Mirror Symmetry Constructions for Quasi-smooth Calabi-Yau Hypersurfaces in Weighted Projective Spaces

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arXiv: 2006.04465

SETUP

NON-DEGENERATE AFFINE HYPERSURFACE

$$Z_\Delta := \{ x \in \mathbb{T}^d \mid \underbrace{f_\Delta(x)}_{= \sum_{m \in A} a_m t^m} = 0 \} \subset \mathbb{T}^d \cong (\mathbb{C}^*)^d$$

LAURENT POLYNOMIAL

a_m ↑ SUFFICIENTLY GENERAL $a_m \in \mathbb{C}$

FIXED NEWTON POLYTOPE

$$\Delta = \mathrm{CONV}(A) \subseteq M_{\mathbb{R}} \cong \mathbb{R}^d$$

$$:\Leftrightarrow \Big\{ \sum_{m \in A \cap \theta} a_m t^m = 0 \Big\} \text{ SMOOTH } \forall \theta \leq \Delta .$$

Interest

Birational Geometry of Z_Δ

Theorem [Khovanskii '78] $p_g(Z_\Delta) = |\Delta^\circ \cap M|$

+ [Ishii '99] $p_g(Z_\Delta) \neq 0 \implies \kappa(Z_\Delta) \geq 0$

Definition

$\Delta \subseteq M_{\mathbb{R}}$ Canonical Fano Polytope

$:\iff \Delta$ d-dim. convex lattice polytope

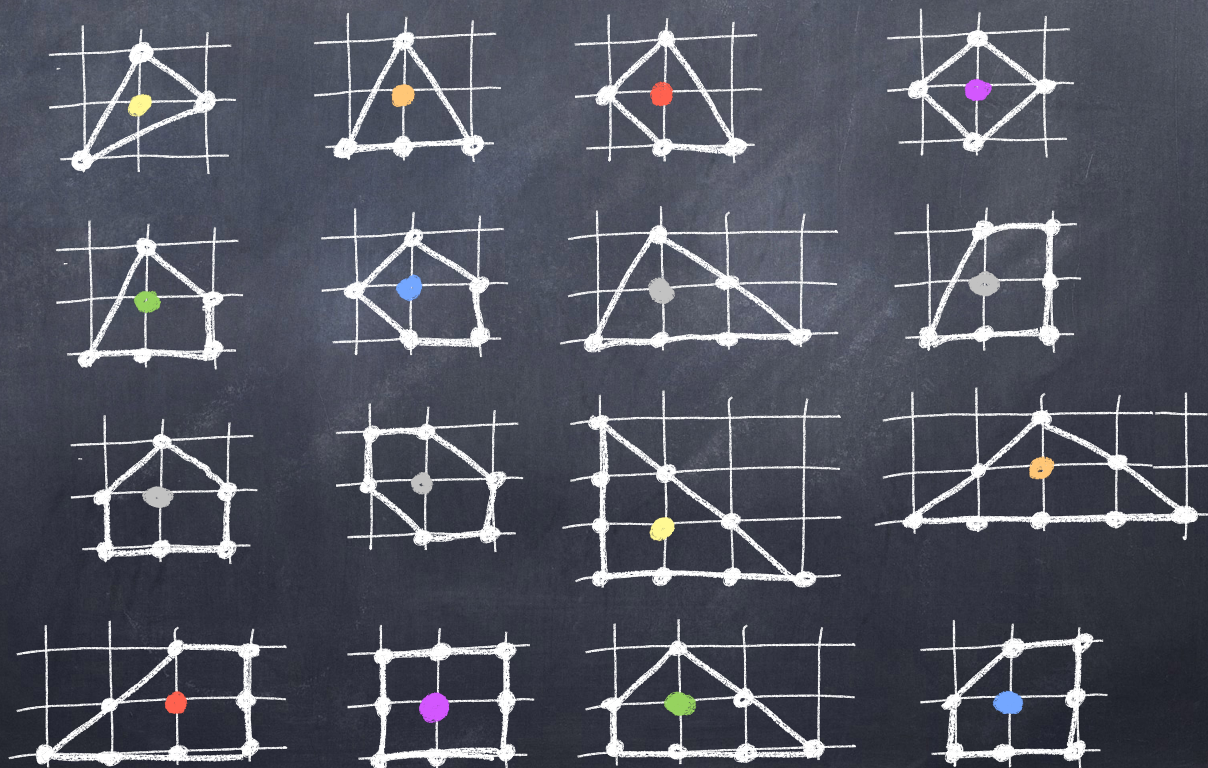
with $|\Delta^\circ \cap M| = 1$.

Remark $p_g(Z_\Delta) = 1 \iff \Delta$ canonical Fano polytope

CLASSIFICATION OF CANONICAL FANO POLYT. UP TO DIMENSION 3

- DIM 1 : #1

- DIM 2 : #16 = # REFLEXIVE POLYGONS



CLASSIFICATION OF CANONICAL FANO POLYT. UP TO DIMENSION 3

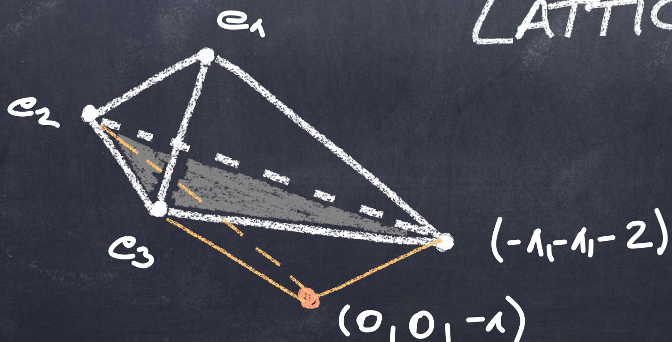
- DIM 3 : # 674688 $\supseteq$ # REFLEXIVE POLYT.
  = # 4319 [KS '98]

[KASPRYZK '10] CLASSIFICATION OF ALL 3-DIM. TORIC FANO VARIETIES W.A.W. CANONICAL SINGULARITIES

DEFINITION

$\Delta \subseteq M_{\mathbb{R}}$ REFLEXIVE $:\Leftrightarrow$ Δ AND DUAL Δ^* LATTICE POLYT.

EXAMPLE



DEFINITION

X CALABI-YAU VARIETY $:\Longleftrightarrow$ X NORMAL PROJECTIVE VARIETY W.A.W. CANONICAL SINGULARITIES AND TRIVIAL CANONICAL CLASS.

THEOREM [BATYREV '17]

$\Delta \subseteq M_{\mathbb{R}}$ CANONICAL FANO POLYTOPE. THEN:

$Z_\Delta \sim X$ CALABI-YAU $\Longleftrightarrow$ $\mathrm{CONV}(\Delta^* \cap N)$ CANONICAL FANO P.

$\Longleftrightarrow$ $\Delta^{FI} = \{0\}$.

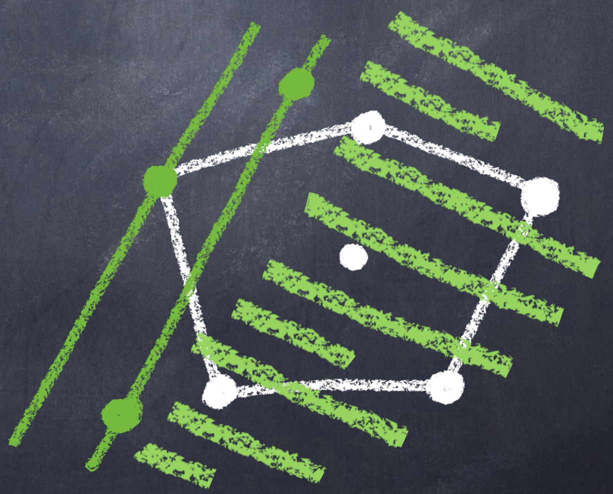
DEFINITION

$\Delta^{FI} \subseteq \Delta$ FINE INTERIOR OF Δ GIVEN AS

$$\Delta^{FI} := \bigcap_{0 \neq n \in N} \left\{ x \in M_{\mathbb{R}} \,\middle|\, \langle x, n \rangle \geq \underbrace{\mathrm{ORD}_\Delta(n)}_{= \min_{y \in \Delta} \langle y, n \rangle} + 1 \right\}$$

PROPERTIES

- $\Delta^{FI} \subseteq \Delta^{\circ}$
- Δ^{FI} RATIONAL POLYTOPE [MAYBE $\emptyset$]
- $\mathrm{CONV}(\Delta^{\circ} \cap M) \subseteq \Delta^{FI}$



Theorem [Batyrev '17]

$\Delta \subseteq M_{\mathbb{R}}$ canonical Fano polytope. Then:

$Z_\Delta \sim X$ Calabi-Yau $\iff$ $\Delta^{FI} = \{0\}$.

Classification of canonical Fano polyt. with $\Delta^{FI} = \{0\}$ up to dim. 3

- Dim 1 : #1
- Dim 2 : #16 = # canonical Fano P. = # reflexive polygons
- Dim 3 : #665599 $\subseteq$ #674688 = #C.F.P. $\subseteq$ #4319 = # reflexive P.

[Batyrev - Kasprzyk - S. '19] arXiv: 1911.12048

Question How to obtain evidence for Mirror Symmetry?

Tool Topological Mirror Symmetry Test for <u>smooth</u> $(d-1)$-dim. Calabi-Yaus [small version]

$$e(V) = (-1)^{d-1} \underbrace{e(V^*)}_{:= \sum_{p,q} (-1)^{p+q} h^{p,q}(V^*)}$$

Euler Number

Problem Non-smooth Mirror Symmetry Partner

SOLUTION STRINGY VERSION

X PROJECTIVE $\mathbb{Q}$-GORENSTEIN VARIETY
W.A.W. LOG-TERMINAL SINGULARITIES.

$\rho : Y \longrightarrow X$ DESING. WITH $K_Y = \rho^* K_X + \sum_{i=1}^{s} \underbrace{a_i}_{> -1} D_i$

SMOOTH IRRED. DIVS
IN EXCEPT. LOCUS
WITH ONLY SNC.

$$D_J := \begin{cases} Y, & J = \emptyset \\ \bigcap_{j \in J} D_j, & J \subseteq I := \{1, \dots, s\}. \end{cases}$$

$D_J \subseteq Y$ SMOOTH

DEFINITION STRINGY EULER NUMBER

$$e_{str}(X) := \sum_{\emptyset \subseteq J \subseteq I} e(D_J) \prod_{j \in J} \left(\frac{-a_j}{a_j + 1} \right) \in \mathbb{Q}.$$

THEOREM [BATYREV '17]

$\Delta \subseteq M_{\mathbb{R}}$ CANONICAL FANO POLYTOPE S.T.

$Z_\Delta \sim X$ CALABI-YAU. THEN

$$e_{\mathrm{str}}(X) = \sum_{k=1}^{d} (-1)^{k-1} \sum_{\substack{\theta \leq \Delta \\ \dim(\theta)=k}} \mathrm{Vol}_k(\theta) \cdot \mathrm{Vol}_{d-k}(\sigma_\theta \cap \Delta^*).$$

NORMALIZED VOLUME

REMARK GENERALIZATION OF [BATYREV-DAIS '96]

INSPIRATION [VAFA '98]

$X_w \subset \mathbb{P}(\overline{w})$ QUASI-SMOOTH CALABI-YAU HYPERSURFACE OF DEGREE w IN WEIGHTED PROJECTIVE SPACE.

THEN ORBIFOLD EULER NUMBER

$$e_{ORB}(X_w) = \frac{1}{w} \sum_{l,r=0}^{w-1} \prod_{\substack{0 \le i \le d \\ lq_i,\, rq_i \in \mathbb{Z}}} \left(1 - \frac{1}{q_i}\right),$$

WHERE $\overline{w} := (w_0, \ldots, w_d)$, $w := \sum_{i=0}^{d} w_i$, $q_i := \frac{w_i}{w}$ $\forall i$.

Terminology

$\overline{w} = (w_0, \dots, w_d) \in \mathbb{Z}^{d+1}_{>0}$ Weight vector.

$\mathbb{P}(\overline{w})$ Weighted projective space, i.e., quotient of $\mathbb{C}^{d+1} \setminus \{0\}$ by $\mathbb{C}^*$-action

$$\lambda \cdot (z_0, \dots, z_d) := (\lambda^{w_0} z_0, \dots, \lambda^{w_d} z_d) \ \forall \lambda \in \mathbb{C}^*.$$

Assume $\overline{w}$ well-formed, i.e.,

$$\mathrm{GCD}(w_0, \dots, w_{i-1}, w_{i+1}, \dots, w_d) = 1 \ \forall i.$$

$\overline{w}$ has IP-property $:\Longleftrightarrow$ $\Delta(\overline{w})$ d-dim. lattice polytope with $(1, \dots, 1) \in \Delta(\overline{w})^\circ$, where

$$\Delta(\overline{w}) := \mathrm{conv}\left\{ (u_0, \dots, u_d) \in \mathbb{Z}^{d+1}_{\geq 0} \,\middle|\, \sum w_i u_i = \sum w_i \right\}.$$

Construction of first Mirror Symmetry Partner being a Calabi-Yau Hypersurf.

Setup

Let $\overline{w} = (w_0, \dots, w_d)$ be a weight vector with IP-property and $Z_{\overline{w}} \subset T_{\overline{w}}$ a non-deg. affine hypersurface defined by a Laurent polynomial $f_{\overline{w}}$ with Newton polytope $\Delta^*_{\overline{w}} := \operatorname{conv}(\{v_0, \dots, v_d\})$, where $v_0, \dots, v_d$ generate the character lattice of $T_{\overline{w}} := \left\{ (x_0, \dots, x_d) \in (\mathbb{C}^*)^{d+1} \,\middle|\, \prod_{i=0}^{d} x_i^{w_i} = 1 \right\}$ and $\sum_{i=0}^{d} w_i v_i = 0$.

THEOREM [BATYREV - S. '20]

LET $\overline{w} = (w_0, \ldots, w_d)$ BE A WEIGHT VECTOR WITH IP-PROPERTY AND $Z_{\overline{w}} \subset \mathbb{T}_{\overline{w}}$ A NON-DEG. AFFINE HYPERSURFACE DEFINED BY A LAURENT POLYNOMIAL $f_{\overline{w}}$ WITH NEWTON POLYTOPE $\Delta^*_{\overline{w}} = \operatorname{CONV}(\{v_0, \ldots, v_d\})$. THEN $X^*_{\overline{w}} := \overline{Z_{\overline{w}}} \subset \mathbb{P}^\vee(\overline{w})$ IS A CALABI-YAU HYPERSURFACE IN THE $\mathbb{Q}$-GORENSTEIN TORIC VARIETY $\mathbb{P}^\vee(\overline{w})$ ASSOCIATED TO THE SPANNING FAN OF $[(\Delta^*_{\overline{w}})^*]$.

THEOREM [BATYREV - S. '20]

LET $\overline{w} = (w_0, \ldots, w_d)$ BE A WEIGHT VECTOR WITH IP-PROPERTY AND $Z_{\overline{w}} \subset \mathbb{T}_{\overline{w}}$ A NON-DEG. AFFINE HYPERSURFACE DEFINED BY A LAURENT POLYNOMIAL $f_{\overline{w}}$ WITH NEWTON POLYTOPE $\Delta^*_{\overline{w}} = \mathrm{conv}(\{v_0, \ldots, v_d\})$. THEN $X^*_{\overline{w}} = \overline{Z_{\overline{w}}} \subset \mathbb{P}^v(\overline{w})$ IS A CALABI-YAU HYPERSURFACE WITH

$$e_{str}(X^*_{\overline{w}}) = (-1)^{d-1} \frac{1}{w} \sum_{l,r=0}^{w-1} \overline{\prod_{\substack{0 \le i \le d \\ lq_i, rq_i \in \mathbb{Z}}}} \left(1 - \frac{1}{q_i}\right).$$

Construction of Second Mirror Symmetry Partner Being a Calabi-Yau Hypersurf.

Definition

$\bar{w}$ TRANSVERSE $:\Longleftrightarrow$ $\exists$ QUASI-SMOOTH (!) CALABI-YAU HYPERSURFACE $X_w \subset \mathbb{P}(\bar{w})$ OF DEGREE w.

QUASI-SMOOTH $:\Longleftrightarrow$ ONLY SINGULAR POINT OF $X_w = \{W(z) = 0\}$ IS 0.

Proposition [Skarke '96]

TRANSVERSE $\Longrightarrow$ IP-PROPERTY.

THEOREM [BATYREV - S. '20]

LET $\overline{w} = (w_0, \ldots, w_d)$ BE A TRANSVERSE WEIGHT VECTOR AND $Z_{\overline{w}} \subset \mathbb{T}_{\overline{w}}$ A NON-DEG. AFFINE HYPERSURFACE DEFINED BY A LAURENT POLYNOMIAL $f_{\overline{w}}$ WITH NEWTON POLYTOPE $\Delta^*_{\overline{w}} = \mathrm{CONV}(\{v_0, \ldots, v_d\})$. THEN $X^*_{\overline{w}} = \overline{Z_{\overline{w}}} \subset \mathbb{P}^{\vee}(\overline{w})$ IS A CALABI-YAU H. AND $\exists$ A QUASI-SMOOTH CALABI-YAU HYPERSURFACE $X_w \subset \mathbb{P}(\overline{w})$ OF DEGREE w WITH

$$e_{\mathrm{str}}(X^*_{\overline{w}}) = (-1)^{d-1} e_{\mathrm{orb}}(X_w).$$

Mirror Construction [Batyrev - S. '20]

Let $\overline{w} = (w_0, \ldots, w_d) \in \mathbb{Z}_{>0}^{d+1}$ be a transverse weight vector.

Then mirrors of quasi-smooth Calabi-Yau hypersurfaces $X_w \subset \mathbb{P}(\overline{w})$ can be obtained as Calabi-Yau compactifications $X_{\overline{w}}^*$ of non-degenerate affine hypersurfaces $Z_{\overline{w}} \subset \mathbb{T}_{\overline{w}}$ with Newton polytope $\Delta_{\overline{w}}^* = \mathrm{conv}(\{v_0, \ldots, v_d\})$.

THANK you!