Compatibility degree of cluster complexes

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Introduction

I will introduce a new function on the set of pairs of cluster variables, which we call the compatibility degree (of cluster complexes). The compatibility degree which is dealt in this poster is a generalization of the "classical" compatibility degree introduced by Fomin and Zelevinsky. Our compatibility degree is defined by using the principal coefficients.

Previous Studies

Definition

- Quiver Mutation:
  Let Q be a quiver without loops and 2-cycles and j be a vertex of Q. The *quiver mutation* $\mu_j(Q)$ at j is obtained from Q as follows:
  1) reverse all arrows incident with j.
  2) for each subquiver $i \longrightarrow j \longrightarrow k$, add new arrow $i \longleftarrow k$.
  3) remove all 2-cycles.
- Seed, Cluster:
  A *seed* is a pair $(Q, \mathbf{x})$ of a quiver $Q = (Q_0.Q_1, s, t)$ and a *cluster* $\mathbf{x}$, where
  - Q is a quiver with n vertices;
  - $\mathbf{x} = (x_i)_{i\in Q_0}$ is a free generating sets of the rational function field in n indeterminates.
- Seed Mutation:
  For a vertex j of Q, the *seed mutation* $\mu_j(Q, \mathbf{x})$ is $(Q', \mathbf{x}')$, where
  - $Q' = \mu_j(Q)$;
  - $x'_i = \begin{cases} x_i & \text{if } i \neq j \\ \dfrac{\prod_{i\to j\in Q} x_i^{b_{ij}} + \prod_{j\to k\in Q} x_k^{b_{jk}}}{x_j} & \text{if } i = j, \end{cases}$ where b_{ij} is the number of arrows from i to j.

Example:

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \end{tikzcd} \overset{\mu_2}{\longmapsto} \begin{tikzcd} 1 \arrow[rr, bend left] & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}$$

$$(x_1, x_2, x_3) \longmapsto \left(x_1, \frac{x_1 + x_3}{x_2}, x_3\right)$$

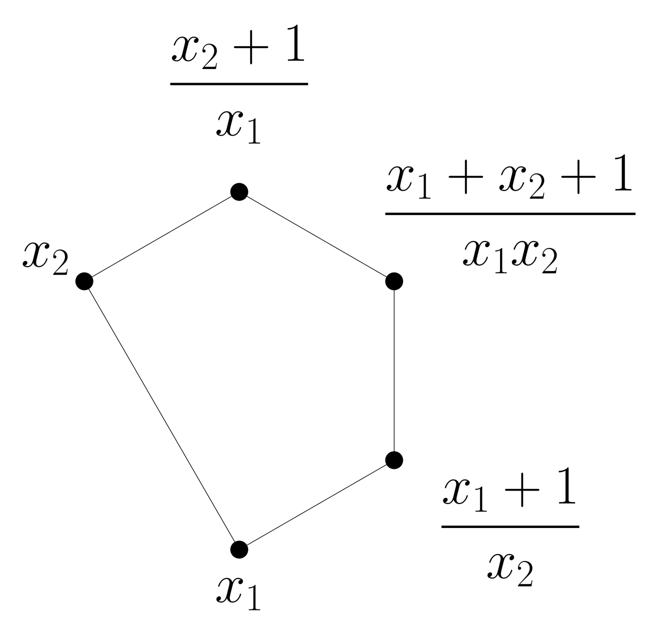
Definition

Let $\mathcal{X}(Q)$ be a set of all cluster variables, which are elements in clusters, obtained by mutations starting from $(Q, \mathbf{x})$. A *cluster complex* $\Delta(Q)$ is the simplicial complex whose ground set is $\mathcal{X}(Q)$ and whose simplices are some elements in a cluster.

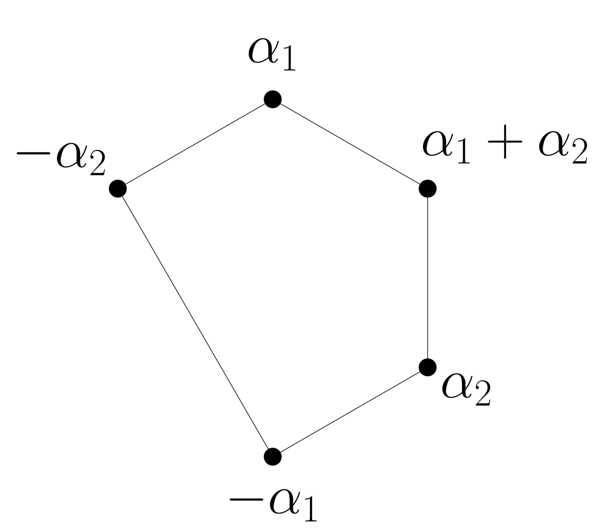
Example: $Q = \ 1 \longrightarrow 2$, $\mathbf{x} = (x_1, x_2)$, Φ: A_2 type, $\Gamma_\Phi = \ 1 \frac{\quad}{} 2$

$$(x_1, x_2) \overset{1}{\mapsto} \left(\tfrac{x_2+1}{x_1}, x_2\right) \overset{2}{\mapsto} \left(\tfrac{x_2+1}{x_1}, \tfrac{x_1+x_2+1}{x_1x_2}\right) \overset{1}{\mapsto} \left(\tfrac{x_1+1}{x_2}, \tfrac{x_1+x_2+1}{x_1x_2}\right) \overset{2}{\mapsto} \left(\tfrac{x_1+1}{x_2}, x_1\right) \overset{1}{\mapsto} (x_2, x_1) \cdots,$$

$\Delta(Q)$



$\Delta(\Phi)$



(For $\Delta(\Phi)$, see below.)

Definition

- Let Φ be a finite root system, $\{\alpha_i\}_{i\in\{1,\dots,n\}}$ simple roots of Φ, and $\Phi_{\geq -1}$ the union of simple negative roots and positive roots of Φ. We define the *(classical) compatibility degree*
  $$(\cdot \parallel \cdot) \colon \Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$$
  as the following condition:
  - $(-\alpha_i \parallel \beta) = (\beta \parallel -\alpha_i) = \max([\beta : \alpha_i], 0)$,
  - $(\alpha \parallel \beta) = (s_\pm(\alpha) \parallel s_\pm(\beta))$ (except for the above case)

  where $[\beta : \alpha_i]$ is the coefficient integer of α_i in the expansion of β, s_+, s_- are compositions of reflections of all simple roots that are not adjacent to each other. In particular, when $(\alpha \parallel \beta) = 0$, we say that α and β are *compatible*.
- We define a *generalized associahedron* $\Delta(\Phi)$ of Φ as a simplicial complex whose simplices are sets of mutually compatible roots in $\Phi_{\geq -1}$.

Theorem (Fomin-Zelevinsky, 2003)

If Q is bipartite quiver and the Dynkin graph $\Gamma(\Phi)$ of Φ is a graph of Q, then $\Phi_{\geq -1} \to \mathcal{X}(Q)$, $\sum d_i\alpha_i \mapsto \dfrac{f(x_1, \dots, x_n)}{\prod x_i^{d_i}}$ *is a bijection, and it induces* $\Delta(\Phi) \simeq \Delta(Q)$.

Motivation

How do we define the compatibility degree of cluster complexes in terms of cluster algebra?

$$\begin{tikzcd} \Phi_{\geq -1} \arrow[r, leftrightarrow, "\text{bijection}"] \arrow[d, "\text{cl. compatibility deg.}"'] & \mathcal{X}(Q) \arrow[d, "\substack{\text{mutation structure} \\ \text{compatibility deg.?}}"] \\ \Delta(\Phi) \arrow[r, leftrightarrow, "\text{isomorphism}"] & \Delta(Q) \end{tikzcd}$$

Our Result (Fu-G., 2019)

For any Q, the "*compatibility degree*" $(\cdot \parallel \cdot) \colon \mathcal{X}(Q) \times \mathcal{X}(Q) \to \mathbb{Z}_{\geq 0}$ has the following property:

- for any cluster variables x, x', we have $(x \parallel x') = (x' \parallel x)$,
- for cluster variables x, x', there exists a cluster $\mathbf{x}$ such that both x and x' is in $\mathbf{x}$ if and only if $(x \parallel x') = (x' \parallel x) = 0$,
- if Q is a bipartite Dynkin quiver, then $(\cdot \parallel \cdot)$ corresponds with the classical compatibility degree through the canonical bijection between $\mathcal{X}(Q)$ and $\Phi_{\geq -1}$.

There are more properties of the compatibility degree which classical one has. In the rest of this poster, we introduce the construction of the compatibility degree.

Definition

For $(Q, \mathbf{x})$, we define $(\overline{Q}, \overline{\mathbf{x}})$ as follows:

- $\overline{Q}$ is a quiver whose vertex set is the union of all vertices $\{1, \dots, n\}$ of Q and their copies $\{1^*, \dots, n^*\}$ and whose arrow set is the union of all arrows of Q and each one arrow from i^* to i for any $i \in \{1, \dots, n\}$.
- $\overline{\mathbf{x}}$ is a $2n$-tuple $(x_1, \dots, x_n, y_1, \dots, y_n)$ of the rational function field in n indeterminates.

We define seeds with the *principal coefficient* at $(Q, \mathbf{x})$ as seeds which is given by mutating at elements in $\{1, \dots, n\}$ repeatedly starting from $(\overline{Q}, \overline{\mathbf{x}})$ (when we mutate $(\overline{Q}, \overline{\mathbf{x}})$, we regard y_i as x_{i^*}).

Definition

For $x, x' \in \mathcal{X}(Q)$,

(1) Take seeds $(Q_t, \mathbf{x}_t)$ containing cluster variable x and $(Q_s, \mathbf{x}_s)$ containing cluster variable x'. We assume $x = x_{i;t}$ and $x' = x_{j;s}$.

(2) Calculate the expansion of $x' = x_{j;s}$ by $\mathbf{x}_t = \{x_{1;t}, \dots, x_{n;t}\}$. Here, we take the principal coefficient at $(Q_t, \mathbf{x}_t)$. We assume
$$x' = \frac{y_1^{f_1} \cdots y_n^{f_n} x_{1;t}^{a_1} \cdots x_{n;t}^{a_n} + \cdots}{x_{1;t}^{d_1} \cdots x_{n;t}^{d_n}}.$$

(3) Define the *compatibility degree* of x and x' as $(x \parallel x') = f_i$.

Example: $Q = \ 1 \longrightarrow 2$, $\mathbf{x} = (x_1, x_2)$. How to calculate $\left(\frac{x_2+1}{x_1} \middle\| \frac{x_1+1}{x_2}\right)$:

$$\begin{array}{ccccccc} 1 \longrightarrow 2 & \overset{\mu_1}{\mapsto} & 1 \longleftarrow 2 & \overset{\mu_2}{\mapsto} & 1 \longrightarrow 2 & \overset{\mu_1}{\mapsto} & 1 \longleftarrow 2 \\ (x_1, x_2) & \mapsto & \left(\frac{x_2+1}{x_1}, x_2\right) & \mapsto & \left(\frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1x_2}\right) & \mapsto & \left(\frac{x_1+1}{x_2}, \frac{x_1+x_2+1}{x_1x_2}\right) \end{array}$$

$$1 \longrightarrow 2 \ \overset{\mu_1}{\mapsto} \ \begin{tikzcd} 1 & 2 \arrow[l] \\ 1^* \arrow[u] & 2^* \arrow[u] \end{tikzcd} \ \overset{\mu_2}{\mapsto} \ \begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 1^* \arrow[u] & 2^* \arrow[ul] \end{tikzcd} \ \overset{\mu_1}{\mapsto} \ \begin{tikzcd} 1 \arrow[d] \arrow[dr] & 2 \arrow[l] \\ 1^* \arrow[ur] & 2^* \end{tikzcd}$$

$$(x'_1, x'_2) \ \mapsto \ \left(x'_1, \frac{y_2 + x'_1}{x'_2}\right) \ \mapsto \ \left(\frac{y_1 y_2 x'_2 + y_2 + x_1}{x'_1 x'_2}, \frac{y_2 + x'_1}{x'_2}\right)$$

Therefore, we have $\left(\frac{x_2+1}{x_1} \middle\| \frac{x_1+1}{x_2}\right) = 1$. On the other hand, the classical one of corresponding roots is

$$(\alpha_1 \parallel \alpha_2) = (s_1(-\alpha_1) \parallel s_1(\alpha_1 + \alpha_2)) = (-\alpha_1 \parallel \alpha_1 + \alpha_2) = 1.$$

Therefore, the compatibility degree corresponds with classical one.

Remark

In [FG19], cluster algebras of *skew-symmetrizable* type are also dealt, and the compatibility degree of cluster complexes of skew-symmetrize type is also given like as in the same way as of quiver type.

References

[FZ] Fomin, S., Zelevinsky, A., *Cluster algebras II: Finite type classification*. Invent. math. 154, 63–121 (2003)
[FG19] Fu, C., Gyoda, Y., *Compatibility degree of cluster complexes*. arXiv:1911.07193.