$ are closed.

(2) $\exists$ commutative diagram:

$$\begin{array}{ccccc} & & X(T_{r-1})_{\mathbb{Q}} & \xrightarrow{\psi_{\mathcal{F}}^*} & X(T_m)_{\mathbb{Q}} \\ & & \downarrow & & \downarrow \cong \\ \mathrm{Pic}(\overline{\mathcal{M}}_{\Sigma,\mathcal{F}})_{\mathbb{Q}} & \xrightarrow{\sim} & \mathrm{Pic}^{T_m}(\overline{V}_{\Sigma,\mathcal{F}})_{\mathbb{Q}} & \xrightarrow{\overline{\varphi}_{\mathcal{F}}^*} & \mathrm{Pic}^{T_m}(U_{\Sigma})_{\mathbb{Q}} \end{array}$$

surjective by (*)

closedness of orbits in $V_{\Sigma,\mathcal{F}}$

$\rightsquigarrow$ $\overline{\mathcal{M}}_{\Sigma,\mathcal{F}}$ is $\mathbb{Q}$-factorial

Rem. 1. For $n \ge 3$, $\tilde{\varphi}_{\Sigma,\mathcal{F}_{max}}$ is not small in general.

2. If $\tilde{\varphi}_{\Sigma,\mathcal{F}}$ is an iso. and $X \to \mathbb{C}^n/G$ is projective, then $\mathcal{M}_{\Sigma,\mathcal{F}}$ is the same as $\mathcal{M}_{\theta}$ for some $\theta \in \Theta$.

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§ Related Questions

1. For given $X = X(\Sigma)$, can we choose $\mathcal{F}$ so that $\tilde{\varphi}_{\Sigma,\mathcal{F}}$ is an isomorphism?

2. When $\tilde{\varphi}_{\Sigma,\mathcal{F}}$ is an isomorphism, does the natural functor
$$\begin{array}{cccc} \Phi_{\mathcal{F}} : & D^b(\mathrm{coh}\, X) & \longrightarrow & D^b(\mathrm{coh}^G \mathbb{C}^n) \\ & \Psi & & \Psi \\ & \mathcal{O}_x & \longmapsto & \mathcal{F}_x \end{array}$$
give an equivalence?

3. For non-projective $X(\Sigma) \to \mathbb{C}^n/G$, is there a rep.-theoretic characterization of points of $V_{\Sigma,\mathcal{F}}$?