Simple-minded systems in cluster categories and singularity categories

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Simple-minded systems in cluster categories

In this poster, we study a class of simple-like objects called simple-minded systems in cluster categories and singularity categories.

Simple-like objects

Derived categories	Simple-minded collections
Cluster categories	Simple-minded systems

k: algebraically closed field. A : finite dimensional k-algebra.

In derived category $\mathcal{D}^{\mathrm{b}}(A)$, the set S_A of simple A-modules satisfies:
- (Schur's lemma) $\dim_k \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(A)}(X,Y) = \delta_{X,Y}, \forall\, X,Y \in S_A$.
- (negative extension vanishing) $\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(A)}(X,Y[-i]) = 0, \forall\, X,Y \in S_A$ and $i \geq 1$.
- (generating condition) $\mathcal{D}^{\mathrm{b}}(A) = \mathsf{Filt}(S_A[-j] \mid j \in \mathbb{Z})$.

A set S in $\mathcal{D}^{\mathrm{b}}(A)$ satisfies the conditions above is called a simple-minded collection (or SMC).

The notion of simple-minded system is analogous to SMC. It has been study by Riedtmann, Koenig-Liu, Dugas, Coelho Simões-Pauksztello, ...

$\mathcal{T}$: a k-linear triangulated category. d: be a positive integer.

A set $S \subset \mathrm{indec}\mathcal{T}$ is called a d-simple-minded system (or d-SMS) $\overset{\text{Def.}}{\Longleftrightarrow}$
- $\dim_k \mathrm{Hom}_{\mathcal{T}}(X,Y) = \delta_{X,Y}, \forall\, X,Y \in S$.
- $\mathrm{Hom}_{\mathcal{T}}(X,Y[-i]) = 0, \forall\, X,Y \in S$ and $d-1 \geq i \geq 1$.
- $\mathcal{T} = \mathsf{Filt}(S[-j] \mid 0 \leq j \leq d-1)$.

Proposition [Riedtmann]

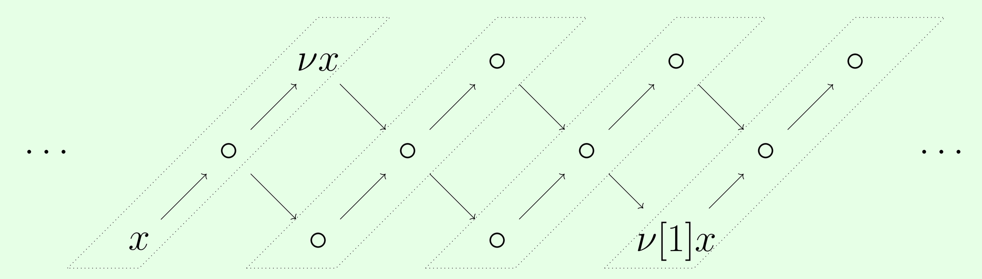
A: finite-dimensional self-injective k-algebra $\Longrightarrow$
- $\{$simple A-modules$\}$ is a 1-SMS in the singularity category $\mathcal{D}_{\mathrm{sg}}(A)$.

Remark. *d-SMSs often appear naturally in $(-d)$-Calabi-Yau (CY) triangulated categories.*

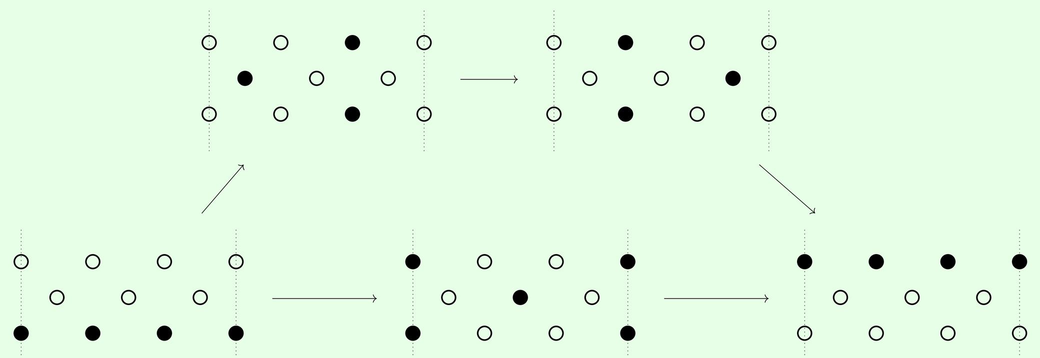
- The $(-d)$-cluster category $\mathcal{C}_{-d}(kQ)$ of kQ is defined as the orbit category $\mathcal{D}^{\mathrm{b}}(kQ)/\nu[d]$ for a Dynkin quiver Q, where ν is the Serre functor.
- $\mathcal{C}_{-d}(kQ)$ is a $(-d)$-CY triangulated category by Keller [K].

Example

Let $Q := 1 \to 2 \to 3$.
- The Auslander-Reiten (AR) quiver of $\mathcal{D}^{\mathrm{b}}(kQ)$ is $\mathbb{Z}Q$.



- The AR quiver of $\mathcal{C}_{-1}(kQ)$ is the residue quiver $\mathbb{Z}Q/\nu[1]$. And there are five 1-SMSs in $\mathcal{C}_{-1}(kQ)$:



Theorem A [IJ]

The number of d-SMSs in $\mathcal{C}_{-d}(kQ)$ is the positive Fuss-Catalan number:

$$C_d^+(W) := \prod_{i=1}^{n} \frac{dh + e_i - 1}{e_i + 1},$$

where n is the rank of W, h is its Coxeter number, and $e_1, \ldots, e_n$ are its exponents. The following tables give the specific formula of $C_d^+(W)$.

Q	A_n	D_n	E_6
$C_d^+(W)$	$\frac{1}{n+1}\binom{(d+1)n+d-1}{n}$	$\frac{(2d+1)n-2d-2}{n}\binom{(n-1)(d+1)-1}{n-1}$	$\frac{d(2d+1)(3d+1)(4d+1)(6d+5)(12d+7)}{30}$

Q	E_7	E_8
$C_d^+(W)$	$\frac{d(3d+1)(3d+2)(9d+2)(9d+4)(9d+5)(9d+8)}{280}$	$\frac{d(3d+1)(5d+1)(5d+2)(5d+3)(15d+8)(15d+11)(15d+14)}{1344}$

d-self-injective differential graded (dg) algebras

Recall that for a finite dimensional self-injective algebra, Proposition shows that simple modules forms a 1-SMS in singularity category. We introduce d-self-injective dg algebra to generalize it.

Let A be a dg k-algebra and $d \geq 1$. $DA := \mathrm{Hom}_k(A,k)$.

- A: d-self-injective $\overset{\text{Def.}}{\Longleftrightarrow}$ $A^{>0} = 0, \dim \mathrm{H}^{\bullet}(A) < \infty$ and $A \cong DA[d-1]$ in $\mathcal{D}(A)$.
- The singularity category $\mathcal{D}_{\mathrm{sg}}(A)$ is defined as $\mathcal{D}^{\mathrm{b}}(A)/\,\mathsf{per}(A)$.

Example

$A = k[X]/(X^{n+1}), n \geq 1$: dg k-algebra with $\deg X = -d \leq 0$ and 0 differential $\Longrightarrow$
- A is $(nd+1)$-self-injective.

The natural morphism $A \to \mathrm{H}^0(A)$ of dg algebras induces a fully faithful functor $\mathrm{mod}\,\mathrm{H}^0(A) \to \mathcal{D}^{\mathrm{b}}(A)$. We define simple dg A-modules as the image of simple $\mathrm{H}^0(A)$-modules (concentrated in degree 0).

Theorem B [J1]

A: d-self-injective dg k-algebra $\Longrightarrow$
- $\{$simple dg A-modules$\}$ is a d-SMS in $\mathcal{D}_{\mathrm{sg}}(A)$.

Let A be a d-self-injective dg k-algebra with $A \cong DA[d-1]$ in $\mathcal{D}(A \otimes_k A^{\mathrm{op}})$. Then $\mathcal{D}_{\mathrm{sg}}(A)$ is $(-d)$-CY and this category is often equivalent to a cluster category.

Cluster categories and singularity categories

Q : Dynkin quiver. $A = kQ \oplus D(kQ)[d-1], d \geq 1$: trivial extension dg k-algebra with 0 differential $\Longrightarrow$
- $\mathcal{D}_{\mathrm{sg}}(A) \simeq \mathcal{C}_{-d}(kQ)$ by Keller [K].

Our main theorem gives a converse of Theorem B in the following sense.

Main Theorem [J1]

C: d-SMS in $\mathcal{C}_{-d}(kQ) \Longrightarrow$
- $\exists$ a d-self-injective dg k-algebra A and a triangle equivalence $F : \mathcal{D}_{\mathrm{sg}}(A) \xrightarrow{\simeq} \mathcal{C}_{-d}(kQ)$ such that $\{$simple dg A-modules$\} = C$.

In the proof of Main Theorem, we need the following reduction process introduced in [J2].

Simple-minded reductions of triangulated categories

$\mathcal{T}$: Krull-Schmidt triangulated category. R : pre-SMC (SMC without generating condition) of $\mathcal{T}$. The SMC reduction of $\mathcal{T}$ w.r.t R is the Verdier quotient $\mathcal{U} = \mathcal{T}/\,\mathsf{thick}(R)$.

Example

$Q := 1 \to 2 \to 3$. $A := kQ \oplus D(kQ)[d] \Longrightarrow$
- $\{S_1\}$ is a pre-SMC of $\mathcal{D}^{\mathrm{b}}(A)$.
- $\mathcal{U} = \mathcal{D}^{\mathrm{b}}(A)/\,\mathsf{thick}(S_1)$ is triangle equivalent to $\mathcal{D}^{\mathrm{b}}(A')$, where $A' = kQ' \oplus \mathcal{D}(kQ')[d]$ with $Q' = 2 \to 3$.

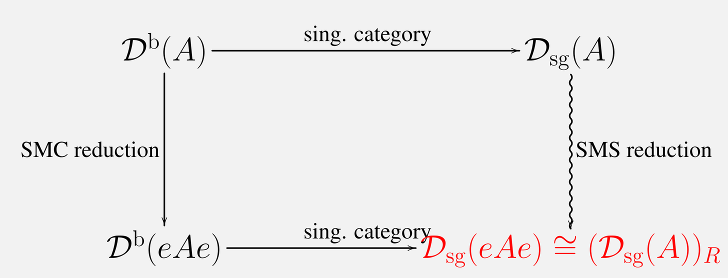
Basic property of SMC reduction [J2]

Under mild conditions, there is a bijection

$$\{\text{SMCs in } \mathcal{T} \text{ contain } R\} \overset{1:1}{\longleftrightarrow} \{\text{SMCs in } \mathcal{U}\}.$$

- Coelho Simões and Pauksztello introduced SMS reduction in negative CY triangulated category.
- SMS reduction is the shadow of SMC reduction.

$A = kQ \oplus D(kQ)[d]$ for a Dynkin quiver A. R : simple dg A-module $\Longrightarrow$
There exists an idempotent $e \in A$ such that the following maps commute.



This diagram plays an important role in the proof of Main Theorem.

References

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