Combinatorial mutations of polygons via dimer models

(joint work with A. Higashitani, arXiv:1903.01636)

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Mirror symmetry for Fano manifolds (cf. Coates - Corti - Galkin - Golyshev - Kasprzyk)

- X: n-dimensional Fano manifold
- X is expected to correspond to a certain Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

The regularised quantum period of X $=$ The classical period π_f of f

mirror partner of X

- There are so many mirror partners of X.

$\rightsquigarrow$ want to understand the relationship between mirror partners.

Mutations of f [Akhtar - Coates - Galkin - Kasprzyk]

$$f \overset{\text{mut.}}{\sim} g \implies \pi_f = \pi_g$$

Thus, mutating a mirror partner f of X, we can obtain a lot of mirror partners of X.

$$
\begin{tikzcd}
\text{Laurent polynomials :} & f \arrow[r, leftrightarrow, "\text{mutation}"] \arrow[d, squiggly] & g \arrow[d, squiggly] \\
\text{Newton polytopes :} & P := \mathrm{Newt}(f) \arrow[r, leftrightarrow, "\text{combinatorial mutation}"] \arrow[d, squiggly] & \mathrm{Newt}(g) =: Q \arrow[d, squiggly] \\
\text{toric varieties :} & X_P \arrow[r, leftrightarrow] & X_Q
\end{tikzcd}
$$

I will focus here (especially dim 2).

"deformation equiv." in the following sense

Thm [Ilten]

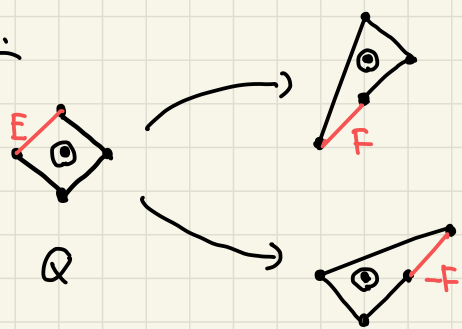
There exists a flat family $\mathcal{X} \to \mathbb{P}^1$ s.t. $\mathcal{X}_0 \cong X_P$ and $\mathcal{X}_\infty \cong X_Q$.

Combinatorial mutations (dim 2)

- Q : lattice polygon, $0 \in Q$ • E : edge of Q
- $w \in \mathbb{Z}^2$: primitive inner normal vector for an edge E.
- $h_{max} := \max\{\langle w, v\rangle \mid v \in Q\}$, $h_{min} = \min\{\langle w, v\rangle \mid v \in Q\}$
- $v_E \in \mathbb{Z}^2$: primitive lattice vector with $\langle w, v_E\rangle = 0$, $F := \mathrm{Conv}\{0, v_E\}$

Roughly,
the combinatorial mutation of Q w.r.t. w and F, denoted by $\mathrm{mut}_w(Q, F)$,
is given by removing $-h_{min}$ primitive segments from E and
adding $h_{max} F$ to the "opposite side".

e.g.



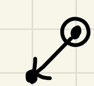
$\mathrm{mut}_w(Q, F)$

$\updownarrow$ unimodular equiv.

$\mathrm{mut}_w(Q, -F)$

$w = (1, -1)$, $v_E = (-1, -1)$

$h_{max} = 1$, $h_{min} = -1$

$F = \mathrm{Conv}\{(0,0), (-1,-1)\} =$ 

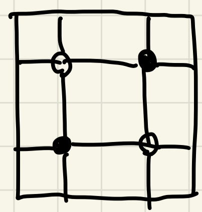
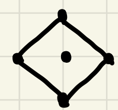
Lattice polygons arising from dimer models

(cf. Hanany et. al., Gulotta, Ishii-Ueda, etc.)

- Any lattice polygon can be obtained from a "dimer model".

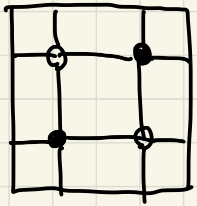
  - A dimer model Γ is a finite bipartite graph described on the real 2-torus T.

e.g.

Γ_1 :  $\rightsquigarrow$  : $\Delta(\Gamma_1)$

  - For any lattice polygon Q, there is a dimer model Γ s.t. $Q = \Delta(\Gamma)$.

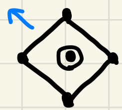
Remark Such a dimer model is not unique.



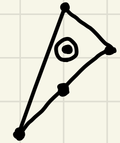
Γ_1

$\rightsquigarrow$

$-w = (-1,1)$



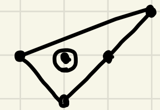
$Q = \Delta(\Gamma_1)$



$\mathrm{mut}_w(Q,F)$

$\exists\, \Gamma_2$ s.t.

$\Delta(\Gamma_2) = \mathrm{mut}_w(Q,F)$



$\mathrm{mut}_w(Q,-F)$

$\exists\, \Gamma_2'$ s.t.

$\Delta(\Gamma_2') = \mathrm{mut}_w(Q,-F)$

where $w = (1,-1)$, $F = \mathrm{Conv}\{(0,0),(-1,-1)\}$

Expectation There is an operation satisfying

$$\Gamma_1 \to \Gamma_2, \qquad \Gamma_1 \to \Gamma_2'$$

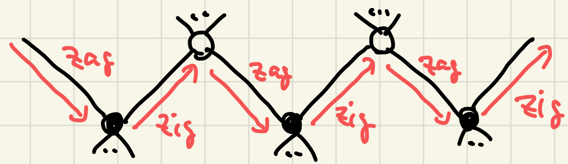
$\boxed{\text{I will introduce "deformations of a dimer model" that realize this expectation}}$

Note There is the operation called "the mutation of dimer models" but this operation does not change the associated polygon.

Construction of lattice polygons from dimer models

Can lift on the universal cover $\mathbb{R}^2 \to T$

Def A path z on a dimer model is called a zigzag path if it makes a maximum turn to the right on a white node and — " — to the left on a black node.

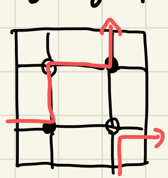


the slope of z

• z is a 1-cycle on T $\rightsquigarrow$ determines the element $[z] \in H_1(T) \cong \mathbb{Z}^2$

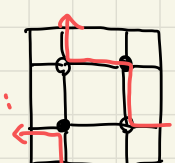
e.g. Zigzag paths on Γ_1

z_1:



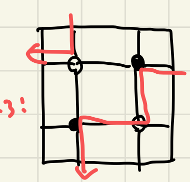
$[z_1] = (1,1)$

z_2:



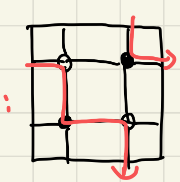
$[z_2] = (-1,1)$

z_3:



$[z_3] = (-1,-1)$

z_4:



$[z_4] = (1,-1)$

<u>Remark</u>

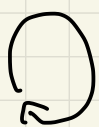
This does not change the slope.

- In the rest, we assume that $\nexists$ 2-valent nodes in a dimer model.

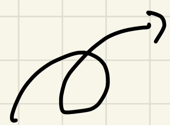
$\Longrightarrow$ length of z : $\ell(z) = \#(\text{zigs}) + \#(\text{zags})$

- Also, we assume that a dimer model is consistent.

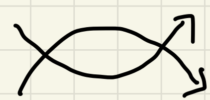
<u>i.e.</u> A dimer model does not have zigzag paths with



homologically trivial



self-intersection on the universal cover



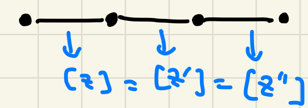
intersect each other on the universal cover in the same direction more than once.

For a consistent dimer model Γ

- Consider $[z] =: (a, b) \in \mathbb{Z}^2 \rightsquigarrow (a,b)/\sqrt{a^2+b^2} \in S^1$

  $\rightsquigarrow$ Define the cyclic order of slopes of zigzag paths along S^1.

- define the zigzag polygon $\Delta(\Gamma)$ satisfying the followings:
  - {Outer normal vectors of side segments} = {Slopes of zigzag paths}



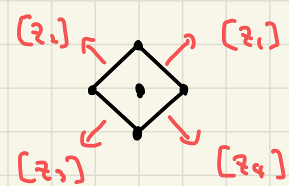
  - The cyclic order of the slopes along $\Delta(\Gamma)$ = the above cyclic order

- $\Delta(\Gamma)$ is determined up to translations.

e.g. The zigzag polygon $\Delta(\Gamma_1)$

$[z_1] = (1,1)$, $[z_2] = (-1,1)$

$[z_3] = (-1,-1)$, $[z_4] = (1,-1)$



Deformations of dimer models

Def A zigzag path z is called type I if

$\tilde{z}$ intersects with any other zigzag paths on $\mathbb{R}^2$ at most once.

on the univ. cover

Def (Deformation data)

(1) $\mathcal{Z}_v = \{z_1, \cdots, z_n\}$: type I zigzag paths with $[z_1] = \cdots = [z_n] =: v$

Lem

$\Longrightarrow \ell(z_1) = \cdots = \ell(z_n)$

(2) Choose $r > 0$ zigzag paths in $\mathcal{Z}_v$ with $p := \frac{\ell(z_1)}{2} - r > 0$

Using these, we introduce the operations which I call

the deformation of Γ at zig and the deformation of Γ at zag

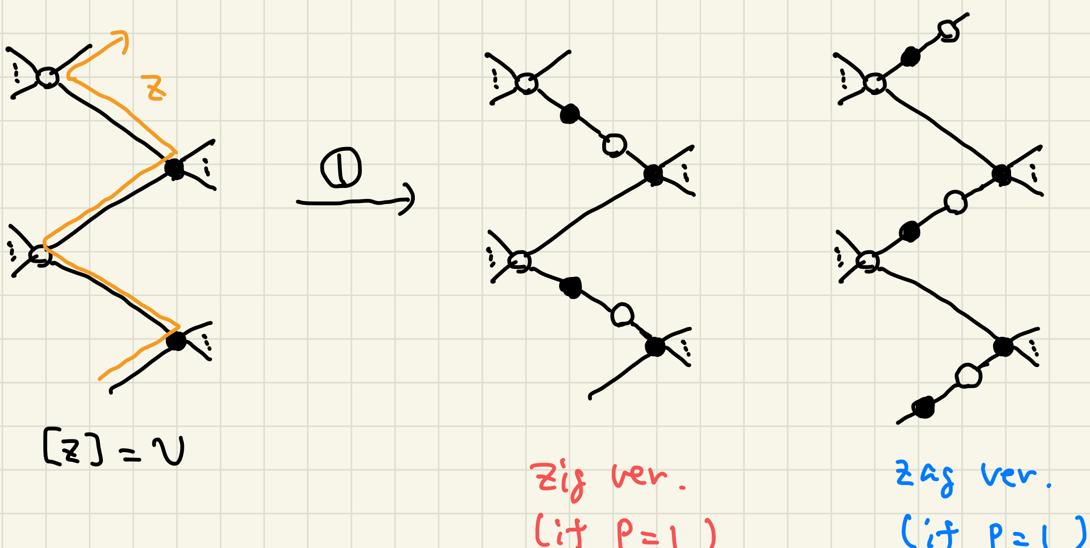
Today, I will discuss the case of $r=1$, thus $p = \ell(z)/2 - 1$

If $r>1$, we need the additional data (and unfortunately it is complicated)

Def (Deformation at zig (resp. zag))

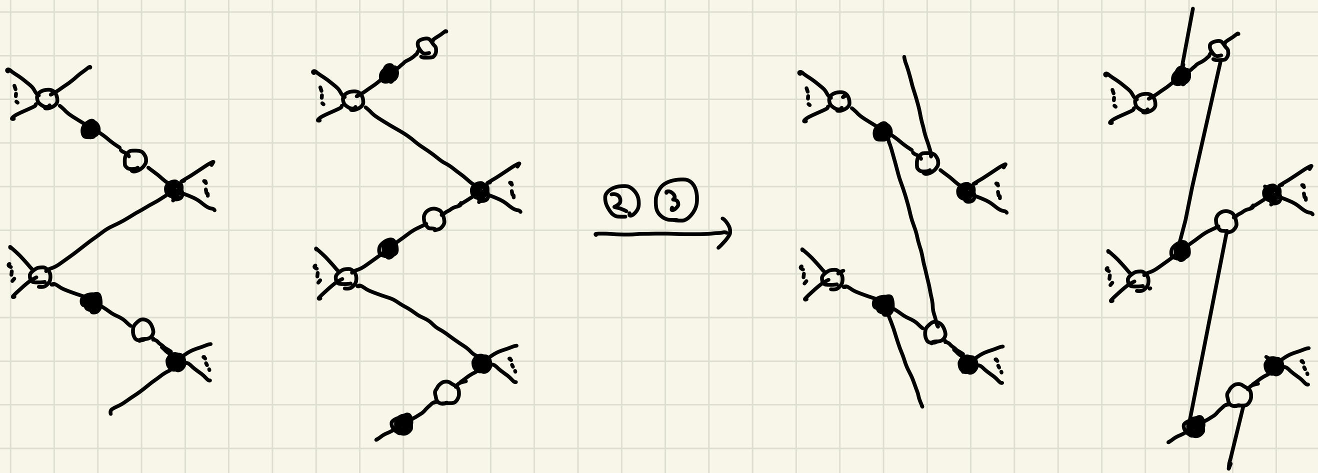
For the chosen zigzag path $z \in \mathcal{Z}_v$ on a cosistent dimer model Γ,

① insert p black nodes and p white nodes in each zig (resp. zag) of z



② remove all zags (resp. zigs) of Z

③ create the new zigzag path Z' so that $[Z'] = -\nu$ and zigzag paths passing through zigs (resp. zags) of Z as zag (resp. zig) will be preserved.



zig ver. (if $p=1$) zag ver. (if $p=1$) zig ver. (if $p=1$) zag ver. (if $p=1$)

④ If $\exists$ 2-valent nodes, remove them.

We will denote the resulting dimer model by $V_p^{zig}(\Gamma, z)$ (resp. $V_p^{zag}(\Gamma, z)$)

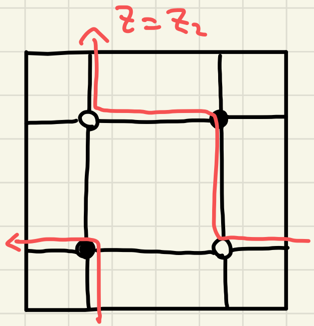
and call the deformation of Γ at zig of z (resp. zag of z)

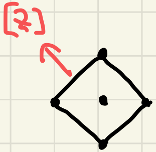
Thm (Higashitani – N.)

$\boxed{\text{The dimer models } V_p^{zig}(\Gamma, z) \text{ and } V_p^{zag}(\Gamma, z) \text{ are consistent.}}$

Example (zig version)

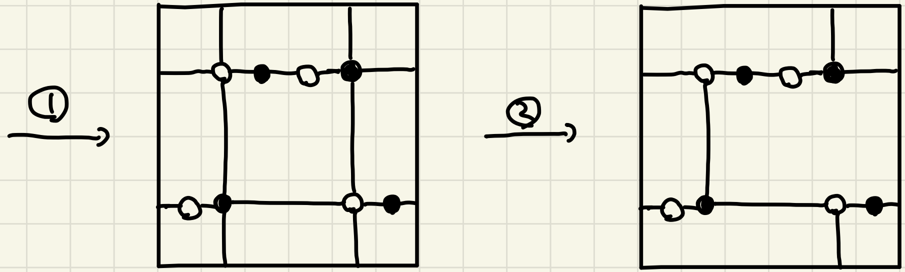
Γ_1

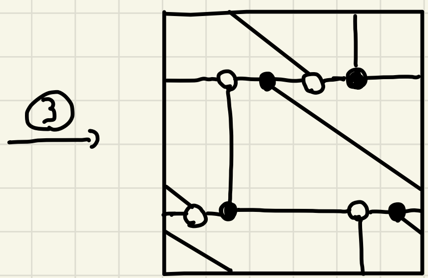




$\ell(z) = 4\ ,\ P = 1$

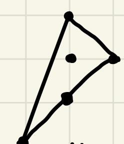
$v = [z] = (-1, 1)$





$V_p^{zig}(\Gamma_1, z)$

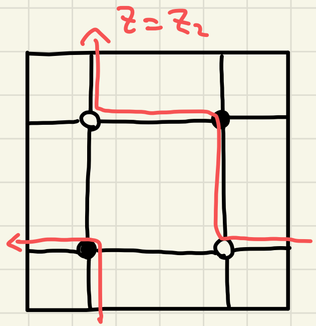
zigzag polygon

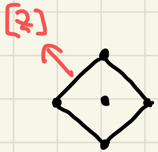


$= \mathrm{Mut}_{(-v)}(Q, F)$

Example (Zag version)

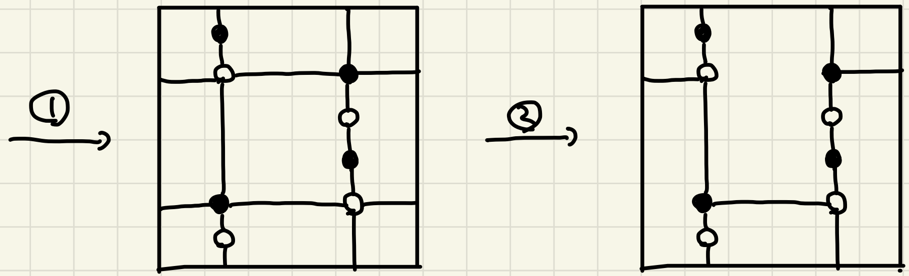
Γ_1

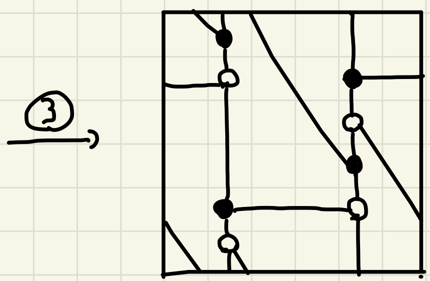




$\ell(z) = 4$, $p = 1$

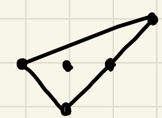
$v = [z] = (-1, 1)$





$V_p^{zag}(\Gamma_1, z)$

zigzag polygon



$\shortparallel$

$\mathrm{mut}_{(-v)}(Q, -F)$

Thm (Higashitani - N.)

Γ : consistent dimer model.

$$\Delta(V_p^{zig}(\Gamma, z)) = \mathrm{mut}_w(\Delta(\Gamma), F)$$

$$\Delta(V_p^{zag}(\Gamma, z)) = \mathrm{mut}_w(\Delta(\Gamma), -F)$$

where

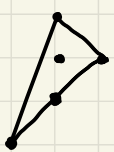
Deformation data	Mutation data
$[z] = v$	$w = -v$
$r = 1$	$h_{min} = -r$
p today	$h_{max} = p$

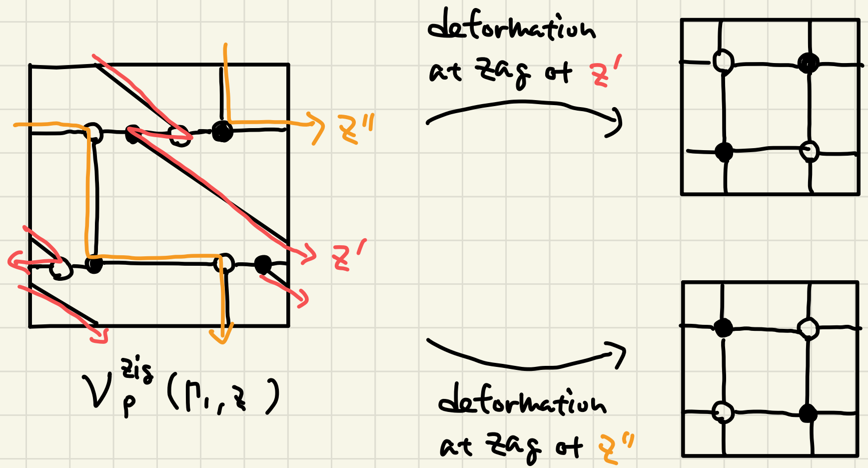
Rem For $r > 1$, we can obtain a similar result,

but we need some additional data.

Remark

If we take appropriate deformation data, we can see that these deformations are mutually inverse in the following sense.



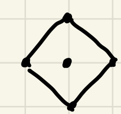


$V_p^{zig}(\Gamma_1, z)$

Γ_1

Γ_1^{op}

$\Delta(\Gamma_1) = \Delta(\Gamma_1^{op})$



$\mathcal{Z}_{(1,-1)} = \{z', z''\}$