Cohomology rings of toric bundles

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Plan: Find "good" descriptions of the cohomology ring $H^*(X, \mathbb{R})$ for certain classes of manifolds X.

1) Recap: smooth projective toric case
2) Toric bundles
3) Applications

§1. Smooth projective toric case

- $T \cong (\mathbb{C}^*)^n$ algebraic torus
- M = character lattice of T
- $N = \mathrm{Hom}(M, \mathbb{Z})$ dual lattice

$X = X_\Sigma$ smooth projective toric variety with corresponding fan $\Sigma \subseteq N_\mathbb{R}$

How to compute $H^*(X_\Sigma, \mathbb{R})$?

Many answers exist:

* Stanley – Reisner description
* Minkowski weights
* Volume polynomial
* .....

Volume polynomial approach:

Let $D \subseteq X_\Sigma$ Cartier divisor.

Then $D = D_1 - D_2$ for <u>ample</u> divisors $D_1, D_2 \subseteq X_\Sigma$ $\rightsquigarrow$ D_i correspond to polytopes $P_i \subseteq M_\mathbb{R}$

Leads to

$$\mathcal{P}_\Sigma^+ = \{ \text{polytopes } P \subseteq M_\mathbb{R} \text{ whose normal fan coarsen } \Sigma \} \xleftrightarrow{\cancel{1:1}} \left(\begin{array}{c} \text{ample divisors} \\ \text{on } X_\Sigma \end{array} \right)$$

$\rightsquigarrow$ I mean "corresponds to", not bijection.

$\mathcal{P}_\Sigma^+$ is a cone w.r.t. Minkowski addition & scaling

$$\rightsquigarrow \mathcal{P}_\Sigma = \text{vector space}(\mathcal{P}_\Sigma^+)$$
$$= \{ \text{virtual polytopes } P_1 - P_2 \text{ for } P_1, P_2 \in \mathcal{P}_\Sigma^+ \}$$
$$= \{ \text{piecewise linear functions on } \Sigma \}$$
$$\xleftrightarrow{\cancel{1:1}} (\text{Cartier divisors on } X_\Sigma)$$

see above

Need 2 more ingredients:

"algebraic ingredient":

$$H^*(X_\Sigma, \mathbb{R}) = \bigoplus_{i=0}^{n} H^{2i}(X_\Sigma, \mathbb{R})$$

graded commutative with

1) $H^0(X_\Sigma, \mathbb{R}) \cong H^{2n}(X_\Sigma, \mathbb{R}) \cong \mathbb{R}$

2) $H^{2i}(X_\Sigma, \mathbb{R}) \times H^{2(n-i)}(X_\Sigma, \mathbb{R}) \longrightarrow \mathbb{R}$

non-degenerate (Poincaré duality)

3) generated in degree 2:

$$\mathbb{R}[H^2] \cong H^*(X_\Sigma, \mathbb{R})$$

Any ring satisfying 1, 2, 3 can be constructed as

$$\mathrm{Diff}(H^2) / \mathrm{Ann}(\varrho) \cong H^*(X_\Sigma, \mathbb{R})$$

* $\mathrm{Diff}(H^2)$ ring of differential operators with constant coefficients.
* $\varrho : H^2 \to H^{2n} \cong \mathbb{R}$

$x \mapsto x^n$ polynomial

"convex geometric ingredient":

BKK Theorem:

For all $P \in \mathcal{P}_\Sigma$

$$D_P^n = n! \ \mathrm{vol}(P)$$

All together gives:

$$H^*(X_\Sigma, \mathbb{R}) \cong \mathrm{Diff}(\mathcal{P}_\Sigma) / \mathrm{Ann}(\mathrm{vol})$$

§2. Toric bundles

Like before $T \cong (\mathbb{C}^*)^n$ algebraic torus
Principal torus bundle

$$\begin{array}{ccc} T & \to & E \\ & & \downarrow p \\ & & B \end{array}$$

B closed compact orientable manifold of (real) dimension k.

Example: (horospherical homogeneous spaces)

G connected reductive group $/\mathbb{C}$
$H \leq G$ closed subgroup with $\mathcal{U} \leq H$
maximal unipotent

$$\begin{array}{ccc} \overbrace{N_G(H)/H}^{\text{algebraic torus}} & \longrightarrow & G/H \\ & & \downarrow \\ & & \underbrace{G/N_G(H)}_{\text{generalised flag variety}} \end{array}$$

For any T-toric variety X_Σ, let E_Σ be the associated fibre bundle

$$\begin{array}{ccc} X_\Sigma & \longrightarrow & E_\Sigma = (E \times X_\Sigma)/T \\ & & \downarrow \\ & & B \end{array}$$

Suppose X_Σ is smooth & projective

<u>Question:</u> How to compute $H^*(E_\Sigma, \mathbb{R})$?

<u>Remark:</u> Sankaran & Uma generalised the Stanley-Reisner description to this case (2003)

<u>Leray-Hirsch theorem:</u>

As vector spaces

$$H^*(E_\Sigma, \mathbb{R}) \cong H^*(B, \mathbb{R}) \otimes H^*(X_\Sigma, \mathbb{R})$$

<u>Recall:</u> $\mathcal{P}_\Sigma \twoheadrightarrow H^2(X_\Sigma, \mathbb{R})$

If $x_1, \dots, x_r$ basis of $\mathcal{P}_\Sigma$, then

$$\underset{\text{Surjection}}{\mathbb{R}[x_1, \dots, x_r] \twoheadrightarrow H^*(X_\Sigma, \mathbb{R})}$$

Together with Leray-Hirsch:

$$\underset{\text{Surjection}}{H^*(B, \mathbb{R})[x_1, \dots, x_r] \twoheadrightarrow H^*(E_\Sigma, \mathbb{R})}$$

What is the kernel?

Need 2 more ingredients

"algebraic ingredient":

$$H^*(E_\Sigma, \mathbb{R}) = \bigoplus_{i=0}^{2n+k} H^i(E_\Sigma, \mathbb{R})$$

graded commutative with

1) multiplicative identity in $H^0(E_\Sigma, \mathbb{R})$
2) $H^0(E_\Sigma, \mathbb{R}) \cong H^{2n+k}(E_\Sigma, \mathbb{R}) \cong \mathbb{R}$
3) pairing $H^i(E_\Sigma, \mathbb{R}) \times H^{2n+k-i}(E_\Sigma, \mathbb{R}) \to \mathbb{R}$
non-degenerate (Poincaré duality)

$\rightsquigarrow$ algebraic description of the kernel of

$$\Phi: \underbrace{H^*(B, \mathbb{R})}_{\text{graded}} [\underset{\uparrow}{x_1}, \dots, \underset{\uparrow}{x_r}] \twoheadrightarrow H^*(E_\Sigma, \mathbb{R})$$

degree 2

$$R := H^*(B, \mathbb{R})[x_1, \dots, x_r] = \bigoplus_{i \geq 0} R^i$$

graded commutative

define linear function:

$$\ell: R^{2n+k} \longrightarrow \mathbb{R}$$

$$x \longmapsto \ell(x) = \Phi(x) \in H^{2n+k}(E_\Sigma, \mathbb{R}) \cong \mathbb{R}$$

extend by 0 to all of R.

(such linear functions are called $(2n+k)$-homogeneous)

The kernel of the surjection Φ accepts the following algebraic description:

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$$I_\ell = \{ a \in R : \ \ell(a \cdot b) = 0 \ \text{ for all } b \in R \}$$
$$= \{ b \in R : \ \ell(a \cdot b) = 0 \ \text{ for all } a \in R \}$$

$$\rightsquigarrow \quad H^*(E_\Sigma, \mathbb{R}) \cong H^*(B, \mathbb{R})[x_1, \dots, x_r] / I_\ell$$

What is I_ℓ?

It suffices to compute

$$\ell(\gamma \cdot x_{i_1} \cdots x_{i_s})$$

for $\gamma \in H^{k+2(n-s)}(B, \mathbb{R})$ and $i_1, \dots, i_s \in \{1, \dots, r\}$

$$(\ \deg(\gamma \cdot x_{i_1} \cdots x_{i_s}) = k + 2n).$$

<u>"convex geometric ingredient":</u>

First note that $\ell(\gamma \cdot x_{i_1} \cdots x_{i_s})$ is a polynomial on $\mathcal{P}_\Sigma$ of degree s for fixed γ.

$\rightsquigarrow$ By polarisation, enough to compute

$$\ell(\gamma \cdot x^s) \quad \text{for } x \in \mathcal{P}_\Sigma.$$

<u>idea:</u> express this number as an integral ("volume")

$\lambda \in M$ character of T

$\rightsquigarrow$ $\mathcal{L}_\lambda = E \times_T \mathbb{C}_\lambda$ line bundle on B

$\rightsquigarrow$ obtain linear map

$$c : M \longrightarrow H^2(B, \mathbb{Z}); \ \lambda \longmapsto c_1(\mathcal{L}_\lambda)$$

(Chern class)

Fix $\gamma \in H^{k+2(n-s)}(B, \mathbb{R})$.

Define function

$$f_\gamma : M_{\mathbb{R}} \to \mathbb{R}; \quad x \mapsto f_\gamma(x) = \gamma \cdot \underbrace{c(x)^{s-n}}_{\in H^k(B,\mathbb{R}) \cong \mathbb{R}}$$

Let $I_\gamma : \mathcal{P}_\Sigma \to \mathbb{R}$

$$I_\gamma(\Delta) = \int_\Delta f_\gamma(x)\, dx$$

(integral makes sense on $\mathcal{P}_\Sigma^+$, extend to all of $\mathcal{P}_\Sigma$)

<u>BKK-type theorem</u> (—, Khovanskii, Monin):

$$s!\, I_\gamma(\Delta) = (s-n)!\ \ell(\gamma \cdot \Delta^s)$$

<u>Special case:</u> If $B = pt$, then $H^*(B,\mathbb{R}) \cong \mathbb{R}$

$*$ $f_\gamma(x) = 1$

$$\Rightarrow \quad I_\gamma(\Delta) = \int_\Delta f_\gamma(x)\, dx = \mathrm{vol}(\Delta)$$

$$* \quad \ell(\underbrace{\gamma}_{\substack{\text{const.,}\\ \text{say } 1}} \cdot \Delta^s) = \ell(\Delta^s) = \begin{cases} 0, & \text{if } s \neq n \\ D_\Delta^n, & \text{if } n = s \end{cases}$$

$$\rightsquigarrow \quad \underbrace{s!\ I_\gamma(\Delta)}_{n!\ \mathrm{vol}(\Delta)} = \underbrace{(s-n)!\ \ell(\gamma \cdot \Delta^s)}_{0! \cdot D_\Delta^n}$$

(classical BKK theorem)

All together gives

$$H^*(E_\Sigma, \mathbb{R}) \cong H^*(B,\mathbb{R})[x_1,\dots,x_r] \,/\, I_\ell$$

where $\ell : H^*(B,\mathbb{R})[x_1,\dots,x_r] \longrightarrow \mathbb{R}$

$(k+2n)$ – homogeneous linear form given by

$$\ell(\gamma \cdot x_{i_1} \cdots x_{i_s}) = I_\gamma(x_{i_1},\dots,x_{i_s})$$

§3. Applications

original motivation:

compute ring of conditions $C^*(G/H)$ of horospherical homogeneous spaces G/H.

De Concini & Procesi showed:

$$C^*(G/H) = \varinjlim_{\Sigma} H^*(G/H \times_{N_G(H)/H} X_\Sigma, \mathbb{R})$$

There exists a natural object

$$\mathcal{P} = \varinjlim_{\Sigma} \mathcal{P}_\Sigma$$

Space of (arbitrary) virtual polytopes

$\rightsquigarrow$ $C^*(G/H) = \left(H^*(G/N_G(H), \mathbb{R}) \otimes \mathrm{Sym}(\mathcal{P})\right) / I_\ell$

where $\ell(\gamma \otimes \Delta_{i_1} \cdots \Delta_{i_s}) = I_\gamma(\Delta_{i_1},\dots,\Delta_{i_s})$

Relation to string polytopes:

For simplicity, consider principal torus bundle

$$\begin{tikzcd} \underset{\text{maximal torus}}{T \cong B/U} \arrow[r] & SL_n/U \arrow[d] \\ & SL_n/B \end{tikzcd}$$

U maximal unipotent

B Borel with $U \subseteq B$

Λ^+ monoid of dominant weights

$M =$ character lattice of T

Gelfand – Zetlin polytopes

$GZ : \Lambda^+ \to \mathcal{P}^+_{\mathbb{R}^N}$ for $N = \frac{n(n-1)}{2}$

linear family of polytopes

(note: this map is the restriction of a

linear map)

We construct a "lifted family":

define cone

$$\mathcal{C} := \{ Q \in \mathcal{P}^+_{M_\mathbb{R}} : Q \subseteq \operatorname{cone}(\Lambda^+) \}$$

$$\widetilde{GZ} : \mathcal{C} \to \mathcal{P}^+_{M_\mathbb{R} \oplus \mathbb{R}^N}$$

$$\widetilde{GZ}(Q) = \{ (\lambda, x) \in M_\mathbb{R} \oplus \mathbb{R}^N : \lambda \in Q, \\ y \in GZ(\lambda) \}$$

denote composition of (linear) lifted family $\widetilde{GZ} : \mathcal{C} \to \mathcal{P}^+_{M_{\mathbb{R}} \oplus \mathbb{R}^N}$ with volume polynomial on $\mathcal{P}^+_{M_{\mathbb{R}} \oplus \mathbb{R}^N}$ by $\widetilde{vol}$.

<u>Thm.</u> (–, Khovanskii, Monin):

Torus bundle $E = SL_n / U$

$X = X_\Sigma$ smooth projective T-toric variety

Then

$$H^{*/2}(E_\Sigma, \mathbb{R}) \cong \mathrm{Diff}(\mathcal{P}_\Sigma) / \mathrm{Ann}(\widetilde{vol}).$$