On Reid's recipe for non Abelian groups

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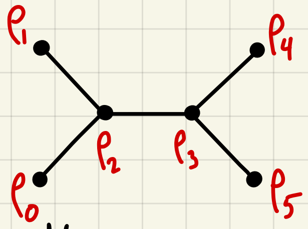
McKay Correspondence

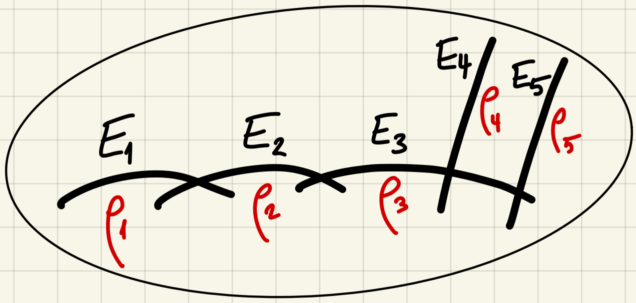
• $G \subset SL(2,\mathbb{C})$ finite. <u>Ex</u>: $G = D_5$ Dihedral group of order 12.

$$\mathrm{Irr}\, D_5 = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$$

minimal resolution $Y \xrightarrow{\ \pi\ } \mathbb{C}^2/D_5$



McKay quiver of D_5



E : Exc. divisor

$$E = \bigcup_{i=1}^{5} E_i, \quad E_i \simeq \mathbb{P}^1$$

$$E_0 := -E_{fund}$$

McKay Correspondence: $E_i \xleftrightarrow{\text{recipe}} \rho_i$

• $G \subset SL(3,\mathbb{C})$ finite

crepant resolution $Y \simeq \boxed{G\text{-Hilb}\,\mathbb{C}^3} \xrightarrow{\ \pi\ } \mathbb{C}^3/G$ [BKR]

"distinguished" from the McKay Corr. point of view

[Reid]

<u>if G Abelian</u> it has a recipe! (with plenty of applications to other related problems)

It remains <u>open</u> for most of non Abelian subgroups of $SL(3,\mathbb{C})$

<u>Plan</u>: Explain Reid's recipe for the Abelian group $\frac{1}{12}(1,7,4)$ and show 2 non Abelian examples:

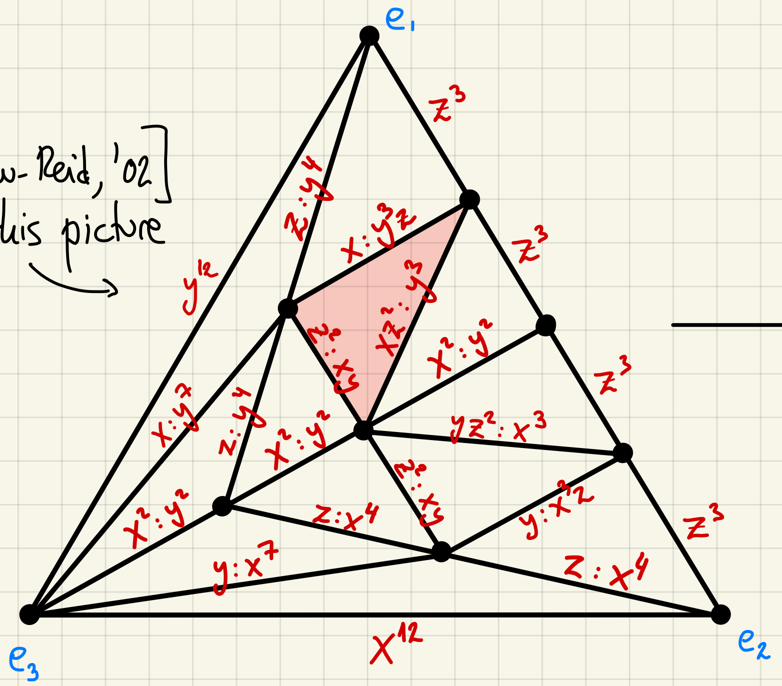
Dihedral $\langle \frac{1}{12}(1,7,4), \beta \rangle$ and (part of) the Trihedral $\langle \frac{1}{13}(1,3,9), T \rangle$

$\boxed{\text{Example}}$ $A = \frac{1}{12}(1,7,4) = \left\langle \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^7 & 0 \\ 0 & 0 & \varepsilon^4 \end{pmatrix} \;\middle|\; \varepsilon = e^{\frac{2\pi i}{12}} \right\rangle \subset SL(3, \mathbb{C})$

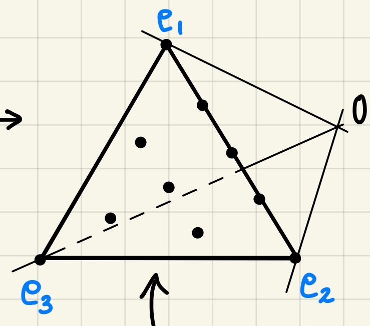
Abelian group of order 12, $A \curvearrowright \mathbb{C}^3$ by $\begin{cases} x \longmapsto \varepsilon x \\ y \longmapsto \varepsilon^7 y \\ z \longmapsto \varepsilon^4 z \end{cases}$

Crepant resolution $Y = A\text{-Hilb}\,\mathbb{C}^3 \xrightarrow{\quad \pi \quad} \mathbb{C}^3/A$

See [Craw-Reid, '02] to get this picture



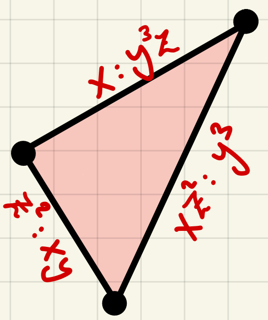
Affine toric singularity



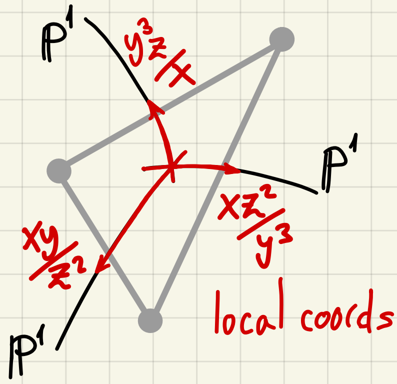
junior simplex Δ

Elementary triangle $\longleftrightarrow$ $\mathcal{U} \overset{\text{open}}{\subset} A\text{-Hilb}\,\mathbb{C}^3$, $\mathcal{U} \simeq \mathbb{C}^3$

T



$\xleftrightarrow{\text{dual}}$



$\mathcal{U}_T \cong \mathbb{C}^3_{\frac{xy}{z^2}, \frac{y^3z}{x}, \frac{xz^2}{y^3}}$

Thus :

- $A\text{-Hilb}\,\mathbb{C}^3$ covered by 12 copies of $\mathbb{C}^3$
- Excep. locus $\left[\begin{array}{l} \text{Interior edge} \longleftrightarrow \mathbb{P}^1 \\ \text{Interior vertex} \longleftrightarrow \text{compact divisor in } \pi^{-1}(0) \end{array} \right.$

$$12 = |A| \overset{\shortparallel}{} \#\,\mathrm{Irr}\,A \overset{\shortparallel}{} e(Y) \overset{\shortparallel}{} \#\,\text{opens}$$

Representation theory of $\frac{1}{12}(1,7,4) = \left\langle \underbrace{\begin{pmatrix} \varepsilon & & \\ & \varepsilon^7 & \\ & & \varepsilon^4 \end{pmatrix}}_{\alpha} \;\middle|\; \varepsilon = e^{2\pi i/12} \right\rangle$

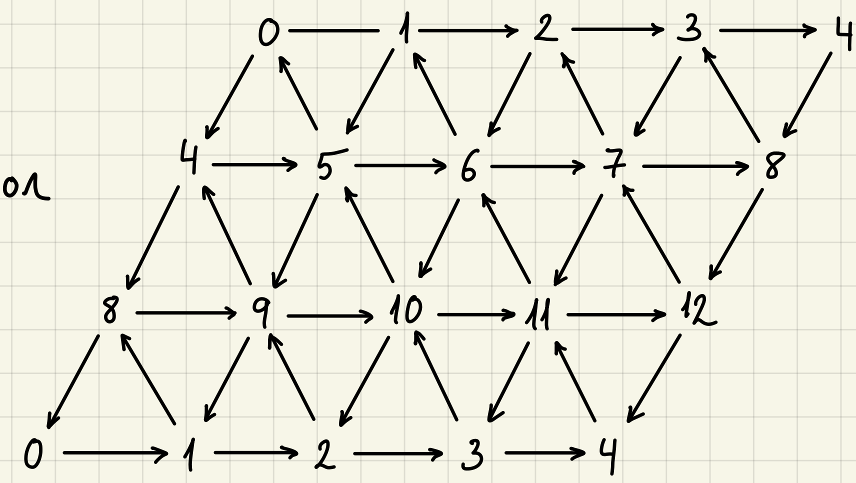
Irreducible reps: $\rho_i : A \longrightarrow \mathbb{C}$, $\rho_i(\alpha) = \varepsilon^i$, $i = 0, \dots, 11$.

$A \curvearrowright \mathbb{C}^3$ $\begin{cases} x \mapsto \varepsilon x \\ y \mapsto \varepsilon^7 y \\ z \mapsto \varepsilon^4 z \end{cases}$ produces the CM $\mathbb{C}[x,y,z]^G$-modules $\boxed{S_\rho}$ (coinvariant algebras)

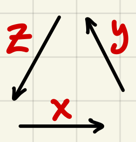
Irr A	α	S_ρ	Irr A	α	S_ρ
ρ_0	1	$1, x^{12}, y^{12}, z^3, \dots$	ρ_6	ε^6	$x^6, y^6, x^2z, \dots$
ρ_1	ε	$x, y^7, x^6y, \dots$	ρ_7	ε^7	$x^7, y, x^3z, \dots$
ρ_2	ε^2	$x^2, y^2, x^7y, \dots$	ρ_8	ε^8	$x^8, y^8, z^2, xy, \dots$
ρ_3	ε^3	$x^3, y^9, yz^2, \dots$	ρ_9	ε^9	$x^9, y^3, x^2y, \dots$
ρ_4	ε^4	$x^4, y^4, z, \dots$	ρ_{10}	ε^{10}	$x^{10}, y^{10}, x^3y, \dots$
ρ_5	ε^5	$x^5, y^{11}, xz, \dots$	ρ_{11}	ε^{11}	$x^{11}, y^5, yz, \dots$

① Maps between ρ_i and between S_{ρ_i} are given by the McKay quiver with relations (Q,R):

Drawn on a torus



"decoration"

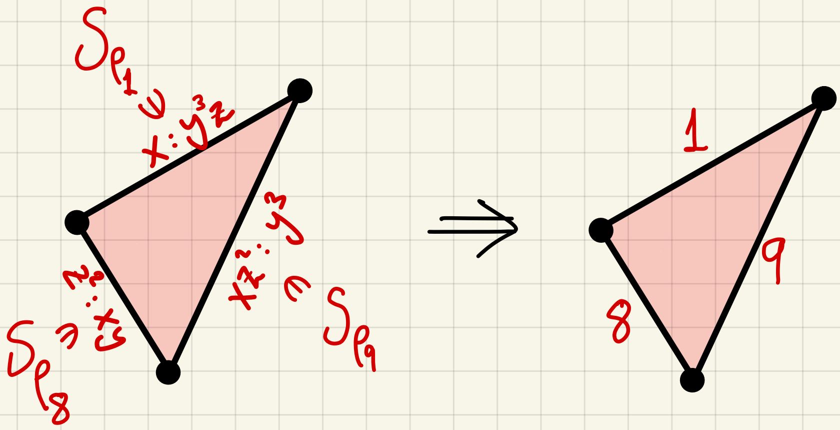


② G-Hilb $\mathbb{C}^3 \simeq \mathcal{M}_\theta(Q,R)$ moduli of θ-stable (generic) repr. of (Q,R).

Reid's Recipe for A-Hilb $\mathbb{C}^3$ ['96]

- **MARKING CURVES**

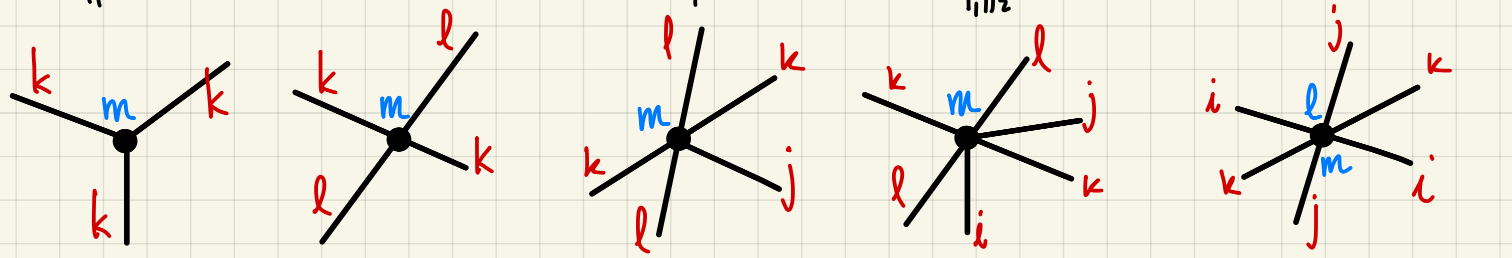
Monomials in ratios of the $\mathbb{P}^1$'s in A-Hilb $\mathbb{C}^3$ belong to the same module S_ρ:



- **MARKING COMPACT DIVISORS**

Every interior vertex in A-Hilb $\mathbb{C}^3$ has valency 3, 4, 5 or 6 and we have the 5 following posibilities:

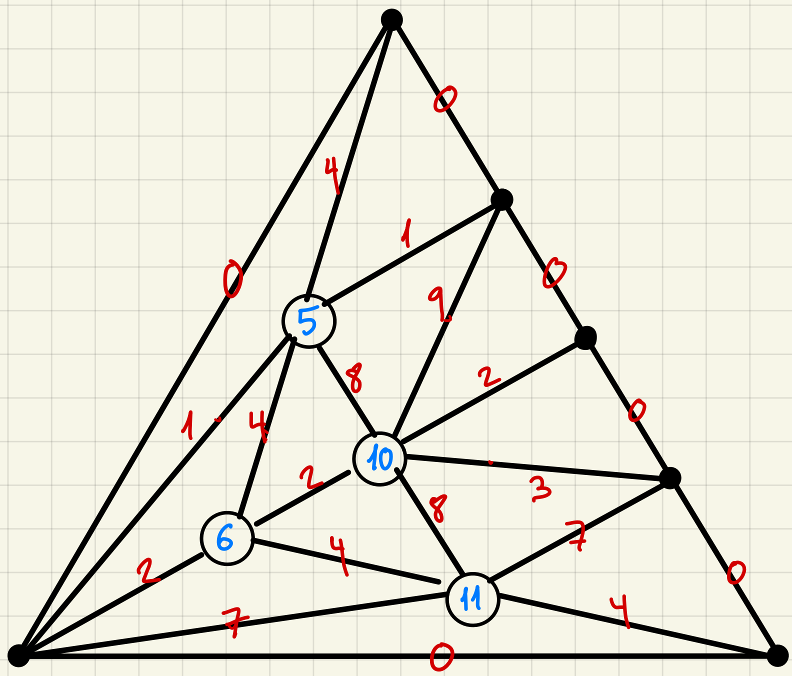
$\mathbb{P}^2$ | $\mathbb{F}_r$ | $Bl_p\mathbb{F}_r$ | $Bl_{p_1,p_2}\mathbb{F}_r$ | dP_6



$\chi_m = \chi_k \otimes \chi_k$ | $\chi_m = \chi_k \otimes \chi_\ell$ | $\chi_m = \chi_k \otimes \chi_\ell$ | $\chi_m = \chi_k \otimes \chi_\ell$ | $\chi_\ell \otimes \chi_m = \chi_i \otimes \chi_j \otimes \chi_k$

For $A = \frac{1}{12}(1,7,4)$

$$\chi_1 \otimes \chi_4 = \chi_5$$
$$\chi_2 \otimes \chi_4 = \chi_6$$
$$\chi_2 \otimes \chi_8 = \chi_{10}$$
$$\chi_4 \otimes \chi_7 = \chi_{11}$$



Moreover,

<u>Fine moduli of G-clusters</u>

Universal sheaf

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "p"'] \arrow[dr] & & \\ G\text{-Hilb}\,\mathbb{C}^3 = Y \arrow[dr, "\pi"'] & & \mathbb{C}^3 \arrow[dl] & \ni Z(y) \ \text{G-orbit} \\ & \mathbb{C}^3/G & & \end{tikzcd}$$

$$Y \ni y$$

[Fibre of y

$\simeq H^0(Z(y), \mathcal{O}_{Z(y)})$

$\underset{G\text{-mod}}{\simeq} \mathbb{C}[G]$ Reg. Rep

$\simeq \bigoplus_{\rho \in \mathrm{Irr}G} \rho_i^{\dim \rho_i}$]

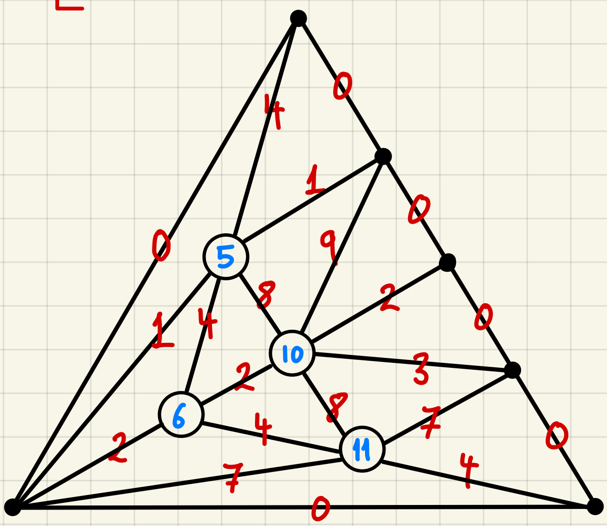
Now $R := p_* \mathcal{O}_{\mathcal{Z}} = \bigoplus_{\rho \in \mathrm{Irr}G} R_k \otimes \rho_k$ has rank $= |G|$ and

$R_k \in \mathrm{Pic}(Y)$ are the <u>tautological bundles</u> $(\mathrm{rank}\, R_k = \dim \rho_k)$

In this setup, for every interior vertex $v \in \Sigma$ (compact div. in Y)

<u>Reid's recipe implies</u>:

[Recipe relation between χ_k at v $\rightsquigarrow$ Relation in $\mathrm{Pic}(Y)$ between corresponding R_k]



<u>Recipe</u>		<u>Relations in Pic(Y)</u>
$\chi_5 = \chi_1 \otimes \chi_4$		$R_5 = R_1 \otimes R_4$
$\chi_6 = \chi_2 \otimes \chi_4$	$\rightsquigarrow$	$R_6 = R_2 \otimes R_4$
$\chi_{10} = \chi_2 \otimes \chi_8$		$R_{10} = R_2 \otimes R_8$
$\chi_{11} = \chi_4 \otimes \chi_7$		$R_{11} = R_4 \otimes R_7$

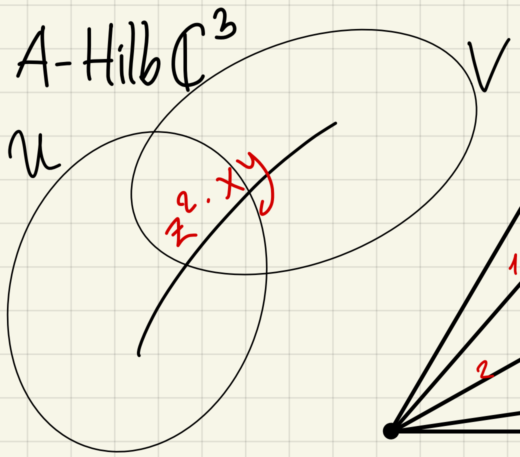
$\circledast$

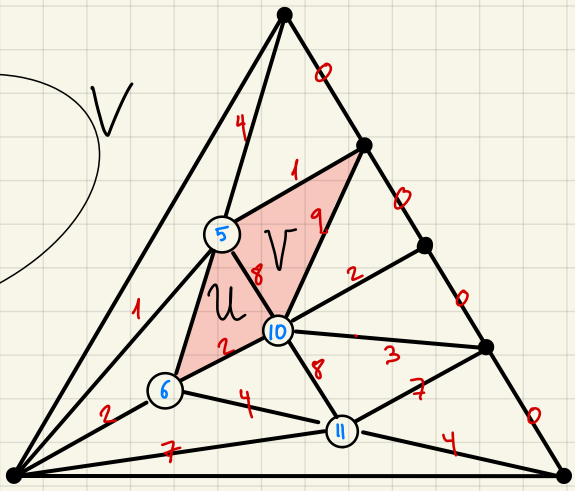
$$\Big(\#\underset{\substack{\| \\ |G|-1}}{\text{non-triv. } R_k}\Big) - \Big(\#\underset{\substack{\| \\ \#\text{ Compact divisors}}}{\text{Relations } \circledast}\Big) = \underset{\substack{\| \\ e(Y)}}{|G|}-1-b_4(Y) = \underset{= \mathrm{rank}\,\mathrm{Pic}(Y)}{b_2(Y)} \Longrightarrow \text{All non-trivial relations in } \mathrm{Pic}(Y)$$

Last ingredients: A-graphs and socles

Γ : A-graph (or A-set) = Basis of the fibres of R

> So, Γ must have $\dim \rho_i$ elements in each S_{ρ_i}





basis of sections of R_{11} at $\mathcal{U}$

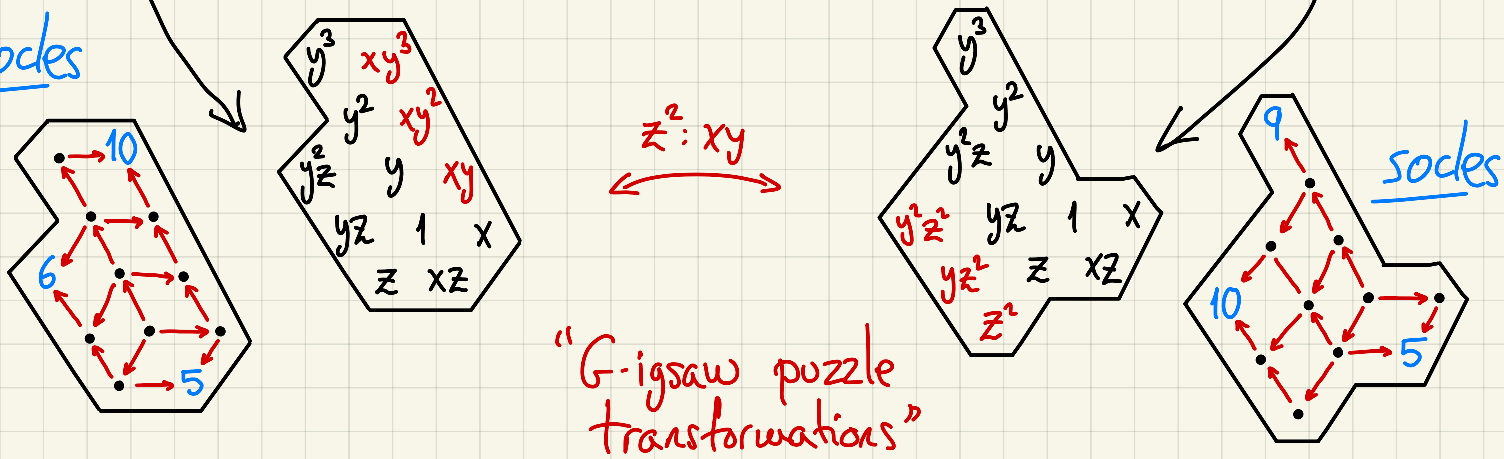
ρ_i	0	1	2	3	4	5	6	7	8	9	10	11
$\mathcal{U}$	1	x	y^2	xy^2	z	xz	y^2z	y	xy	y^3	xy^3	yz
V	1	x	y^2	yz^2	z	xz	y^2z	y	z^2	y^3	y^2z^2	yz

$z^2 : xy$

A-graph for $\mathcal{U}$

A-graph for V

socles



socles

"G-igsaw puzzle transformations"

Obs:

- If ρ_m marks a curve $\Rightarrow$ $\deg R_m|_C = 1$
- If ρ_m marks vertex v $\Rightarrow$ $m \in$ socle of every triangle containing v

Some remarks

- [Craw '05] Everything works for any Abelian subgroup $A \subset SL(3,\mathbb{C})$.

  Moreover, in the Abelian case the recipe leads to:

$$\boxed{\{\text{Irreducible rep. of } G\} \xleftrightarrow{\text{1-to-1}} \text{basis of } H^*(Y,\mathbb{Z})}$$

- Derived and Geometric Reid's recipe

[Cautis-Logvinenko, Craw, Bocklandt, Quintero-Velez, Tapia, Heuberger]

Extension of the correspondence in the language of derived categories for the Abelian and dimer model case.

($\rightsquigarrow$ See Liana Heuberger's talk)

- Both approaches depend heavily on the "well known" explicit description of either $A\text{-Hilb}\,\mathbb{C}^3$ or the dimer model combinatorics (both toric).

  To extend Reid's recipe to the non-Abelian case we need the explicit knowledge of $G\text{-Hilb}\,\mathbb{C}^3$.
  (challenging but fun calculations!)

$\boxed{\text{Example}}$ Dihedral group of order 24

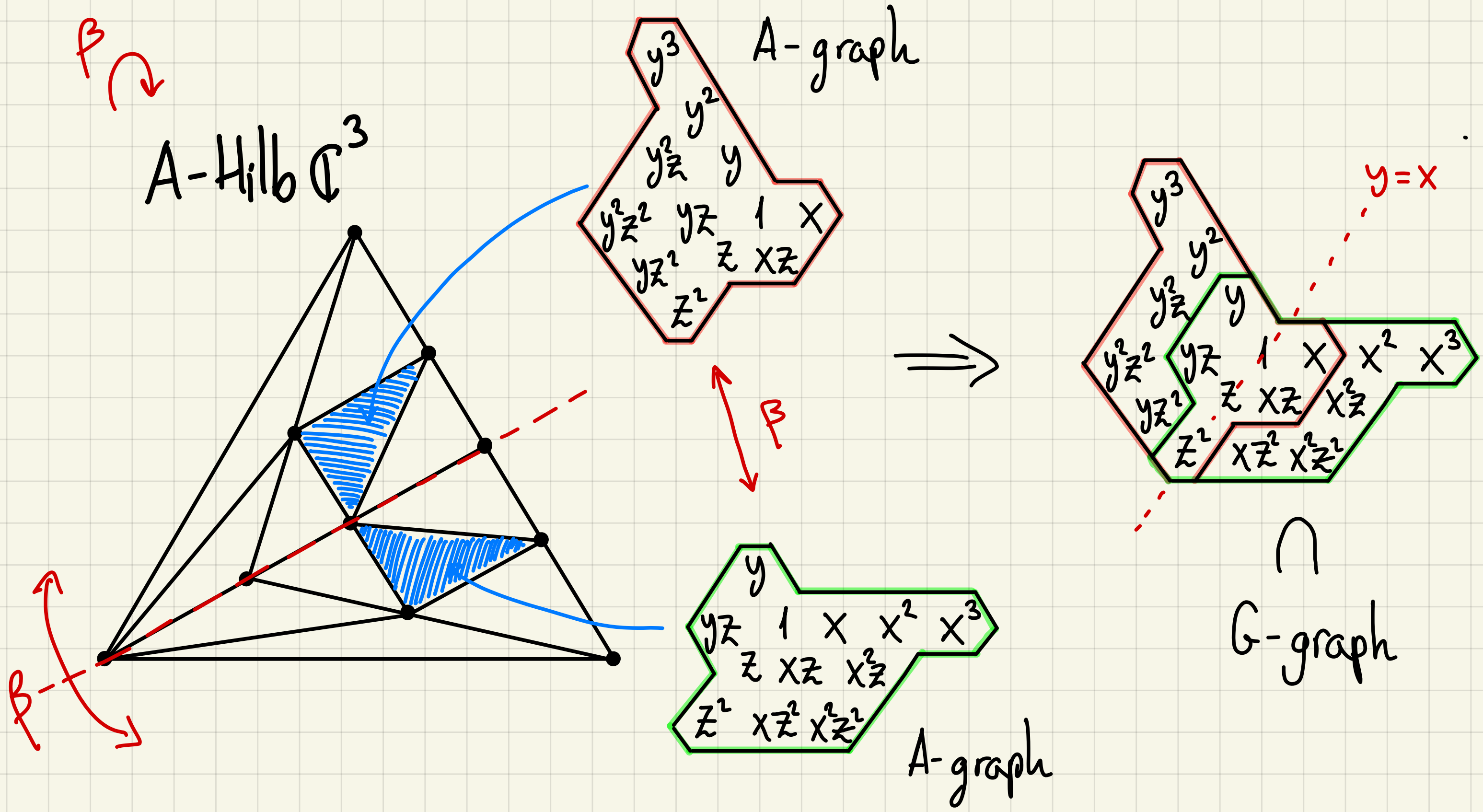
$$G = \Big\langle \underbrace{\tfrac{1}{12}(1,7,4)}_{=A},\ \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{=\beta} \Big\rangle \subset SL(3,\mathbb{C})$$

$$\beta : \begin{cases} x \longmapsto y \\ y \longmapsto -x \\ z \longmapsto z \end{cases}$$

Two facts are helping us in this case:

1. The normal subgroup $A = \langle \alpha \rangle = \frac{1}{12}(1,7,4) \underset{\text{index } 2}{\lhd} G$ produces lots of symmetry:

- G-graphs are β-invariant, i.e., symmetric w.r.t. $y=x$



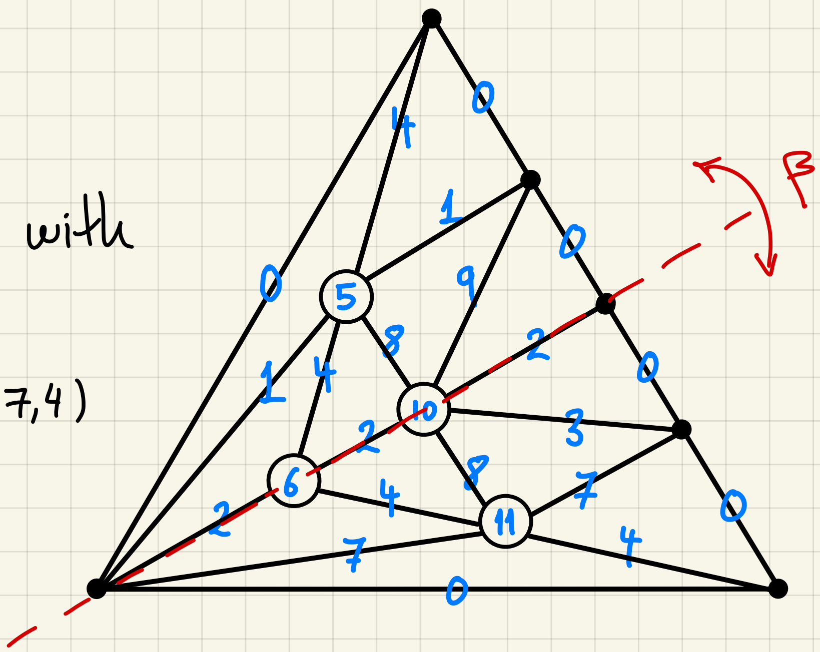
(Careful! $G\text{-Hilb}\,\mathbb{C}^3 \not\cong \underbrace{G/A}_{\langle\bar\beta\rangle \cong \mathbb{Z}/2\mathbb{Z}}\text{-Hilb}\,(A\text{-Hilb}\,\mathbb{C}^3)$ in general)

• $\beta \circlearrowright \mathrm{Irr}\,A$ to form $\mathrm{Irr}\,G$:

		α	
$\circlearrowright$	ρ_0	1	$1,\ x^{12},\ y^{12},\ z^3, \dots$
$\circlearrowright$	ρ_2	ε^2	$x^2,\ y^2, \dots$
$\circlearrowright$	ρ_4	ε^4	$x^4,\ y^4,\ x^2y^2,\ z, \dots$
	$\vdots$	$\vdots$	$\vdots$
$\circlearrowright$	ρ_{10}	ε^{10}	$x^{10},\ y^{10},\ x^2z^2,\ y^2z^2, \dots$
	ρ_1	ε	$x,\ y^7,\ y^3z, \dots$
	ρ_7	ε^7	$x^7,\ y,\ x^3z, \dots$
	ρ_5	ε^5	$x^5,\ y^{11},\ xz, \dots$
	ρ_{11}	ε^{11}	$x^{11},\ y^5,\ yz, \dots$
	ρ_3	ε^3	$x^3,\ y^9,\ xy^2, \dots$
	ρ_9	ε^9	$x^9,\ y^3,\ x^2y, \dots$

	α	β	
ρ_{0+}	1	1	$1,\ x^{12}+y^{12}, \dots$
ρ_{0-}	1	-1	$x^{12}-y^{12}, \dots$
ρ_{2+}	ε^2	1	$\boxed{x^2+y^2=(+)}$
ρ_{2-}	ε^2	-1	$\boxed{x^2-y^2=(-)}$
ρ_{4+}	ε^4	1	$x^4+y^4,\ x^2y^2,\ (+)^2,\ (-)^2, \dots$
ρ_{4-}	ε^4	-1	$x^4-y^4=(+)(-), \dots$
$\vdots$			
ρ_{10+}	ε^{10}	1	$(+)^3(-)^2,\ z^2(+), \dots$
ρ_{10-}	ε^{10}	-1	$(+)^4(-),\ z^2(-), \dots$
V_1	$\begin{pmatrix}\varepsilon & \\ & \varepsilon^7\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}x\\ y\end{pmatrix}, \begin{pmatrix}y^7\\ x^7\end{pmatrix}, \begin{pmatrix}yz(+)\\ xz(+)\end{pmatrix}, \dots$
V_5	$\begin{pmatrix}\varepsilon^5 & \\ & \varepsilon^{11}\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}x^5\\ y^5\end{pmatrix}, \begin{pmatrix}y^{11}\\ x^{11}\end{pmatrix}, \begin{pmatrix}xz\\ yz\end{pmatrix}, \dots$
V_9	$\begin{pmatrix}\varepsilon^9 & \\ & \varepsilon^3\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}x^9\\ y^9\end{pmatrix}, \begin{pmatrix}xy^2\\ x^2y\end{pmatrix}, \begin{pmatrix}y(+)\\ x(+)\end{pmatrix}, \dots$

<u>Obs</u> : The action agrees with Reid's Recipe on $A\text{-}\mathrm{Hilb}\,\mathbb{C}^3$ for $A = \frac{1}{12}(1,7,4)$



② G is a group of type (B) in the Yau-Yu classification of finite subgroups of $SL(3,\mathbb{C})$. They are constructed as follows:

Take $\overline{G} \subset GL(2,\mathbb{C})$ and define:

$$G = \left\langle \begin{pmatrix} g & 0 \\ 0 & \det(g)^{-1} \end{pmatrix} \;\middle|\; g \in \overline{G} \right\rangle$$

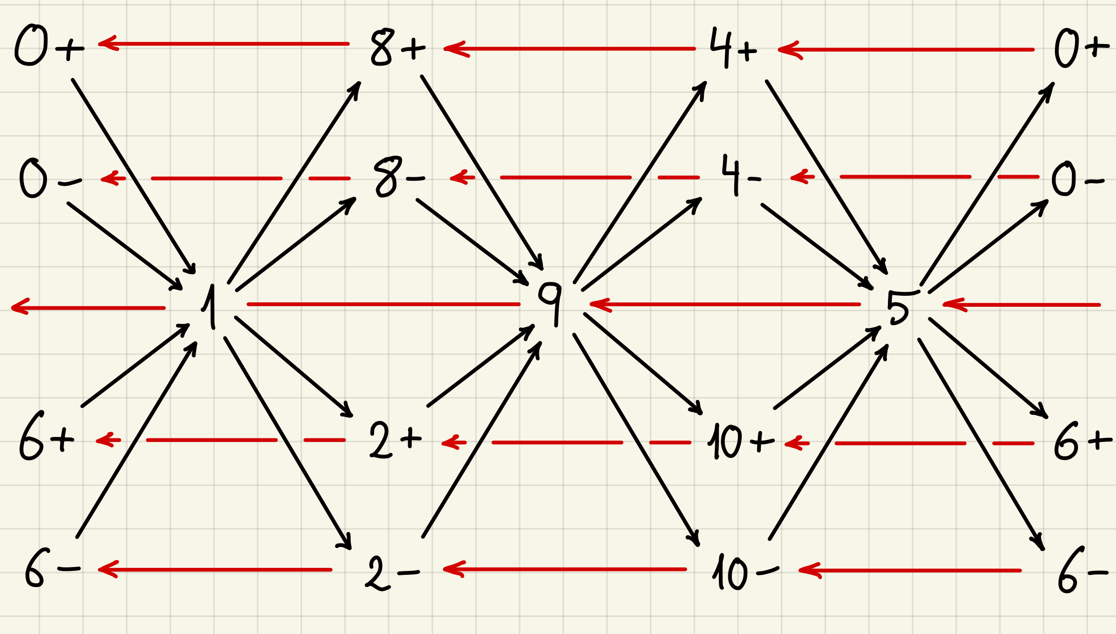
In our case,

$$G = \left\langle \begin{pmatrix} \boxed{\begin{matrix} \varepsilon & 0 \\ 0 & \varepsilon^7 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \varepsilon^4 \end{pmatrix}, \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \;\middle|\; \varepsilon = e^{\frac{2\pi i}{12}} \right\rangle$$

Riemenschneider notation

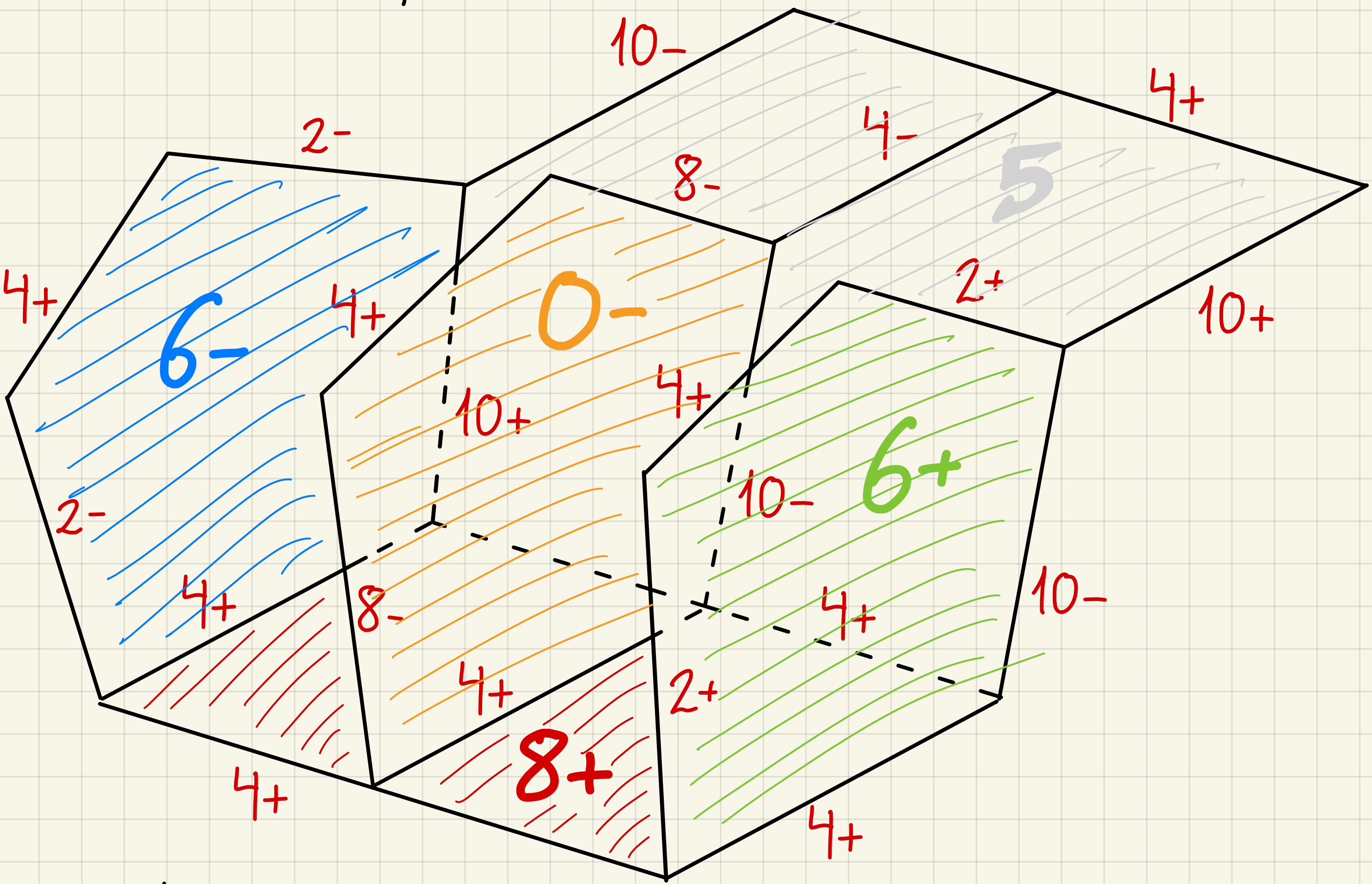
$$\begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^7 & 0 \\ 0 & 0 & \varepsilon^4 \end{pmatrix} = \frac{1}{12}(\boxed{1,7},4) \Rightarrow \overline{G} = \left\langle \frac{1}{12}(1,7), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \simeq \mathbb{D}_{5,2}$$

which implies that the McKay quiver of G can be obtained from the McKay quiver of $\overline{G}$ by adding certain $\xrightarrow{z}$ arrows:



(McKay quiver of $\overline{G} = \left\langle \frac{1}{12}(1,7), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \simeq \mathbb{D}_{5,2}$ in black)

Then, we can draw the exceptional divisor E in $G\text{-Hilb}\,\mathbb{C}^3$ as follows:



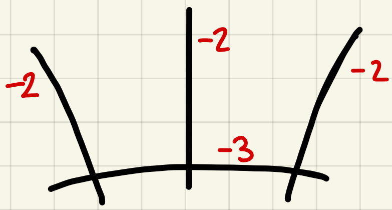
Observations:

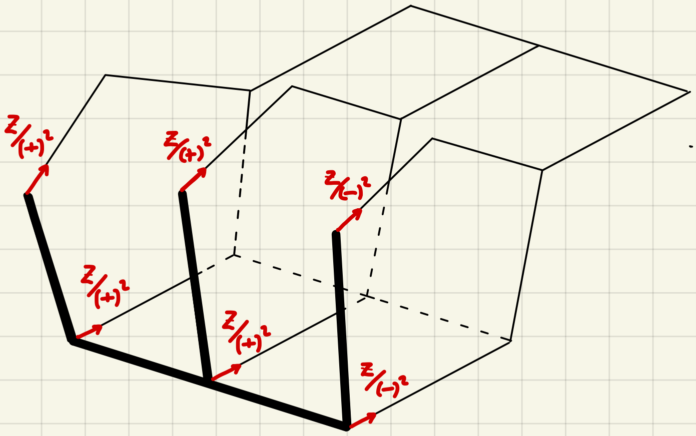
① $Y = G\text{-Hilb}\,\mathbb{C}^3$ it is covered by $\#\operatorname{Irr} G = e(Y) = 15$ open affine subsets U_i where either $U_i \simeq \mathbb{C}^3$ or $U_i \simeq \mathbb{C}^4/f$.

② Every irreducible representation (except V_1 and V_9) appear as either:

- marking a line in E
- marking a divisor in E

(3) Recall that $\overline{G} = \mathbb{D}_{5,2} \subset GL(2,\mathbb{C})$ is a subgroup of G and then:

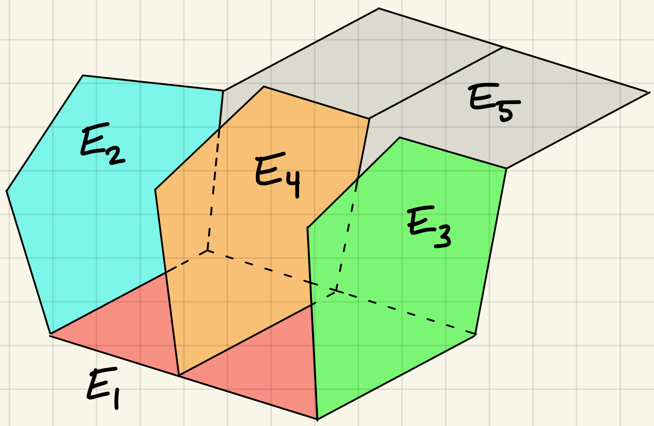




$$\mathbb{D}_{5,2}\text{-Hilb}\,\mathbb{C}^2 \overset{z=0}{\subset} G\text{-Hilb}\,\mathbb{C}^3$$

(4) G-Hilb $\mathbb{C}^3$ has 5 irreducible compact divisors

$$\pi^{-1}(0) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$



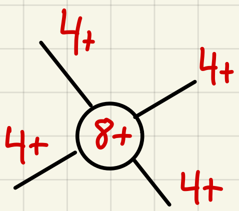
Actually, G has:

5 conjugacy classes of age 2 $\overset{\text{1-to-1}}{\underset{\text{[Ito-Reid]}}{\longleftrightarrow}}$ Irreducible exceptional compact divisors

5 Relations ⊛ in Pic(Y)

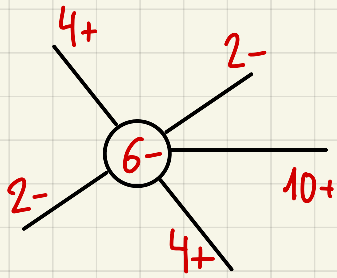
For every compact irreducible divisor $E_i \subset \pi^{-1}(0)$ there exists a relation in Pic(Y):

For E_1:



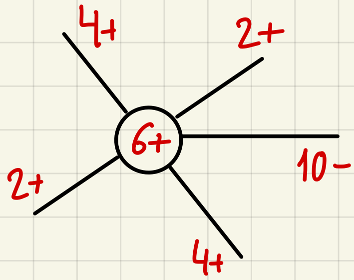
$\Longrightarrow$ $\boxed{R_{8+} = R_{4+} \otimes R_{4+}}$

For E_2:



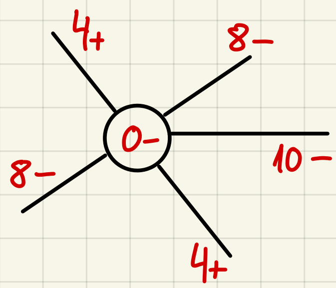
$\Longrightarrow$ $\boxed{R_{6-} = R_{4+} \otimes R_{2-}}$

For E_3:



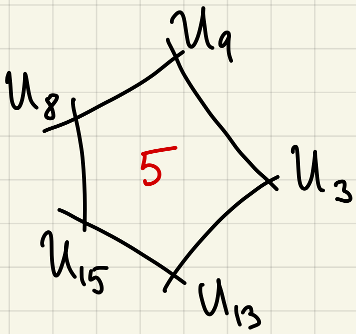
$\Longrightarrow$ $\boxed{R_{6+} = R_{4+} \otimes R_{2+}}$

For E_4:



$\Longrightarrow$ $\boxed{R_{0-} = R_{4+} \otimes R_{8-}}$

For E_5:



$\longleftarrow$ Affine covering of E_5 (dual picture)

Elements in V_5 lie in the socle of every $U_i \subset E_5$

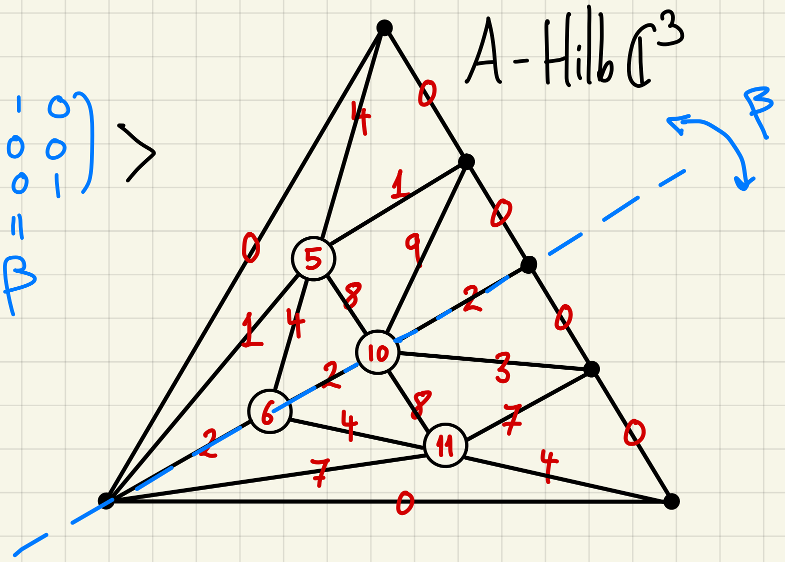
$\Longrightarrow$ V_5 marks E_5 and $\boxed{\det R_5 = \det R_9 \otimes R_{4+}}$

Some comments :

- Again $\underset{\text{(non-triv)}}{\# \operatorname{Irr} G} - \#\text{Relations} = 14 - 5 = 9 = \operatorname{rank} \operatorname{Pic} Y$

$\Rightarrow$ The relations $\circledast$ generate all relations in $\operatorname{Pic} Y$

- $G = \left\langle \underset{\substack{\parallel \\ A}}{\frac{1}{12}(1,7,4)}, \underset{\substack{\parallel \\ \beta}}{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \right\rangle$



β

__Relations in $\operatorname{Pic}(A\text{-Hilb}\,\mathbb{C}^3)$__ | __Relations in $\operatorname{Pic}(G\text{-Hilb}\,\mathbb{C}^3)$__

$\beta \circlearrowleft R_6 = R_2 \otimes R_4$ $\longrightarrow$ $R_{6-} = R_{4+} \otimes R_{2-}$, $R_{6+} = R_{4+} \otimes R_{2+}$

β: $R_5 = R_1 \otimes R_4$, $R_{11} = R_4 \otimes R_7$ "$\longrightarrow$" $\det R_5 = \det R_9 \otimes R_{4+}$

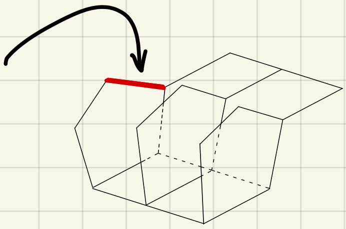
$\beta \circlearrowleft R_{10} = R_2 \otimes R_8$

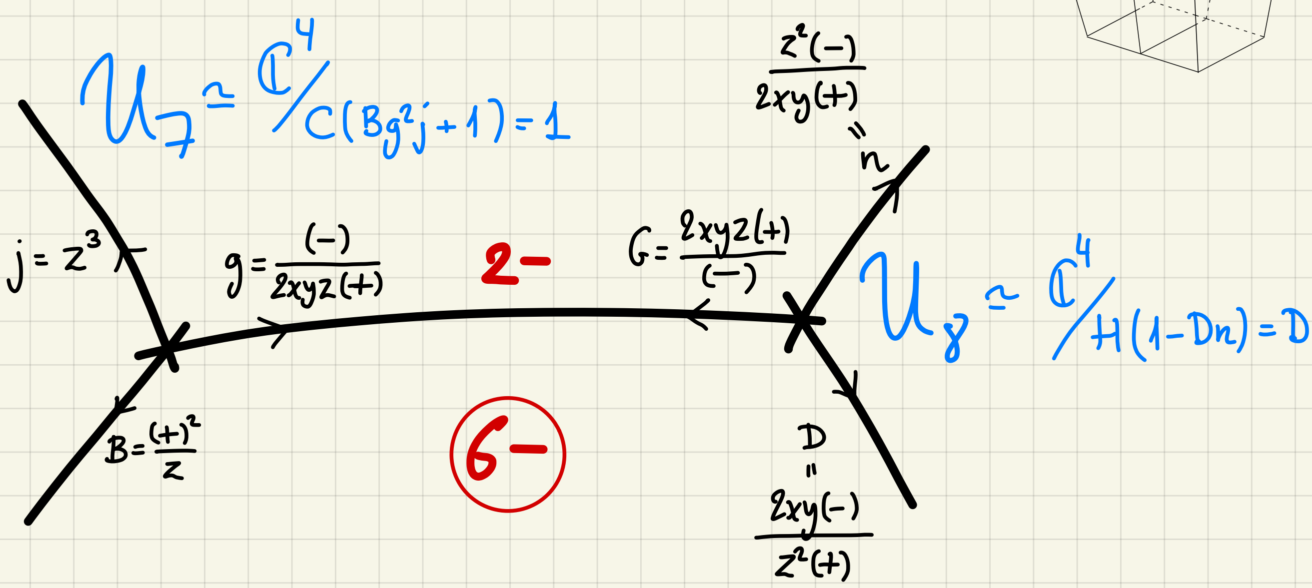
$R_{0-} = R_{4+} \otimes R_{8-}$

$R_{8+} = R_{4+} \otimes R_{4+}$

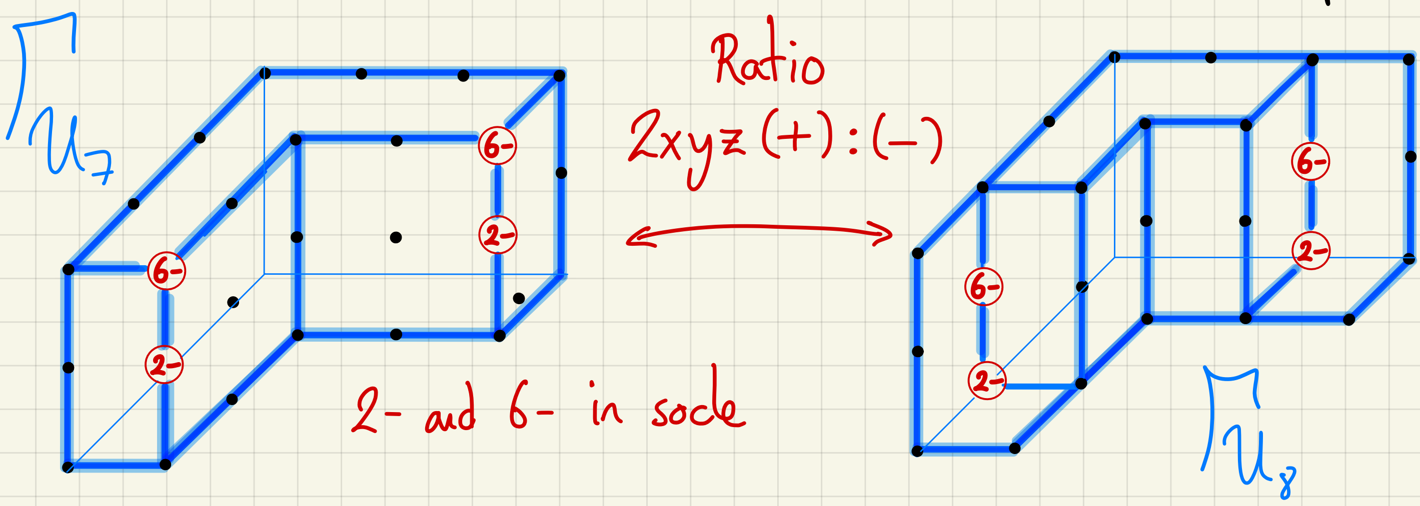
Although $\overbrace{G/A\text{-}\operatorname{Hilb}(A\text{-Hilb}\,\mathbb{C}^3)}^{\text{doesn't have Reid's Recipe}} \neq G\text{-Hilb}\,\mathbb{C}^3$ but we might expect to "induce" relations where they agree.

Looking closer to one of the curves in E





Also have G-igsaw puzzle transf. between the G-graphs :



Basis of the fibres of $\mathcal{R} = \bigoplus_{\mathrm{Irr}\, G} R_k \otimes \rho_k$ over $\mathcal{U}_7$:

0+	1	4+	z	8+	z^2
0−	$2xyz$	4−	$2xyz^2$	8−	$2xy$
V_1	$\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} yz(+) \\ -xz(+) \end{pmatrix}$	V_5	$\begin{pmatrix} xz \\ yz \end{pmatrix}, \begin{pmatrix} yz^2(+) \\ -xz^2(+) \end{pmatrix}$	V_9	$\begin{pmatrix} xz^2 \\ yz^2 \end{pmatrix}, \begin{pmatrix} y(+) \\ -x(+) \end{pmatrix}$
2+	$(+)$	6+	$z(+)$	10+	$z^2(+)$
2−	$2xyz(+)$	6−	$2xyz^2(+)$	10−	$2xy(+)$

$(-)$ in $\mathcal{U}_8$ $\qquad$ $z(-)$ in $\mathcal{U}_8$

$\boxed{\text{Example}}$ Trihedral group of order 39

$$G = \Big\langle \underbrace{\tfrac{1}{13}(1,3,9)}_{=\,A} \,,\; \underbrace{\begin{pmatrix} 0&1&0\\0&0&1\\1&0&0\end{pmatrix}}_{=\,T} \Big\rangle \subset SL(3,\mathbb{C})$$

$T : x \to y \to z \to x \qquad \mathbb{Z}/3\mathbb{Z}$

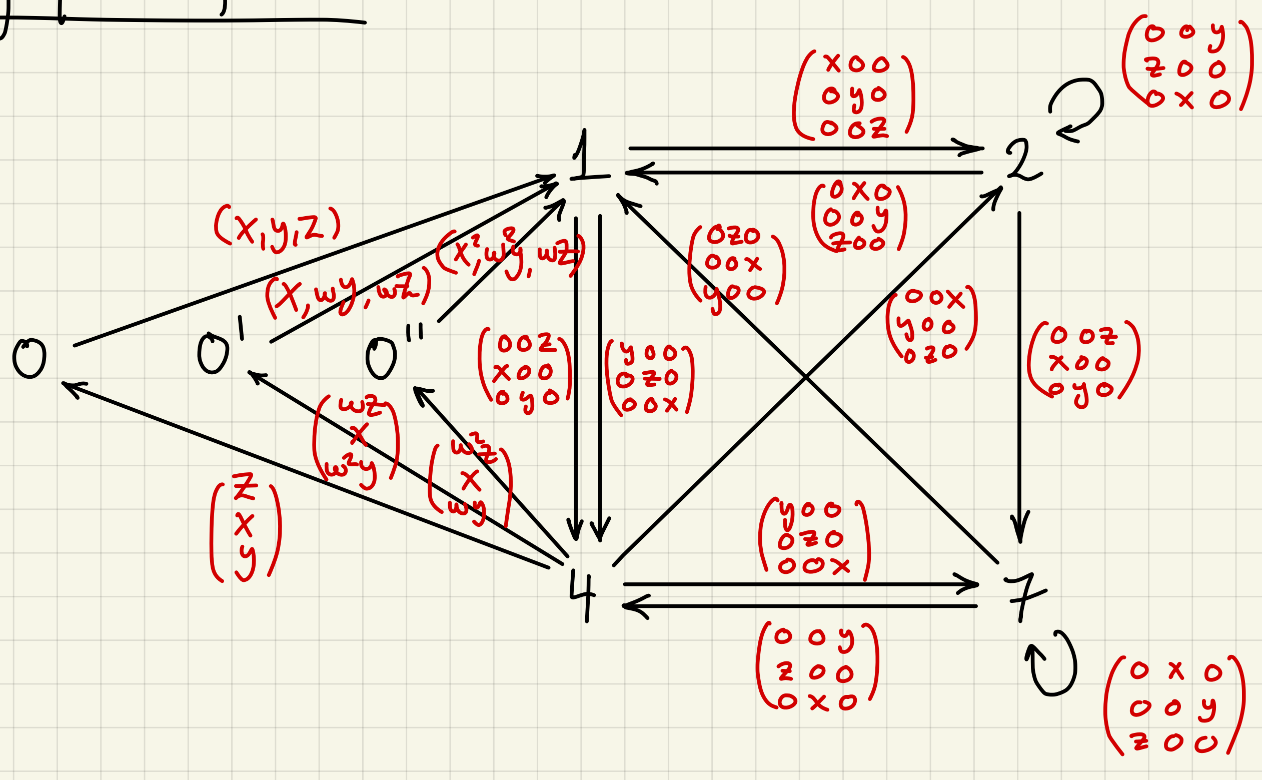
$T \curvearrowright$

Irr A : $\rho_0 \circlearrowleft$, $\quad \rho_1 \to \rho_3 \to \rho_9 \to \rho_1$, $\quad \rho_2 \to \rho_6 \to \rho_5 \to \rho_2$, $\quad \rho_4 \to \rho_{12} \to \rho_{10} \to \rho_4$, $\quad \rho_7 \to \rho_8 \to \rho_{11} \to \rho_7$

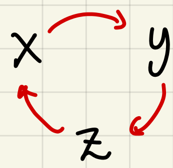
$\rho_0 \mapsto V_0, V_{0'}, V_{0''}$; $\quad \rho_9 \mapsto V_1$; $\quad \rho_5 \mapsto V_2$; $\quad \rho_{10} \mapsto V_4$; $\quad \rho_{11} \mapsto V_7$

Irr G : $V_0 \; V_{0'} \; V_{0''}$ (1 dim $\times 3$), $\quad V_1$ (3 dim), $\quad V_2$ (3 dim), $\quad V_4$ (3 dim), $\quad V_7$ (3 dim)

McKay quiver of G + "decoration"



Boats for $\langle \overset{A}{\overset{\shortparallel}{\tfrac{1}{13}(1,3,9)}}, T \rangle$



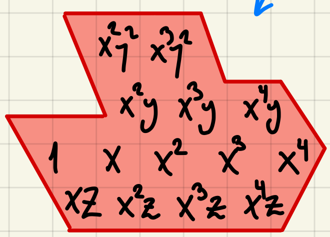
Now G-graphs are $\mathbb{Z}/3\mathbb{Z}$-symmetric by the action of T

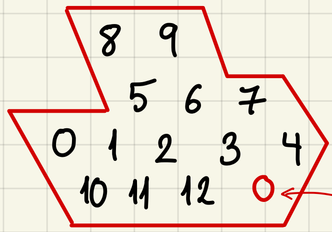
They are nicely drawn using "<u>boats</u>".

For example,

$\overset{1}{\overset{\shortparallel}{\dim \rho_i}}$ elements in each $\rho_i \in \mathrm{Irr}\,A$

The boat B_3 :

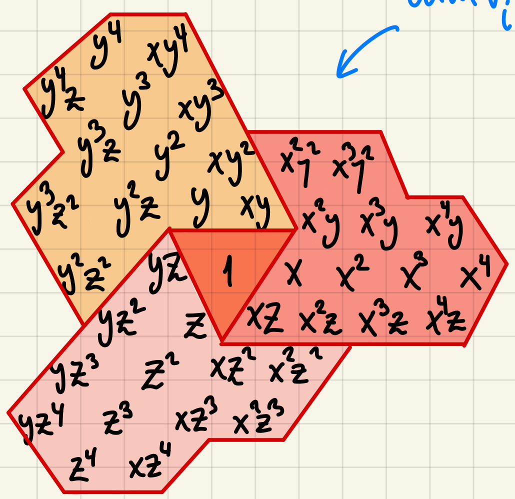




$\rho_0 \to V_0 \ni 1$, $\boxed{V_0'}$, $\boxed{V_0''}$ need elts in this reps.

Corresponds to the G-graph :

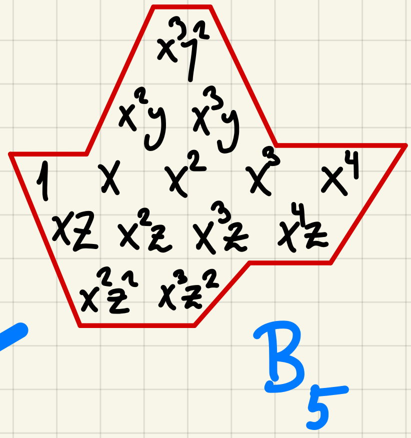
$\dim V_i$ in each $V_i \in \mathrm{Irr}\,G$

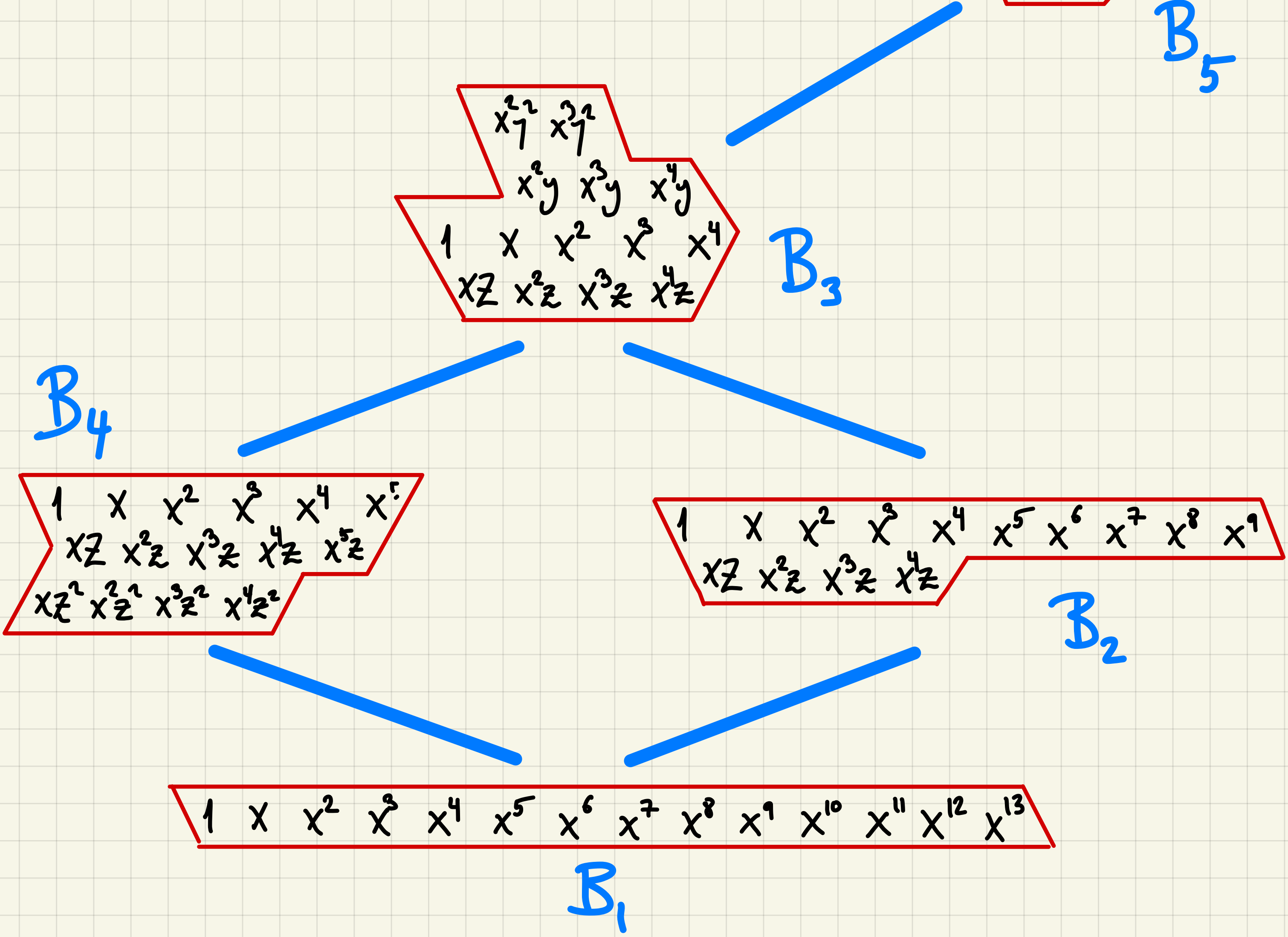


The 3 symmetric boats overlap only at 1

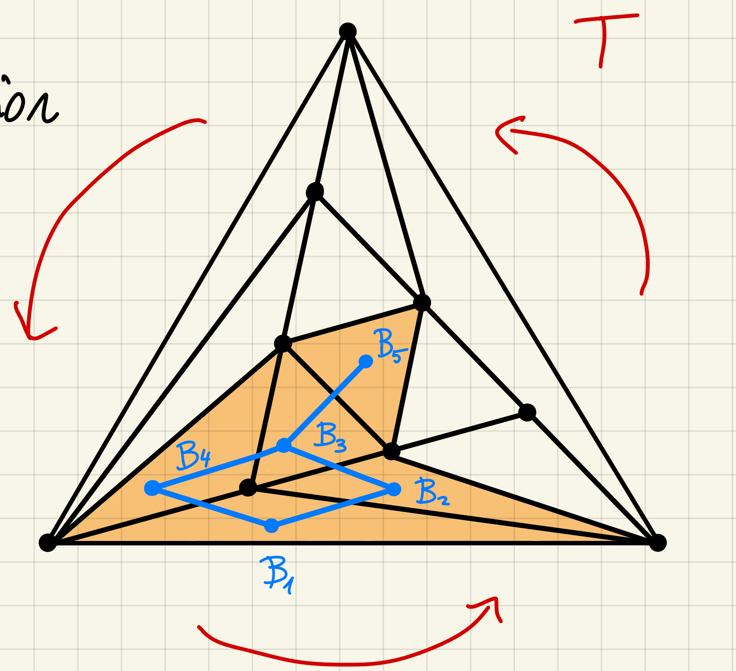
→ For every such boat B there exists an open subset $\mathcal{U}_B \subset G\text{-Hilb}\,\mathbb{C}^3$.

For $G = \langle \frac{1}{13}(1,3,9), T \rangle$ there are 5 boats related by G-igsaw puzzle transformations as follows:





It resembles the construction of T-Hilb(A-Hilb $\mathbb{C}^3$) which in this case is expected to be <u>not</u> isomorphic to G-Hilb $\mathbb{C}^3$

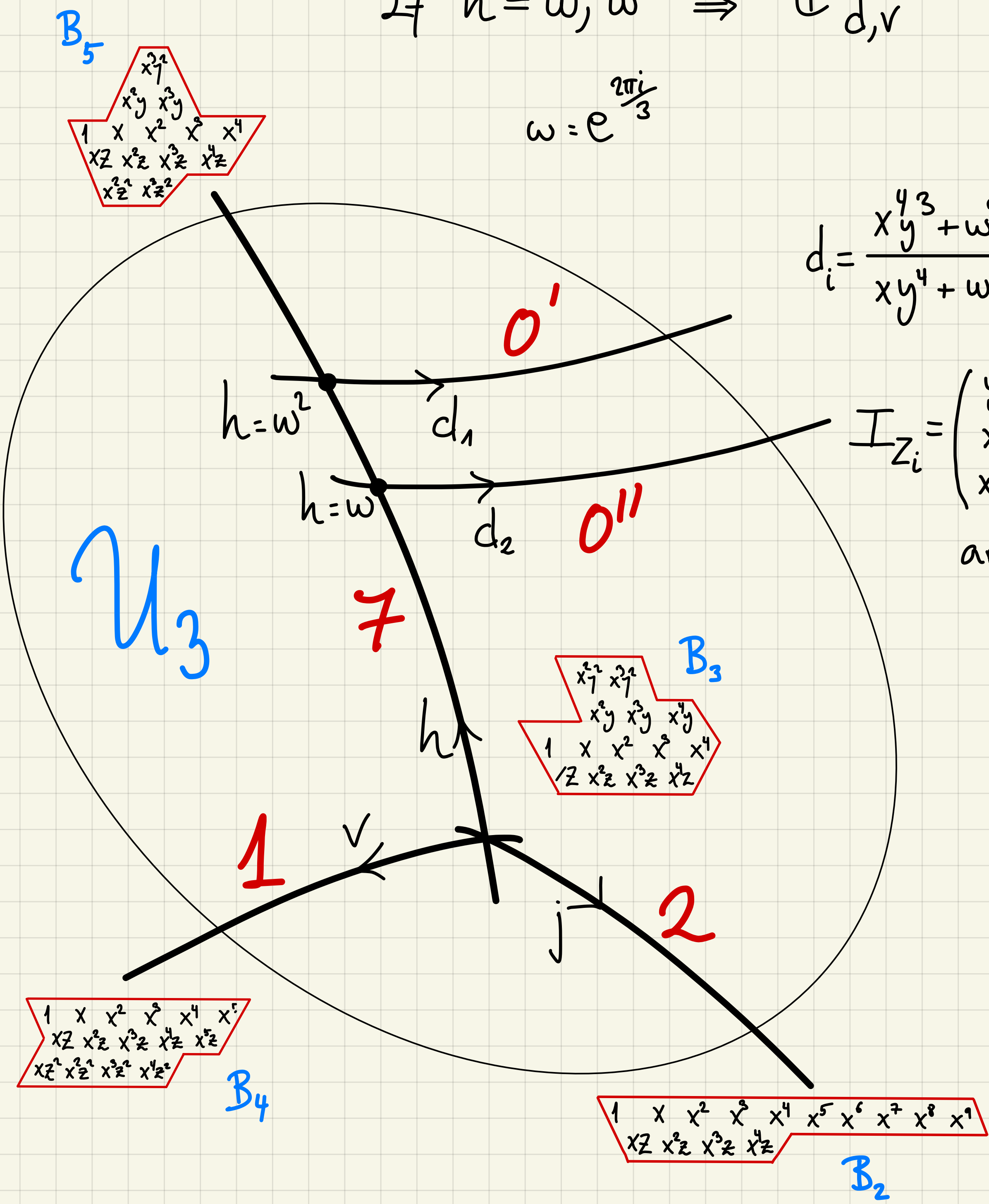


Open set $\mathcal{U}_3$ and markings

$$\boxed{\mathcal{U}_3 \cong \left(d(1+h+h^2) = -hj - v^2 \right)} \subset \mathbb{C}^4_{d,h,v,j}$$

- If $h \neq \omega, \omega^2 \Rightarrow h, j, v$ local coords
- If $h = \omega, \omega^2 \Rightarrow \mathbb{C}^2_{d,v}$

$$\omega = e^{\frac{2\pi i}{3}}$$



O^i

$$d_i = \frac{x^4y^3 + \omega^{2i}y^4z^3 + \omega^i x^3z^4}{xy^4 + \omega^i x^4 z + \omega^{2i} y z^4}$$

$$I_{Z_i} = \begin{pmatrix} yz^3 - \omega^i x^4 \\ x^3z - \omega^i y^4 \\ xy^3 - \omega^i z^4 \end{pmatrix}, xyz$$

are G-clusters

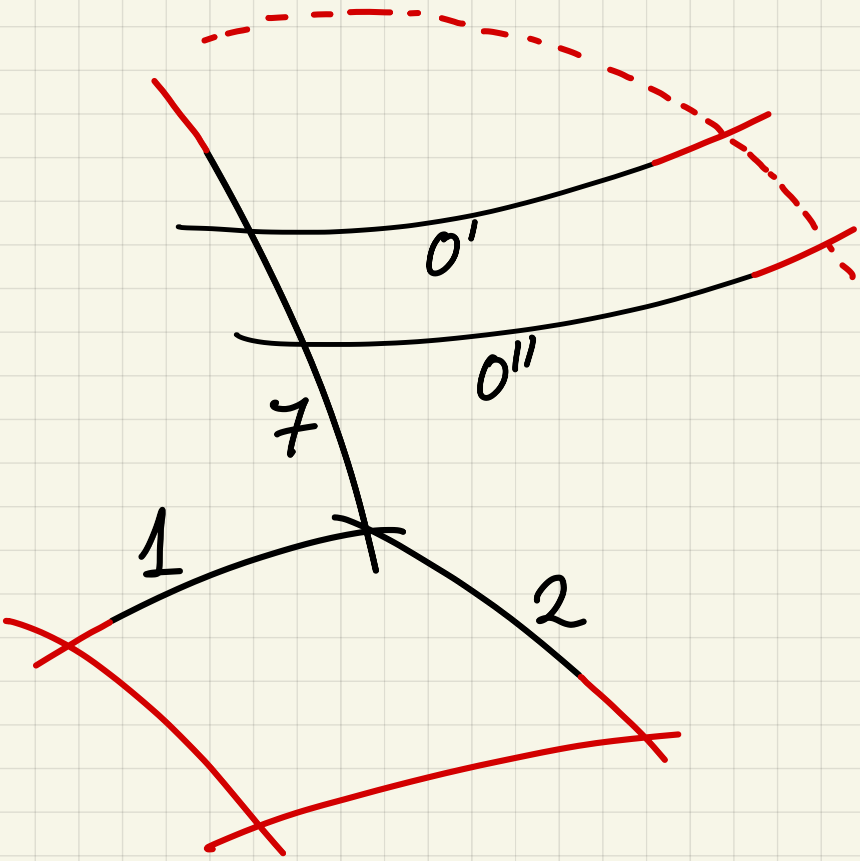
Expectations :

G has $\underset{\underbrace{e(Y)}}{\overset{\|}{7}}$ conjugacy classes : 5 of age 1 & $\underbrace{\text{2 of age 2}}$

$\Downarrow$ 7 open subsets

$\Downarrow$ 2 compact divisors in $\pi^{-1}(0)$

Guessing how $G\text{-Hilb}\,\mathbb{C}^3$ may look like for $G = \left\langle \frac{1}{13}(1,3,9), T \right\rangle$ it would be something similar to:



Parts in red are conjectural

and relations in Pic : $R_{0'} \otimes R_{0''} = \det R_4$

$\det R_1 = \det R_2$