Dense g-vector fans for tame algebras

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This poster is based on the preprint [PY] that is joint work with Pierre-Guy Plamondon (Université Paris-Saclay).

g-vector fan

- Λ : a finite dimensional algebra over an algebraic closed field k.
- $K^b(\operatorname{proj}\Lambda)$: the homotopy category of bounded complexes of finitely generated projective Λ-modules with shift functor Σ.
- $K^{[-1,0]}(\operatorname{proj}\Lambda)$: the full subcategory of $K^b(\operatorname{proj}\Lambda)$ whose objects are complexes concentrated in degrees -1 and 0, i.e. $P^{-1} \to P^0$.

An object $P \in K^{[-1,0]}(\operatorname{proj}\Lambda)$ is presilting if $\operatorname{Hom}_{K^b(\operatorname{proj}\Lambda)}(P, \Sigma P) = 0$. It is silting if, moreover, it generates $K^b(\operatorname{proj}\Lambda)$. We denote by 2-silt Λ the set of isomorphism classes of basic silting objects in $K^{[-1,0]}(\operatorname{proj}\Lambda)$

- $K_0(\operatorname{proj}\Lambda)$: the Grothendieck group of $K^b(\operatorname{proj}\Lambda)$
- $[X]$: the image of an object X in $K_0(\operatorname{proj}\Lambda)$

It is well-known that $K_0(\operatorname{proj}\Lambda)$ is a free abelian group, i.e. $K_0(\operatorname{proj}\Lambda) \simeq \mathbb{Z}^n$.

The **g**-vector of $P \in K^{[-1,0]}(\operatorname{proj}\Lambda)$ is $[P] \in K_0(\operatorname{proj}\Lambda) \simeq \mathbb{Z}^n$.

Theorem ([AIR])

*There is a simplicial polyhedral fan $\mathcal{F}^{\mathbf{g}}(\Lambda)$, called **g**-vector fan of Λ, whose*

- *ray is spanned by the **g**-vector of an indecomposable presilting object of $K^{[-1,0]}(\operatorname{proj}\Lambda)$;*
- *maximal cone is a positive cone spanned by $[S_1], \dots, [S_n]$ for $\bigoplus_{i=1}^n S_i \in$ 2-silt Λ.*

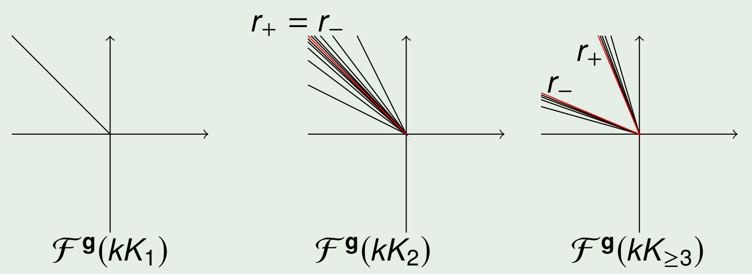
The fan $\mathcal{F}^{\mathbf{g}}(\Lambda)$ is identified with its geometric realization, i.e. $\mathcal{F}^{\mathbf{g}}(\Lambda) \subseteq \mathbb{R}^n$.

Example (g-vector fans)

Let $m \in \mathbb{Z}_{\geq 1}$ and K_m be an m-Kronecker quiver, i.e.

$$K_m := [\, 1 \overset{m}{\rightrightarrows} 2 \,].$$

In particular, K_1 is of type A_2 and K_2 is a Kronecker quiver. The **g**-vector fan $\mathcal{F}^{\mathbf{g}}(kK_m)$ of the path algebra kK_m is well known as follows:



For $m \geq 2$, $\mathcal{F}^{\mathbf{g}}(kK_m)$ contains infinitely many rays converging to the rays $r_\pm$. If $m = 2$, then $r_+ = r_-$. If $m \geq 3$, then $r_+ \neq r_-$ and the interior of the cone spanned by r_+ and r_- is the complement of the closure $\overline{\mathcal{F}^{\mathbf{g}}(kK_m)}$.

A complete list of indecomposable presilting objects in $K^{[-1,0]}(\operatorname{proj} kK_2)$ is given by

$$\Sigma P_1,\ \Sigma P_2,\ P_1,\ H_\pm^m = (P_1^{m\pm 1} \to P_2^m)\ (m \in \mathbb{Z}_{\geq 1})$$

whose **g**-vectors are $(-1,0), (0,-1), (1,0), (-m \mp 1, m)$, respectively. A maximal cone spanned by two adjacent **g**-vectors corresponds to a silting object, and their adjacent cones (or the corresponding silting objects) are related by a "mutation".

It was proved by [A, DIJ] that the following are equivalent:
(1) $\mathcal{F}^{\mathbf{g}}(\Lambda) = \mathbb{R}^n$; (2) $\#$ 2-silt $\Lambda < \infty$.
This naturally leads to study the algebras Λ satisfying $\overline{\mathcal{F}^{\mathbf{g}}(\Lambda)} = \mathbb{R}^n$.

Main result

The algebra Λ is tame if for any dimension vector $\mathbf{d}$, there are $k[t]$-Λ-bimodules $M_1, \dots, M_{m(\mathbf{d})}$ such that

(1) each M_i is free of finite rank as a $k[t]$-module;
(2) all but finitely many indecomposable Λ-modules of dimension vector $\mathbf{d}$ have the form $k[t]/(t-\lambda) \otimes_{k[t]} M_i$ with $i \in \{1, \dots, m(\mathbf{d})\}$ and $\lambda \in k$.

Main theorem

Tame algebras Λ satisfy $\overline{\mathcal{F}^{\mathbf{g}}(\Lambda)} = \mathbb{R}^n$.

Remark that there is a non-tame algebra Λ satisfying $\overline{\mathcal{F}^{\mathbf{g}}(\Lambda)} = \mathbb{R}^n$.

Sketch of the proof

Main tool

For $U, X \in K^b(\operatorname{proj}\Lambda)$, we choose a basis $(f_1, \dots, f_d)$ of the space $\operatorname{Hom}_{K^b(\operatorname{proj}\Lambda)}(U, X)$ and a triangle

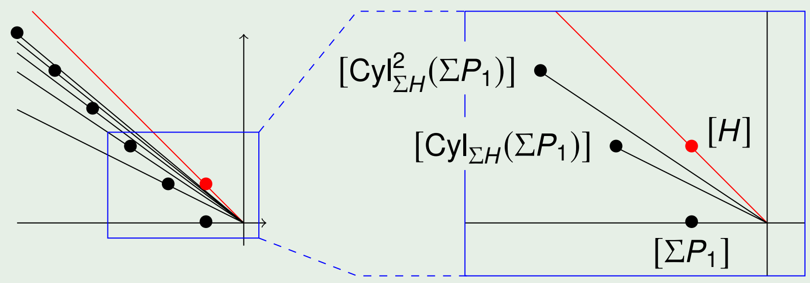
$$\Sigma^{-1}X^d \to \operatorname{Cyl}_X U \to U \xrightarrow{f} X^d, \text{ where } f = [f_1 \cdots f_d]^{\mathrm{T}}.$$

The object $\operatorname{Cyl}_X U$ is the cylinder of U with respect to X.

Example (Idea of the proof)

We consider the path algebra kK_2 of a Kronecker quiver K_2. The half line r_+ is not in $\mathcal{F}^{\mathbf{g}}(kK_2)$. But we can show that it's in $\overline{\mathcal{F}^{\mathbf{g}}(kK_2)}$ as follows: We only need to prove $(-1,1) \subset \overline{\mathcal{F}^{\mathbf{g}}(kK_2)}$. There is a non-presilting object $H = (P_1 \to P_2)$ with $[H] = (-1,1)$. Then it is easy to get $\operatorname{Cyl}_{\Sigma H}^m(\Sigma P_1) = H_+^m$. Since $\operatorname{Cyl}_{\Sigma H}^m(\Sigma P_1)$ is presilting and $[\operatorname{Cyl}_{\Sigma H}^m(\Sigma P_1)] = (-m-1, m)$,

$$(-1,1) \in \overline{\bigcup_{m \geq 1} \mathbb{R}_{\geq 0}[\operatorname{Cyl}_{\Sigma H}^m(\Sigma P_1)]} \subseteq \overline{\mathcal{F}^{\mathbf{g}}(kK_2)}.$$



Let Λ be a tame algebra. We need to show that any $\mathbf{g} \in K_0(\operatorname{proj}\Lambda)$ is contained in $\overline{\mathcal{F}^{\mathbf{g}}(\Lambda)}$. In this case, $\mathbf{g}$ has a decomposition [GLS]

$$\mathbf{g} = \mathbf{g}' + a_1\mathbf{h}_1 + \dots + a_r\mathbf{h}_r,$$

where $a_i \in \mathbb{Z}_{>0}$, $\mathbf{h}_i \neq \mathbf{h}_j$ and

(1) there is a presilting object G in $K^{[-1,0]}(\operatorname{proj}\Lambda)$ with $[G] = \mathbf{g}'$;
(2) there is a non-presilting object H_i in $K^{[-1,0]}(\operatorname{proj}\Lambda)$ with $[H_i] = \mathbf{h}_i$ satisfying some properties (e.g. $\mathrm{H}^0(H_i)$ is a brick);
(3) $\operatorname{Hom}_{K^b(\operatorname{proj}\Lambda)}(X, \Sigma Y) = 0$ for $X, Y = G$ or H_i.

Let G' be the *Bongartz co-completion* of G, defined by the triangle

$$\Lambda \xrightarrow{f} G'' \to G' \to \Sigma\Lambda,$$

where f is a left $(\operatorname{add} G)$-approximation of Λ. Then $G \oplus G' \in$ 2-silt Λ.

Lemma

For $d, m_1, \dots, m_s \in \mathbb{Z}_{>0}$, the object $G'' = G^d \oplus \operatorname{Cyl}_{\Sigma H_s}^{m_s} \cdots \operatorname{Cyl}_{\Sigma H_1}^{m_1} G'$ is presilting in $K^{[-1,0]}(\operatorname{proj}\Lambda)$ with $[G''] = d[G] + [G'] + \sum_{i=1}^s m_i d_i [H_i]$, where $d_i = \dim \operatorname{Hom}_{K^b(\operatorname{proj}\Lambda)}(G', \Sigma H_i)$.

Taking $d = md_1 \cdots d_s$ and $m_i = a_i d / d_i$ for any $m \in \mathbb{Z}_{>0}$, Lemma gives a presilting object G'' with $[G''] = md\mathbf{g} + [G']$. On the other hand,

$$\mathbf{g} \in \overline{\bigcup_{m \geq 1} \mathbb{R}_{\geq 0}(md\mathbf{g} + [G'])}.$$

Therefore, $\mathbf{g} \in \subseteq \overline{\mathcal{F}^{\mathbf{g}}(\Lambda)}$. This finishes the proof of the main theorem.

References

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