Cluster categories from Calabi-Yau algebras

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Subject: Cluster category

A differential graded (dg) algebra Λ over a field k is *n-Calabi-Yau (CY)* if
- Λ is homologically smooth, that is, $\Lambda \in \operatorname{per}\Lambda^e$ where $\Lambda^e = \Lambda^{op} \otimes_k \Lambda$.
- There is an isomorphism $\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)[n] \simeq \Lambda$ in $\mathcal{D}(\Lambda^e)$.

Theorem-Definition (Amiot, [1])

Let Λ be a $(d+1)$-CY dg algebra concentrated in (cohomological) degree ≤ 0 such that $H^0\Lambda$ is finite dimensional, and set

$$\mathcal{C}(\Lambda) := \operatorname{per}\Lambda/\mathcal{D}^b(\Lambda).$$

Then this category is d-CY, and $\Lambda \in \mathcal{C}(\Lambda)$ is a d-cluster tilting object.
We call $\mathcal{C}(\Lambda)$ the cluster cateory of Λ.

Note:
- $\operatorname{per}\Lambda$ =perfect derived category of Λ.
- $\mathcal{D}^b(\Lambda) = \{$dg Λ-modules of finite dimensional total cohomology$\}$.
- $\mathcal{D}^b(\Lambda) \subset \operatorname{per}\Lambda$ by smoothness of Λ.

Popular examples (Keller, [3])

- The Calabi-Yau completion (or the derived preprojective algebra) $\mathbf{\Pi}_{d+1}(A)$ is $(d+1)$-CY.
  - attatched to a (τ_d-finite, finite dimensional) algebra A.
  - The associated cluster category $\mathcal{C}(\mathbf{\Pi}_{d+1}(A)) =: \mathcal{C}_d(A)$ is the d-cluster category of A; d-CY with a d-cluster tilting object. There is an embedding $\mathcal{D}^b(\operatorname{mod} A)/\nu_d \hookrightarrow \mathcal{C}_d(A)$ for $\nu_d = -\otimes^L_A DA[-d]$.
- The Ginzburg dg algebra $\Gamma(Q, W)$ is 3-CY.
  - attatched to a (Jacobi-finite) quiver with potential
  - The associated cluster category $\mathcal{C}(\Gamma(Q,W)) =: \mathcal{C}_{(Q,W)}$ is cluster category for quiver with potential; 2-CY with a 2-cluster titling object.

The definition of general cluster category is somehow abstract.

Aim

Give some descriptions of $\mathcal{C}(\Lambda)$ for some class of CY dg algebras Λ.

Results

We first construct examples of CY dg algebras from ordinary (non-dg) CY algebras.
Recall: a graded (non-dg) algebra R is *n-CY of a-invariant a* if
- R is homologically smooth, that is, $R \in \operatorname{per}^{\mathbb{Z}} R^e$.
- There is an isomorphism $\mathrm{RHom}_{R^e}(R, R^e)(a)[n] \simeq R$ in $\mathcal{D}(\operatorname{Mod}^{\mathbb{Z}} R^e)$.

Example (of non-dg Calabi-Yau algebras)

- Polynomial rings.
- Skew group rings.
- Preprojective algebras.
- Jacobian algebras

We view a graded algebra R as a dg algebra R^{dg} with 0 differentials.

Proposition

Let R be a graded n-CY algebra of a-invariant a. Then R^{dg} is sign-twisted $(n+a)$-CY.

Example

Let $R = k[x_0, \dots, x_d]$ with $\deg x_i = -a_i < 0$.
- As a graded algebra, this is $(d+1)$-CY of a-invariant $a = \sum_{i=0}^d a_i$.
- As a dg algebra R^{dg}, this is $(d+a+1)$-CY.

Toward our main result,

Input

A negatively graded algebra

$$R = \bigoplus_{i \le 0} R_i$$

which is $(d+1)$-CY of a-invariant a, and each R_i is finite dimensional.

An important category associated to a graded algebra R is

$$\operatorname{qgr} R := \operatorname{mod}^{\mathbb{Z}} R/\operatorname{fl}^{\mathbb{Z}} R = \{\text{finitely generated graded modules}\}/\{\text{finite length graded modules}\}.$$

Remark:
- When R is generated in degree -1, then $\operatorname{qgr} R = \operatorname{coh}(\operatorname{Proj} R)$, the category of coherent sheaves over the projective scheme $\operatorname{Proj} R$.
- $\operatorname{qgr} R$ is therefore called the *non-commutative projective scheme* (Artin–Zhang).

Define

$$\begin{aligned} T &= R \oplus \cdots \oplus R(-(a-1)) &&: \text{tilting bundle on } \operatorname{qgr} R \text{ (see below)},\\ A &= \operatorname{End}_{\operatorname{qgr} R}(T) &&: \text{finite dimensional algebra},\\ U &= \operatorname{Hom}_{\operatorname{qgr} R}(T, T(-1)) &&: (A,A)\text{-bimodule},\\ B &= A \oplus U &&: \text{trivial extension algebra.}\end{aligned}$$

Notes:
- When $R = k[x_0, \dots, x_d]$ with $\deg x_i = -1$, then $\operatorname{qgr} R = \operatorname{coh}\mathbb{P}^d$ and T is Beilinson's tilting bundle $\mathcal{O} \oplus \cdots \oplus \mathcal{O}(d)$ on $\mathbb{P}^d$. The above T is its non-commutative analogue (Minamoto–Mori, [4]), and we have a triangle equivalence $\mathcal{D}^b(\operatorname{qgr} R) \simeq \mathcal{D}^b(\operatorname{mod} A)$.
- It is easy to see B is *d*-Gorenstein, i.e. $\operatorname{inj.dim} B \le d$ (Minamoto–Yamaura).

Main Theorem

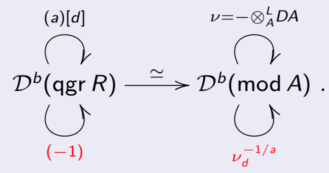
There exists an embedding and a triangle equivalence

$$\mathcal{D}^b(\operatorname{qgr} R)/(-1)[1] \hookrightarrow \mathcal{C}(R^{dg}) \xrightarrow{\simeq} \mathcal{D}_{sg}(B).$$

Proof: The functor taking total complexes $\mathrm{Tot}\colon \mathcal{D}^b(\operatorname{mod}^{\mathbb{Z}} R) \to \operatorname{per} R^{dg}$ induces the first embedding. We use quasi-equivalence of dg orbit categories to show the second equivalence. □

Remark: in terms of finite dimensional algebra A

We can write the orbit category $\mathcal{D}^b(\operatorname{qgr} R)/(-1)[1]$ in terms of A.
- Comparison of Serre functors shows



Therefore we define $\nu_d^{-1/a}$ as the automorphism of $\mathcal{D}^b(\operatorname{mod} A)$ corresponding to (-1) on $\mathcal{D}^b(\operatorname{qgr} R)$, so that $\mathcal{D}^b(\operatorname{qgr} R)/(-1)[1] \simeq \mathcal{D}^b(\operatorname{mod} A)/\nu_d^{-1/a}[1]$.
- We can formally write $\nu_d^{-1/a}[1] = \nu_{d+a}^{-1/a}$, and therefore have an embedding

$$\mathcal{D}^b(\operatorname{mod} A)/\nu_{d+a}^{-1/a} \hookrightarrow \mathcal{C}(R^{dg}).$$

This shows *$\mathcal{C}(R^{dg})$ is a "$\mathbb{Z}/a\mathbb{Z}$-quotient" of the $(d+a)$-cluster category $\mathcal{C}_{d+a}(A)$ of A.*

Examples

We look at polynomial rings $R = k[x_0, \dots, x_d]$ with various gradings.

Easiest Example

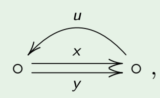
Let

$$R = k[x, y] \text{ with } \deg x = \deg y = -1,$$

which is bimodule 2-CY of a-invariant 2.
- By Proposition, R^{dg} is sign-twisted 4-CY.
- The algebra $A = \operatorname{End}_{\operatorname{qgr} R}(T) = \operatorname{End}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1))$ is the Kronecker quiver

$$\circ \rightrightarrows \circ\ .$$

- The 1-Gorenstein algebra $B = A \oplus U$ is presented by



$xuy = yux,\ uxu = uyu = 0.$

with "commutativity relations" and "$u^2 = 0$".
- By Main Theorem ,there exist equivalences of (twisted) 3-CY categories

$$\mathcal{D}^b(\operatorname{coh}\mathbb{P}^1)/(1)[1] = \mathcal{D}^b(\operatorname{qgr} R)/(-1)[1] \simeq \mathcal{C}(R^{dg}) \simeq \mathcal{D}_{sg}(B).$$

In general we have the following descriptions of A and B.

Proposition

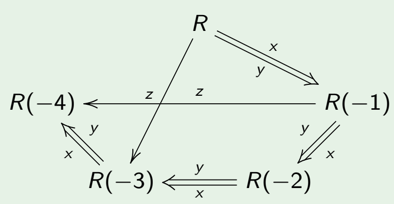
Let $R = k[x_0, \dots, x_d]$ with $\deg x_i = -a_i < 0$ so that R is $(d+1)$-CY of a-invariant $a = \sum_{i=0}^d a_i$. Suppose that $(a_0, \dots, a_d)$ is relatively prime and let G be the cyclic subgroup of $\mathrm{SL}_{d+1}(k)$ generated by $g = \operatorname{diag}(\zeta^{a_0}, \dots, \zeta^{a_d})$ for a primitive a-th root of unity ζ.
1. The finite dimensional algebra A is presented by the quiver which is obtained from the McKay quiver of G by removing the arrows $i \to j$ with $i > j$, where i is the vertex corresponding to the representation $g \mapsto \zeta^i$ $(0 \le i \le a-1)$ of G.
2. The d-Gorenstein algebra B is presented by the quiver obtained by adding to the quiver of A the arrows $u\colon l \to l-1$ for each $1 \le l \le a-1$

Bigger Example

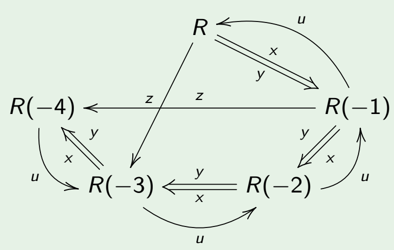
Let

$$R = k[x, y, z] \text{ with } \deg x = -1, \deg y = -1, \deg z = -3,$$

which is bimodule 3-CY of a-invariant 5.
- By Proposition, the dg algebra R^{dg} is 8-CY.
- The finite dimensional algebra $A = \operatorname{End}_{\operatorname{qgr} R}(R \oplus \cdots \oplus R(-4))$ is presented by the quiver below (with the vertex i in the above Proposition corresponding to the summand $R(-i)$)



with commutativity relations.
- The 2-Gorenstein algebra $B = A \oplus U$ is given by



with commutativity relations and $u^2 = 0$.
- By Main Theorem, there exists an embedding and an equivalence of 7-CY categories

$$\mathcal{D}^b(\operatorname{mod} A)/\nu_7^{-1/5} = \mathcal{D}^b(\operatorname{qgr} R)/(-1)[1] \hookrightarrow \mathcal{C}(R^{dg}) \xrightarrow{\simeq} \mathcal{D}_{sg}(B).$$

References

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[4] H. Minamoto and I. Mori, The structure of AS-Gorenstein algebras, Adv. Math. 226 (2011) 4061-4095.