On the stringy E-functions of minimal models

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The McKay correspondence, mutation and related topics
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E-polynomial

Defintion

Let X be a d-dimensional smooth projective variety over $\mathbb{C}$. The generating polynomial of the Hodge numbers $h^{p,q}(X)$ of X

$$E(X; u, v) := \sum_{0 \le p,q \le d} (-1)^{p+q} h^{p,q}(X) u^p v^q$$

we call E-polynomial (or Hodge polynomial) of X. Its value $E(X; 1, 1)$ is the usual Euler number of X.

Hodge-Deligne polynomials

Using the mixed Hodge structure on the cohomology groups with compact supports $H_c^k(X, \mathbb{C})$, one can extend by additivity usual E-polynomials to Hodge-Deligne E-polynomials

$$E(X; u, v) := \sum_{p,q} e^{p,q}(X) u^p v^q,$$

where the coefficients

$$e^{p,q}(X) := \sum_{k \geq 0} (-1)^k h^{p,q}(H_c^k(X, \mathbb{C})).$$

are called Hodge-Deligne numbers of X.

The main motivation for the stringy E-functions

Stringy Hodge numbers of Gorenstein varieties:

One needs *stringy Hodge numbers* $h^{p,q}_{\mathrm{str}}(X)$ of singular Gorenstein projective algebraic varieties X satisfying the equations

$$h^{p,q}_{\mathrm{str}}(X) = h^{p,q}(Y)\ \forall p, q,$$

if X admits a crepant desingularization $\rho : Y \to X$ ($\rho^* K_X = K_Y$).

Topological mirror symmetry test:

X is a d-dimensional Calabi-Yau variety with at worst canonical Gorenstein singularities. If X^* is a Calabi-Yau mirror of X, then

$$h^{p,q}_{\mathrm{str}}(X) = h^{d-p,q}_{\mathrm{str}}(X^*), \ \ \forall p, q.$$

Discrepancies

Let X be a normal irreducible quasi-projective $\mathbb{Q}$-Gorenstein algebraic variety. Take a resolution of singularities of X

$$\rho : Y \to X$$

whose the exceptional locus $\bigcup_{i=1}^{r} D_i$ is a union of smooth irreducible divisors with only normal crossings.
$I := \{1, \ldots, r\}$

$$K_Y = \rho^* K_X + \sum_{i \in I} a_i D_i,$$

The rational numbers $a_i \in \mathbb{Q}$ $(i \in I)$ are called *discrepancies* of divisors D_i.

Singularities in MMP

We consider the following three classes of singularities of $\mathbb{Q}$-Gorenstein algebraic varieties X that appear in the minimal model program:

Definition

Singularities of X are called at worst

- *terminal* if $a_i > 0$, $\forall i \in I$;
- *canonical* if $a_i \geq 0$, $\forall i \in I$;
- *log-terminal* if $a_i > -1$, $\forall i \in I$.

All $\mathbb{Q}$-Gorenstein algebraic varieties X considered in this talk have at worst log-terminal singularities.

Stringy E-function I

Definition (first version)

Let $\rho : Y \to X$ be a resolution and $K_Y = \rho^* K_X + \sum_{i \in I} a_i D_i$. Assume that $a_i > -1$ ($\forall i \in I$). Define for any subset $J \subseteq I$:

$$D_\emptyset := Y, \quad D_J := \bigcap_{j \in J} D_j \quad (\emptyset \neq J \subseteq I).$$

The *stringy E-function* of X is the following rational function

$$E_{\mathrm{str}}(X; u, v) := \sum_{\emptyset \subseteq J \subseteq I} E(D_J; u, v) \prod_{j \in J} \left(\frac{uv - 1}{(uv)^{a_j + 1} - 1} - 1 \right)$$

$$= \sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|} E(D_J; u, v) \prod_{j \in J} \left(\frac{uv - (uv)^{a_j+1}}{(uv)^{a_j + 1} - 1} \right).$$

(a product over $\emptyset$ is assumed to be 1)

Some properties

Theorem (B., 1998)

The stringy E-function of a projective variety X with at worst log-terminal singularities has the following properties:

- $E_{\rm str}(X; u, v) = E_{\rm str}(X; v, u)$ (symmety $u \leftrightarrow v$).
- $E_{\rm str}(X; u, v) = (uv)^d E_{\rm str}(X; u^{-1}, v^{-1})$ if X is projective and $d = \dim X$ (Poincaré duality).
- $E_{\rm str}(X; u, v)$ does not depend on the resolution. In particular, $E_{\rm str}(X; u, v) = E(X; u, v)$ if X is smooth.
- $E_{\rm str}(X; u, v) = E_{\rm str}(Y; u, v)$ if $\rho : Y \to X$ is crepant, i.e., $\rho^* K_X = K_Y$. In particular, $E_{\rm str}(X; u, v) = E(Y; u, v)$ if $\rho : Y \to X$ is a crepant desingularization.
- $E_{\rm str}(X; u, v) = E_{\rm str}(X'; u, v)$ if X and X' are K-equivalent.

Stringy E-function II

Definition (second version)

Let $\rho : Y \to X$ be a resolution and $K_Y = \rho^* K_X + \sum_{i \in I} a_i D_i$. Assume that $a_i > -1$ $(\forall i \in I)$. Define for any subset $J \subseteq I$:

$$D_J^\circ := D_J \setminus \bigcup_{j \notin J} D_j.$$

The *stringy E-function* of X is the following rational function

$$E_{\mathrm{str}}(X; u, v) := \sum_{\emptyset \subseteq J \subseteq I} E(D_J^\circ; u, v)(uv - 1)^{|J|} \prod_{j \in J} \frac{1}{(uv)^{a_j+1} - 1}.$$

The combinatorial meaning of the product

$$\prod_{j\in J}\frac{1}{(uv)^{a_j+1}-1}$$

Using the equation

$$\frac{1}{t-1}=t^{-1}+t^{-2}+\cdots=\sum_{l>0}t^{-l},$$

we obtain

$$\prod_{j\in J}\frac{1}{(uv)^{a_j+1}-1}=\sum_{n\in\mathbb{Z}^{|J|}_{>0}}(uv)^{-\alpha(n)},$$

where $\alpha(x)$ is the linear function $\sum_{j\in J}(a_j+1)x_j$.

Stringy Hodge numbers

Definition

Let X be a d-dimensional irreducible projective variety with at worst Gorenstein canonical singularities. Assume that $E_{\mathrm{str}}(X; u, v)$ is a polynomial. Then the *stringy Hodge numbers* $h^{p,q}_{\mathrm{str}}(X)$ are defined by the equation:

$$E_{\mathrm{str}}(X; u, v) = \sum_{p,q} (-1)^{p+q} h^{p,q}_{\mathrm{str}}(X) u^p v^q.$$

It is more convenient to check the mirror symmetry $h^{p,q}_{\mathrm{str}}(X) = h^{d-p,q}_{\mathrm{str}}(X^*), \ \forall p, q$, using the equivalent single equation

$$E_{\mathrm{str}}(X; u, v) = (-u)^d E_{\mathrm{str}}(X^*; u^{-1}, v).$$

Reflexive polytopes in Mirror Symmetry

$M \cong \mathbb{Z}^d$, $N := \mathrm{Hom}(M, \mathbb{Z})$, $\langle *, * \rangle : M \times N \to \mathbb{Z}$ pairing
$M_{\mathbb{R}} := M \otimes \mathbb{R}$, $N_{\mathbb{R}} := N \otimes \mathbb{R}$ are vector spaces over $\mathbb{R}$.

Definition (B. 1994)

A d-dimensional lattice polytope $P \subset M_{\mathbb{R}}$ containing $0 \in M$ in its interior is called *reflexive* if the *polar dual* polytope

$$P^* := \{y \in N_{\mathbb{R}} \ : \ \langle x, y \rangle \geq -1, \ \forall x \in P\}$$

is a lattice polytope.

The combinatorial duality

If $P \subset M_{\mathbb{R}}$ is reflexive, then $P^* \subset N_{\mathbb{R}}$ is also reflexive and

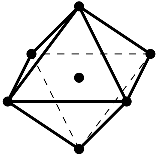
$$\boxed{(P^*)^* = P.}$$

There exists a natural 1-to-1 correspondence between k-dimensional faces $Q \prec P$ and $(d-k-1)$-dimensional dual faces $Q^* \prec P^*$:

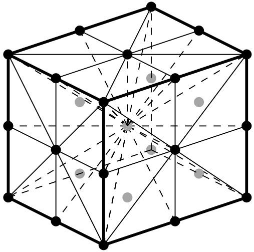
$$Q^* := \{y \in P^* \ : \ \langle x, y \rangle = -1 \ \ \forall x \in Q\}.$$

Examples of the combinatorial polar duality:

$$P := \mathrm{Conv}(\pm e_1, \ldots, \pm e_d) \in \mathbb{R}^d,$$
$$P^* := \{(x_1, \ldots, x_d) \in \mathbb{R}^d \ : \ |x_i| \leq 1 \ (1 \leq i \leq d)\}.$$



P



P^*

The combinatorial duality $P \leftrightarrow P^*$ perfectly agrees with predictions of mirror symmetry for Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties.

Theorem (B., Borisov 1996)

Let $X \subset V_P$ and $X^* \subset V_{P^*}$ be general Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties V_P and V_{P^*} corresponding to a pair of d-dimensional reflexive polytopes (P, P^*). Then one has

$$E_{\mathrm{str}}(X; u, v) = (-u)^{d-1} E_{\mathrm{str}}(X^*; u^{-1}, v),$$

i.e.

$$h^{p,q}_{\mathrm{str}}(X) = h^{d-1-p,q}_{\mathrm{str}}(X^*) \ \ \forall p, q.$$

Idea of proof

- Consider general Calabi-Yau hypersurfaces $X \subset V_P$ in Gorenstein toric Fano varieties V_P as projective compactifications of non-degenerate affine hypersurfaces $Z \subset \mathbb{T}_d$ defined by a Laurent polynomial with a reflexive Newton polytope P.
- Apply the algorithm of Danilov and Khovanskiĭ (1986) for computing the Hodge-Deligne polynomials $E(Z_Q; u, v)$ of affine hypersurfaces $Z_Q \subset \mathbb{T}_{\dim Q}$ for all faces $Q \preceq P$.
- Derive the combinatorial formula for the stringy E-function

$$E_{\mathrm{str}}(X; u, v) = \sum_{\substack{Q \preceq P \\ k = \dim Q \geq 1}} E(Z_Q; u, v)(uv - 1)^{d-k} \sum_{n \in \sigma_Q^\circ} (uv)^{\langle Q, n \rangle}$$

- Use the duality $Q \leftrightarrow Q^*$ and the equality $\sigma_Q = \mathbb{R}_{\geq 0} Q^*$ for $(d - \dim Q)$-dimensional cones σ_Q in the normal fan Σ_P of P.

Non-degenerate affine toric hypersurfaces

Definition

A Laurent polynomial

$$f(\mathbf{t}) = \sum_{m \in A} a_m \mathbf{t}^m \in \mathbb{C}[M] \cong \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$$

with Newton polytope $P = \mathrm{conv}(A) \subset M_{\mathbb{R}} := M \otimes \mathbb{R}$ is called *non-degenerate* if for any k-dimensional face $Q \preceq P$ the affine hypersurface

$$Z_Q := \{ \sum_{m \in A \cap Q} a_m \mathbf{t}^m = 0 \} \subset \mathbb{T}_k.$$

is either empty, or reduced and smooth. The non-degeneracy of $f(\mathbf{t})$ is a Zariski open condition on its coefficients $\{a_m\} \in \mathbb{C}^{|A \cap M|}$.

Minimal models

Definition

A normal projective variety X with at worst $\mathbb{Q}$-factorial terminal singularities is called *minimal model* if the canonical class K_X is a semi-ample $\mathbb{Q}$-Cartier divisor.

Example

A normal projective variety X with at worst $\mathbb{Q}$-factorial terminal singularities and trivial canonical class (Calabi-Yau variety) is always a minimal model.

Remark

If X and X' are two birational minimal models, then

$$E_{\mathrm{str}}(X; u, v) = E_{\mathrm{str}}(X'; u, v).$$

Minimal models of some toric Calabi-Yau hypersurfaces

Let $P \subset M_{\mathbb{R}}$ be any d-dimensional reflexive polytope, Σ_P the normal fan of P in $N_{\mathbb{R}}$. Consider a maximal simplicial refinement $\widehat{\Sigma}$ of Σ_P coming from triangulations of facets of the dual reflexive polytope $P^* \subset N_{\mathbb{R}}$ such that $\widehat{\Sigma}[1] = P^* \cap (N \setminus \{0\})$. Then $\widehat{\Sigma}$ defines a simplicial projective toric variety with at worst terminal singularities and crepant morphisms

$$\rho \ : \ \widehat{V} \to V_P, \ \ \widehat{X} := \rho^{-1}(X) \to X \subset V_P,$$

such that $\widehat{X} \subset \widehat{V}$ is a minimal Calabi-Yau model of a Gorenstein Calabi-Yau hypersurface $X \subset V_P$.

Our goal and natural questions

We want to extend the combinatorial computing of the stringy E-function to arbitrary minimal models $\widehat{Z}$ of non-degenerate hypersurfaces $Z \subset \mathbb{T}_d$ defined by Laurent polynomials f with a given Newton polytope P.

Questions

- For which d-dimensional lattice polytopes P a minimal model $\widehat{Z}$ does exist?
- For which P is the minimal model $\widehat{Z}$ a Calabi-Yau variety?
- How to construct a minimal model $\widehat{Z}$ through the Newton polytope P?
- How to compute the stringy E-function of the minimal model $\widehat{Z}$ through the Newton polytope P?

Fine interior $F(P)$ of a lattice polytope P

$A \subset M$ a finite subset, $P := \mathrm{conv}(A) \subset M_{\mathbb{R}}$ a full-dimensional lattice polytope. Consider the piecewise linear function $\mathrm{ord}_P : N_{\mathbb{R}} \to \mathbb{R}$:

$$\mathrm{ord}_P(y) = \min_{x \in P} \langle x, y \rangle, \quad y \in N_{\mathbb{R}}.$$

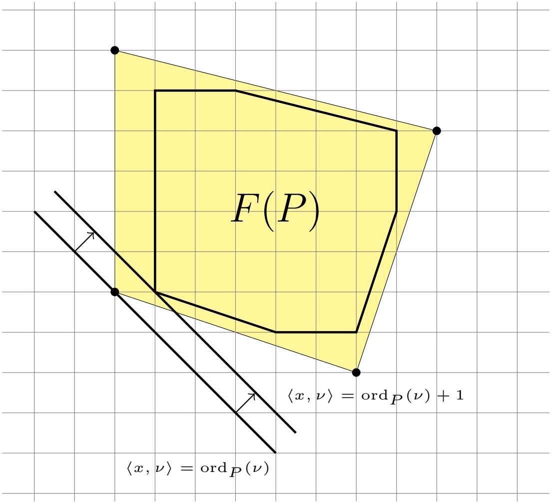
Then $P = \{x \in M_{\mathbb{R}} \ : \ \langle x, \nu \rangle \geq \mathrm{ord}_P(\nu) \quad \forall \nu \in N \setminus \{0\}\}$.

Definition

$F(P) := \{x \in M_{\mathbb{R}} \ : \ \langle x, \nu \rangle \geq \mathrm{ord}_P(\nu) + 1 \quad \forall \nu \in N \setminus \{0\}\}$ is called Fine interior of P.

- Jonathan Fine, *Resolution and completion of algebraic varieties*, Ph.D.-Thesis, University of Warwick, June 1983. (in [Ph.D.-Thesis, §4] $F(P)$ is called heart of P.)

Example: $F(P) = \mathrm{Conv}(P^\circ \cap M)$ if $\dim P = 2$



Theorem of Ishii: Existence of a minimal models

Some results in

- Shihoko Ishii, *The minimal model theorem for divisors of toric varieties*, Tohoku Math. J. (1999), 213-226.

can be reformulated in the following way:

Theorem (Ishii, 1999)

A non-degenerate affine hypersurface $Z \subset \mathbb{T}_d$ defined by a Laurent polynomial $f \in \mathbb{C}[M]$ with Newton polytope P is birational to a minimal model $\widehat{Z}$ if and only if the Fine interior of P is not empty. Moreover, a minimal model $\widehat{Z}$ (if exists) can be obtained as Zariski closure of Z in some torus embedding $\mathbb{T}_d \hookrightarrow \widehat{V}$, in some simplicial projective toric variety $\widehat{V}$ with at worst terminal singularities.

Constructing a minimal model (Ishii)

Definition

The *support of Fine interior* $F(P)$ is the finite set

$$S_F(P) := \{\nu \in N \ : \ \mathrm{ord}_{F(P)}(\nu) = \mathrm{ord}_P(\nu) + 1\}.$$

This is the set of *essential valuations* $\nu \in N$ that contribute to $F(P)$

- Find the set $S_F(P)$ as the set of lattice generators of 1-dimensional cones in the fan $\widehat{\Sigma}$.
- Construct the simplicial fan $\widehat{\Sigma}$ with $\widehat{\Sigma}[1] = S_F(P)$ as a normal fan of some full-dimensional simple polytope $\square(\varepsilon)$ with given facet normals $S_F(P)$ using "puffing up" of the rational polytope $F(P)$.

Calabi-Yau minimal models if P is reflexive (B.1994)

Remark

Let P be a reflexive polytope. Then

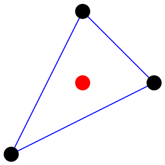
- $F(P) = \{0\}$;
- $S_F(P) = P^* \cap (N \setminus \{0\})$.

The algorithm

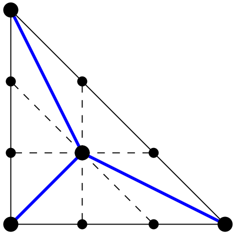
- Take the Zariski closure $\widetilde{Z}$ of Z in the Gorenstein toric Fano variety $\widetilde{V}$ defined by the normal fan of P. We call (Calabi-Yau) $\widetilde{Z}$ *canonical model of Z*;
- Take a maximal projective simplicial subdivision $\widehat{\Sigma}$ of Σ_P with $\widehat{\Sigma}[1] = S_F(P)$. The Zariski closure $\widehat{Z}$ of Z in the toric variety $\widehat{V}$ is a (Calabi-Yau) *minimal model of Z*.

Example: dim = 2

$$f(\mathbf{t}) := t_1 + t_2 + \frac{1}{t_1 t_2}, \quad Z \subset \widetilde{V} := \{z_0^3 - z_1 z_2 z_3 = 0\}$$



$\{0\} = F(P) \subset P$



$P^*, \quad S_F(P)$

Canonical hull of P if $F(P) \neq \emptyset$

Definition

Let P be a lattice polytope with $F(P) \neq \emptyset$. We define *canonical hull* $C(P)$ of P as

$$C(P) := \{x \in M_{\mathbb{R}} \ : \ \langle x, \nu \rangle \geq \mathrm{ord}_P(\nu) \ \ \forall \nu \in S_F(P)\}.$$

Remarks

- The canonical hull $C(P)$ is a rational polytope containing P.
- For any 2-dimensional lattice polytope one has $P = C(P)$.
- A d-dimensional lattice polytope P is reflexive if and only if $F(P) = 0$ and $P = C(P)$.
- In general, $C(P)$ is larger than P.

Non-reflexive lattice polytope P with $F(P) = \{0\}$

Example (B. 2017)

Let P be the d-dimensional lattice simplex with vertices $e_1, \ldots, e_d, e_0 = -e_1 - \cdots - e_{d-1} - 2e_d$, where $d = 2k + 1$ is an odd integer ≥ 3. Then $F(P) = \{0\}$, but P is not reflexive. It follows from [B., arXiv:2006.15825] that the stringy E-function of a Calabi-Yau compactification $X = \widehat{Z}$ of the non-degenerate affine hypersurface

$$Z \;:\; t_1 + \cdots + t_{2k+1} + \frac{1}{t_1 \cdots t_{2k} t_{2k+1}^2} = 0$$

is a polynomial and X admits a Calabi-Yau mirror $X^* \subset \mathbb{P}(1^d, 2)$ satisfying the topological mirror symmetry test.

Calabi-Yau minimal models if $F(P) = \{0\}$

Theorem (B. 2017)

Let $Z \subset \mathbb{T}_d$ be a non-degenerate affine hypersurface with Newton polytope P. Then

- Z is birational to a Calabi-Yau minimal models if and only if $F(P)$ is a single lattice point (in the latter we can assume that $F(P) = \{0\}$).
- If $F(P) = \{0\}$, then the Zariski closure $\widetilde{Z}$ of Z in the $\mathbb{Q}$-Gorenstein canonical toric Fano variety $\widetilde{V}$ corresponding to the normal fan of $C(P)$ is a projective Calabi-Yau variety with at worst canonical singularities.
- A minimal model $\widehat{Z}$ of Z is the Zariski closure in a toric variety $\widehat{V}$ corresponding to a simplicial fan $\widehat{\Sigma}$ with $\widehat{\Sigma}[1] = S_F(P)$ which is a maximal projective crepant partial resolution of $\widetilde{V}$.

The canonical model in case $F(P) \neq \emptyset$

Theorem (B. 2020)

Let $P \subset M_{\mathbb{R}}$ be a d-dimensional lattice polytope with $F(P) \neq \emptyset$. Consider the d-dimensional rational polytope $\widetilde{P} := C(P) + F(P)$. Then the following statements hold:

- All primitive lattice vectors generating 1-dimensional cones of the normal fan $\widetilde{\Sigma}$ of the rational polytope $\widetilde{P}$ are contained in $S_F(P)$ (i.e. $\widetilde{\Sigma}[1] \subseteq S_F(P)$).
- The fan $\widetilde{\Sigma}$ defines a $\mathbb{Q}$-Gorenstein toric variety $\widetilde{V}$ with at worst canonical singularities.
- The Zariski closure $\widetilde{Z}$ of Z in the toric variety $\widetilde{V}$ is a projective $\mathbb{Q}$-Gorenstein hypersurface with at worst canonical singularities.

We call $\widetilde{Z}$ the *canonical model* of non-degenerate affine hypersurface $Z \subset \mathbb{T}_d$.

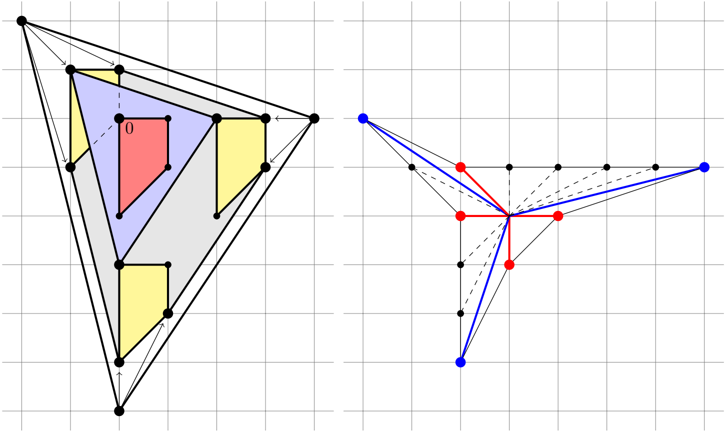
Minimal models

Theorem (B. 2020)

Let $P \subset M_{\mathbb{R}}$ be a d-dimensional lattice polytope with $F(P) \neq \emptyset$. Consider the normal fan $\widetilde{\Sigma}$ of the rational polytope $\widetilde{P} := C(P) + F(P)$. Then the following statements hold:

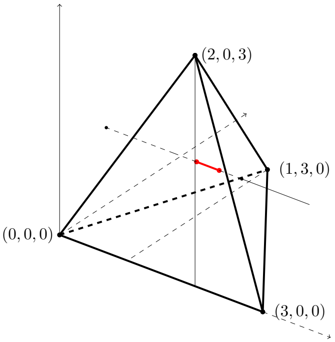
- Any convex maximal simplicial subdivision $\widehat{\Sigma}$ of $\widetilde{\Sigma}$ with $\widehat{\Sigma}[1] = S_F(P)$ defines a crepant morphism $\widehat{V} \to \widetilde{V}$ of the corresponding toric varieties.
- The corresponding simplicial toric variety $\widehat{V}$ has at worst terminal singularities.
- The Zariski closure $\widehat{Z}$ of a non-degenerate affine hypersurface Z in $\widehat{V}$ is a *minimal model* of Z.

Example: $\dim P = 2$



The fan $\widetilde{\Sigma}$ is the coarsest common refinement of the normal fans Σ_P and $\Sigma_{F(P)}$. One has $\widetilde{P} = P + F(P) = F(2P)$.

Example: $\dim P = 3$



The 3-dimensional lattice simplex P has 1-dimensional Fine interior

$$F(P) = \mathrm{Conv}\left((\frac{4}{3}, 1, 1), (\frac{5}{3}, 1, 1) \right).$$

Applications of canonical models

- There exist a unique(!) *canonical model* $\widetilde{Z}$ of any non-degenerate toric affine hypersurface Z with the Newton polytope P if $F(P) \neq \emptyset$.
- The Kodaira dimension of $\widehat{Z}$ equals $\kappa = \min\{d-1, \dim F(P)\}$.
- The *Iitaka fibration* of $\widetilde{Z} \to V_{F(P)}$ is induced by the natural toric morphism $\widetilde{V} \to V_{F(P)}$ (canonical toric Fano fibration).
- Generic *fibers of the Iitaka fibrations* are $(d-1-\kappa)$-dimensional canonical non-degenerate toric hypersurfaces of Kodaira dimension 0.

The stringy E-function of minimal models

Theorem (B., 2020)

Let $Z \subset \mathbb{T}_d$ be a non-degenerate affine hypersurface with the Newton polytope P and $F(P) \neq \emptyset$. Then the stringy E-function of its minimal model $\widehat{Z}$ equals

$$E_{\mathrm{str}}(\widehat{Z}; u, v) = \sum_{\substack{Q \preceq P \\ k = \dim Q \geq 1}} E(Z_Q; u, v) \sum_{\nu \in \sigma_Q^\circ \cap N} (uv - 1)^{d-k} (uv)^{-\alpha(\nu)}.$$

where $\alpha(\nu) := \mathrm{ord}_{F(P)}(\nu) - \mathrm{ord}_P(\nu)$, $E(Z_Q; u, v) \in \mathbb{Z}[u, v]$ is the Hodge-Deligne polynomial of the non-degenerate $(k-1)$-dimensional affine toric hypersurface $Z_Q \subset \mathbb{T}_k$, σ_Q° is the interior of the $(d-k)$-dimensional dual cone $\sigma_Q \in \Sigma_P$.

The stringy E-function of Calabi-Yau hypersurfaces

Corollary (B., 2017)

Let $Z \subset \mathbb{T}_d$ be a non-degenerate affine hypersurface with the Newton polytope P and $F(P) = \{0\}$. Then the stringy E-function of its Calabi-Yau minimal model $\widehat{Z}$ equals

$$E_{\mathrm{str}}(\widehat{Z}; u, v) = \sum_{\substack{Q \preceq P \\ k = \dim Q \geq 1}} E(Z_Q; u, v)(uv - 1)^{d-k} \sum_{\nu \in \sigma_Q^\circ \cap N} (uv)^{\mathrm{ord}_P(\nu)}.$$

The last formula is the best tool for testing Mirror Symmetry for non-degenerate Calabi-Yau hypersurfaces in toric varieties.

Thank you !