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TILTING OBJECTS IN SINGULARITY CATEGORIES AND LEVELLED MUTATIONS

The McKay correspondence, mutation and related topics

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Preprojective algebras and the McKay correspondence

Let $G < SL(2, k)$ be a finite group which acts on $S := k[x, y]$.

Theorem (Reiten–Van den Bergh 89)

There is an equivalence of categories

$$\operatorname{mod}(S * G) \cong \operatorname{mod} \Pi\tilde{Q}_G,$$

where $\tilde{Q}_G$ *is an extended Dynkin quiver associated to* G *via the McKay correspondence and* Π *is the preprojective algebra.*

Theorem (Derived McKay correspondence (Kapranov–Vasserot 00))

Let $\tilde{X}$ *be the minimal resolution of the Kleinian singularity* $X := \operatorname{Spec} S^G$. *There is an equivalence*

$$\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Pi\tilde{Q}_G) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \tilde{X}).$$

Preprojective algebras

- Preprojective algebras were first defined by Gel'fand and Ponomarev to study the representation theory of finite-dimensional hereditary algebras.

Definition

Let $\Lambda := kQ$ be a finite-dimensional algebra. Define $D(-) := \mathrm{Hom}_k(-, k)$. The **preprojective algebra** $\Pi(\Lambda)$ is defined as

$$\Pi(\Lambda) := k\overline{Q} / \left(\sum_{a \in Q_1} (aa^* - a^*a) \right) \cong T_\Lambda \operatorname{Ext}^1_\Lambda(D\Lambda, \Lambda),$$

where $\overline{Q}$ is the double quiver of Q, whose vertices are given by $\overline{Q}_0 = Q_0$ and arrows are given by $\overline{Q}_1 = Q_1 \cup \{a^* : j \to i \mid a : i \to j \in Q_1\}$.

- From the tensor algebra structure, we see that the preprojective algebra comes equipped with a natural grading, obtained by putting the arrows in Q_1 in degree 0 and the other ones in degree 1.

Higher Auslander–Reiten theory

- Many of the central concepts in Auslander–Reiten theory were generalized to a 'higher dimensional setting'.
- In particular, n-representation-infinite algebras are finite-dimensional algebras of global dimension n which enjoy properties analogous to representation-infinite hereditary algebras in the classical theory.
- One can define Higher preprojective algebras in this setting. Let Λ be an n-representation-infinite algebra, then the higher preprojective algebra $\Pi(\Lambda)$ is defined as
$$\Pi(\Lambda) := T_\Lambda \operatorname{Ext}^n(D\Lambda, \Lambda).$$
- As in the classical case, these preprojective algebras are equipped with a natural grading induced from the tensor algebra structure.

Question

Let $G < SL(n, k)$ be a finite group which acts on $S = k[x_1, \ldots, x_n]$.

A generalization of [RVdB89]?

Is the skew-group algebra $S * G$ Morita equivalent to a higher preprojective algebra?

Partial answer

Theorem (Amiot–Iyama–Reiten 15)

Let $G < SL(n,k)$ be a finite cyclic group of order r. Suppose that there exists a generator $g = \frac{1}{r}(a_1, \dots, a_n)$ of G such that

1. $\gcd(a_i, r) = 1$ *and* $0 < a_i < r$ *for all* i*;*
2. $\sum_{i=1}^{n} a_i = r$*.*

Then there exists a grading on $S * G$ endowing it with the structure of a higher preprojective algebra.

Negative answer

Definition

Let $n_1, n_2 \geq 1$ such that $n_1 + n_2 = n$. We say that G **embeds** into $SL(n_1, k) \times SL(n_2, k)$ if G is conjugate to a group in which each element is of the form

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},$$

where $g_1 \in SL(n_1, k)$ and $g_2 \in SL(n_2, k)$.

Theorem (T. 20)

Let G be a finite group which embeds in $SL(n_1, k) \times SL(n_2, k)$. The skew-group algebra $S * G$ is not Morita equivalent to a higher preprojective algebra.

Connection with age

Definition

Let $G < SL(n, \mathbb{C})$ finite and $g \in G$. Choose a representation $g = \frac{1}{r}(a_1, \ldots, a_n)$.
Define the **age** of g by $\mathrm{age}(g) := \frac{1}{r}\sum a_i$.

Lemma

If G embeds in $SL(n_1, k) \times SL(n_2, k)$, then it does not contain any junior element.

Proposition

Let $G < SL(4, \mathbb{C})$ be a finite cyclic group such that S^G is an isolated singularity. Then the singularity $\operatorname{Spec} S^G$ is terminal if and only if $S * G$ does not admit the grading structure of a preprojective algebra.

Graded singularity category

Let R be a graded noetherian Gorenstein ring.

Definition

The **graded singularity category** of R is the Verdier localization

$$\mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(R) := \mathrm{D}^{\mathrm{b}}(\operatorname{gr} R)/\mathrm{D}^{\mathrm{b}}(\operatorname{grproj} R),$$

where $\mathrm{D}^{\mathrm{b}}(\operatorname{grproj} R)$ is the triangulated subcategory consisting of objects that are isomorphic to bounded complexes of projectives.

- It is analogous to the singularity category over algebraic varieties, which reflects the properties of the singularities of X.
- There is a triangle equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}}(R) \simeq \mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(R)$ [Buchweitz 87, Orlov 04].

Tilting objects and the McKay correspondence

Let $G < SL(2,k)$ be a finite group which acts on $S := k[x,y]$ and $R := S^G$ be endowed with a grading induced from the preprojective algebra grading on $S * G \cong_m \Pi\tilde{Q}_G$.

Theorem (Kajiura–Saito–Takahashi 07, Lenzing–de la Pena 11)

The singularity category $\mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(R)$ admits a tilting object. There is a triangle equivalence

$$\mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(R) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, kQ_G),$$

where Q_G is the Dynkin quiver associated to G via the McKay correspondence.

Theorem (AIR 15)

Let Π be a higher preprojective algebra, e be an idempotent satisfying certain axioms and $R := e\Pi e$. There is a triangle equivalence

$$\mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(R) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, \Pi_0/\langle e\rangle).$$

Tilting objects with a different grading

Theorem (Iyama–Takahashi 13)

Let $S = k[x_1, \dots, x_n]$ be graded by putting the variables in degree 1, $G < SL(n,k)$ be finite and $R := S^G$ be an isolated singularity. Let $e = \frac{1}{|G|}\sum_{g\in G} g$. There is a triangle equivalence

$$\mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(R) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{mod}(1-e)\nabla(S^! * G)(1-e)),$$

where $S^!$ is the Koszul dual of S, and ∇ is the Beilinson algebra.

Theorem (Mori–Ueyama 16)

Let S be a noetherian AS-regular Koszul algebra, $G < \operatorname{GrAut} S$ be finite with homological determinant 1 and $R := S^G$ be a non-commutative isolated singularity. Let $e = \frac{1}{|G|}\sum_{g\in G} g$. There is a triangle equivalence

$$\mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(R) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{mod}(1-e)\nabla(S^! * G)(1-e)).$$

The setting

Definition

Let $A = \oplus_{i\geq 0} A_i$ be a noetherian locally finite graded algebra. We say that A is n-**AS-regular** (resp. n-**AS-Gorenstein**) **of Gorenstein parameter** ℓ if $\mathrm{gl.dim}\, A = n$ and $\mathrm{gl.dim}\, A_0 < \infty$ (resp. $\mathrm{inj.dim}_A A = \mathrm{inj.dim}_{A^{\mathrm{op}}} A = n$) and

$$\bigoplus_{i\in\mathbb{Z}} \mathbf{R}\mathrm{Hom}_{\mathrm{Gr}\, A}(A_0, A(i)) \cong (DA_0)(\ell)[-n] \quad \text{in } \mathrm{D}(\mathrm{Gr}\, A_0) \text{ and in } \mathrm{D}(\mathrm{Gr}\, A_0^{\mathrm{op}}).$$

Setting

Let $A = \oplus_{i\geq 0} A_i$ be a locally finite noetherian n-AS-regular algebra of Gorenstein parameter ℓ, with $\ell \geq 1$. Let $e = e^2 \in A$ be such that

a) A/AeA is finite-dimensional;

b) $R := eAe$ is n-AS-Gorenstein of parameter ℓ;

c) $eA_0e \cong k$.

Tilting objects with other gradings?

Question

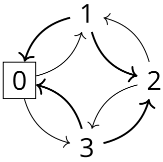
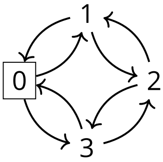
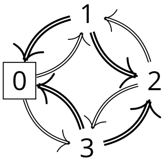
When does $\mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(R)$ admit a tilting object?

Running example

Let $S = k[x_1, \ldots, x_n]$, $G < SL(n, k)$ be finite and $e = \frac{1}{|G|} \sum_{g \in G} g$. Let $A = S * G$ be the skew-group algebra. Then $R := eAe \cong S^G$ and the conditions of the setting are satisfied for many different gradings on A.

- Both previous situations give partial answers to the question. We are interesting in finding other classes where there is a tilting object, for example for the skew-group algebras which do not admit a structure of preprojective algebra.

Example: $A = S * G$, $\mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(R) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$

	G	Q_A	Q_Λ
2-AS Gor. par 1: (AIR15)	$\langle \frac{1}{4}(1,3) \rangle$		$1 \longleftarrow 2 \longrightarrow 3$
2-AS Gor. par 2: (IT13, MU16)	$\langle \frac{1}{4}(1,3) \rangle$		$1' \longleftarrow 2 \longrightarrow 3'$ $1 \longrightarrow 2' \longleftarrow 3$
4-AS Gor. par 2:	$\langle \frac{1}{4}(1,3,1,3) \rangle$		?

Derived category of graded tails

Definition

We define the quotient abelian **category of graded tails**

$$\operatorname{qgr} A := \operatorname{gr} A / \operatorname{tors} A,$$

where $\operatorname{tors} A$ is the full subcategory consisting of all graded finite-dimensional A-modules.

Theorem (Orlov 09)

Let R be an AS-Gorenstein algebra of Gorenstein parameter ℓ. There is a fully faithful functor

$$\Phi : \mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(R) \to \mathrm{D}^{\mathrm{b}}(\operatorname{qgr} R)$$

and a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} R) = \langle \mathrm{q}R, \ldots, \mathrm{q}R(\ell - 1), \Phi(\mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(R)) \rangle$$

Beilinson algebra

Definition

The **Beilinson algebra**, ∇A, is defined as

$$\nabla A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-1} \\ 0 & A_0 & \cdots & A_{\ell-2} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}$$

Theorem (Minamoto–Mori 11)

Let A be a locally finite noetherian n-AS-regular algebra of Gorenstein parameter ℓ. There is a semiorthogonal decomposition and an equivalence of derived categories:

$$\mathrm{D^b}(\mathrm{qgr}\, A) = \langle \mathrm{q}A, \ldots, \mathrm{q}A(\ell - 1) \rangle \cong \mathrm{D^b}(\nabla A).$$

Strategy

Let A be a locally finite noetherian n-AS-regular algebra of Gorenstein parameter ℓ, e be an idempotent as in the setting so that $R = eAe$ is AS-Gorenstein.

- We compare two semi-orthogonal decompositions in $\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} R)$:

$$\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} R) = \langle \mathrm{q}Ae, \dots, \mathrm{q}Ae(\ell - 1)\rangle \cong \mathrm{D}^{\mathrm{b}}(\nabla A) \qquad \text{(Minamoto–Mori)}$$
$$\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} R) = \langle \mathrm{q}eAe, \dots, \mathrm{q}eAe(\ell - 1), \Phi(\mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(R))\rangle \qquad \text{(Orlov)}$$

- We can then hope to obtain a tilting object in $\mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(R)$ by using mutations.
- Unfortunately, mutations do not always preserve tilting objects, so we need more assumptions.

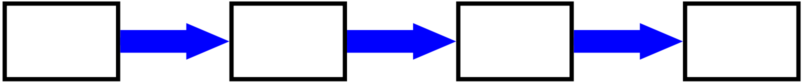
Levelled algebras

Definition

A quiver Q is **ordered** if $Q_0 = \{0, \ldots, m\}$ and there is no arrow $i \to j$ if $j \leq i$.

Definition (Hille)

A quiver Q is **levelled** if Q is ordered and there exists $s : Q_0 \to \{0, \ldots, M\}$ which is surjective, monotonic and there are only arrows $i \to j$ if $s(j) = s(i) + 1$.



Levelled algebras behave well under mutations.

Tilting object

Theorem (T.)
If ∇A is a Koszul levelled algebra, then there is a triangle equivalence

$$\mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(R) \cong \mathrm{D}^{\mathrm{b}}((1-\tilde{e})(\nabla A)^{!}(1-\tilde{e})),$$

where $(-)^{!}$ denotes the Koszul dual and $\tilde{e}$ is the idempotent in $(\nabla A)^{!}$ induced by e.

The theorem recovers the situation of the result by [Iyama–Takahashi, Mori–Ueyama].

Lemma
If S is an AS-regular Koszul algebra, and $G < \mathrm{GrAut}\, S$ is finite, then $\nabla(S * G)$ is a Koszul levelled algebra.

Example

$S = k[x_1, x_2, x_3, x_4]$, $G = \langle \frac{1}{4}(1,3,1,3) \rangle$, A= $S * G$, e is the idempotent corresponding to vertex 0, so that $R = eS * Ge \cong S^G$.

$$Q_{S*G}: \begin{tikzcd}
& 1 \arrow[dl, Rightarrow, bend right] \arrow[dr, Rightarrow] & \\
\boxed{0} \arrow[ur] \arrow[dr, bend right] & & 2 \arrow[ul, bend right] \arrow[dl] \\
& 3 \arrow[ul, Rightarrow] \arrow[ur, Rightarrow, bend right] &
\end{tikzcd} \quad + \text{ commutativity relations.}$$

<u>Grading</u>: The **thick arrows** are in **degree** 1, the others in degree 0.

Then A is a (non-Koszul) 4-AS regular algebra of Gor. parameter 2.

Remark: $S * G$ cannot be endowed with a grading structure of Gor. parameter 1.

Example

$$Q_{S*G}\colon \begin{tikzcd} & 1 \arrow[dl, Rightarrow, bend right] \arrow[dr, Rightarrow, bend left] & \\ \boxed{0} \arrow[ur, Rightarrow, bend right] \arrow[dr, Rightarrow, bend right] & & 2 \arrow[ul, Rightarrow, bend right] \arrow[dl, Rightarrow, bend right] \\ & 3 \arrow[ul, Rightarrow, bend left] \arrow[ur, Rightarrow, bend right] & \end{tikzcd}, \qquad Q_{(\nabla A)^!}\colon \begin{tikzcd} \boxed{0} & 1 \arrow[l, Rightarrow] \arrow[dl, Rightarrow] & \boxed{0'} \arrow[l, Rightarrow] \arrow[dl, Rightarrow] & 1' \arrow[l, Rightarrow] \arrow[dl, Rightarrow] \\ 2 & 3 \arrow[l, Rightarrow] \arrow[ul, Rightarrow] & 2' \arrow[l, Rightarrow] \arrow[ul, Rightarrow] & 3' \arrow[l, Rightarrow] \arrow[ul, Rightarrow] \end{tikzcd}$$

The induced idempotent $\tilde{e}$ is the one corresponding to the vertices in the boxes.

$$Q_{(1-\tilde{e})(\nabla A)^!(1-\tilde{e})}\colon \begin{tikzcd} & 1 \arrow[dl, Rightarrow] & & 3' \arrow[ll] \arrow[dl, Rightarrow] \\ 2 & & 2' \arrow[ul, Rightarrow] \arrow[dl, Rightarrow] & \\ & 3 \arrow[ul, Rightarrow] & & 1' \arrow[ll] \arrow[ul, Rightarrow] \end{tikzcd}$$

There is a triangle equivalence

$$\mathrm{D}^{\mathrm{gr}}_{\mathrm{Sg}}(S^G) \cong \mathrm{D}^{\mathrm{b}}((1-\tilde{e})(\nabla A)^!(1-\tilde{e})).$$