Convexity of Charged Operators in CFTs and the Weak Gravity Conjecture

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String theory provides a large number of different low-energy effective theories which have an ultraviolet completion to quantum gravity

In general, the different theories have different properties, like number of particles, gauge groups, etc..

But there are some (rare) features which, as far as we can tell, are common to all of them

The **Swampland program** is motivated by the existence of such apparently universal features, with an aim to understand if they are always required by any theory of quantum gravity

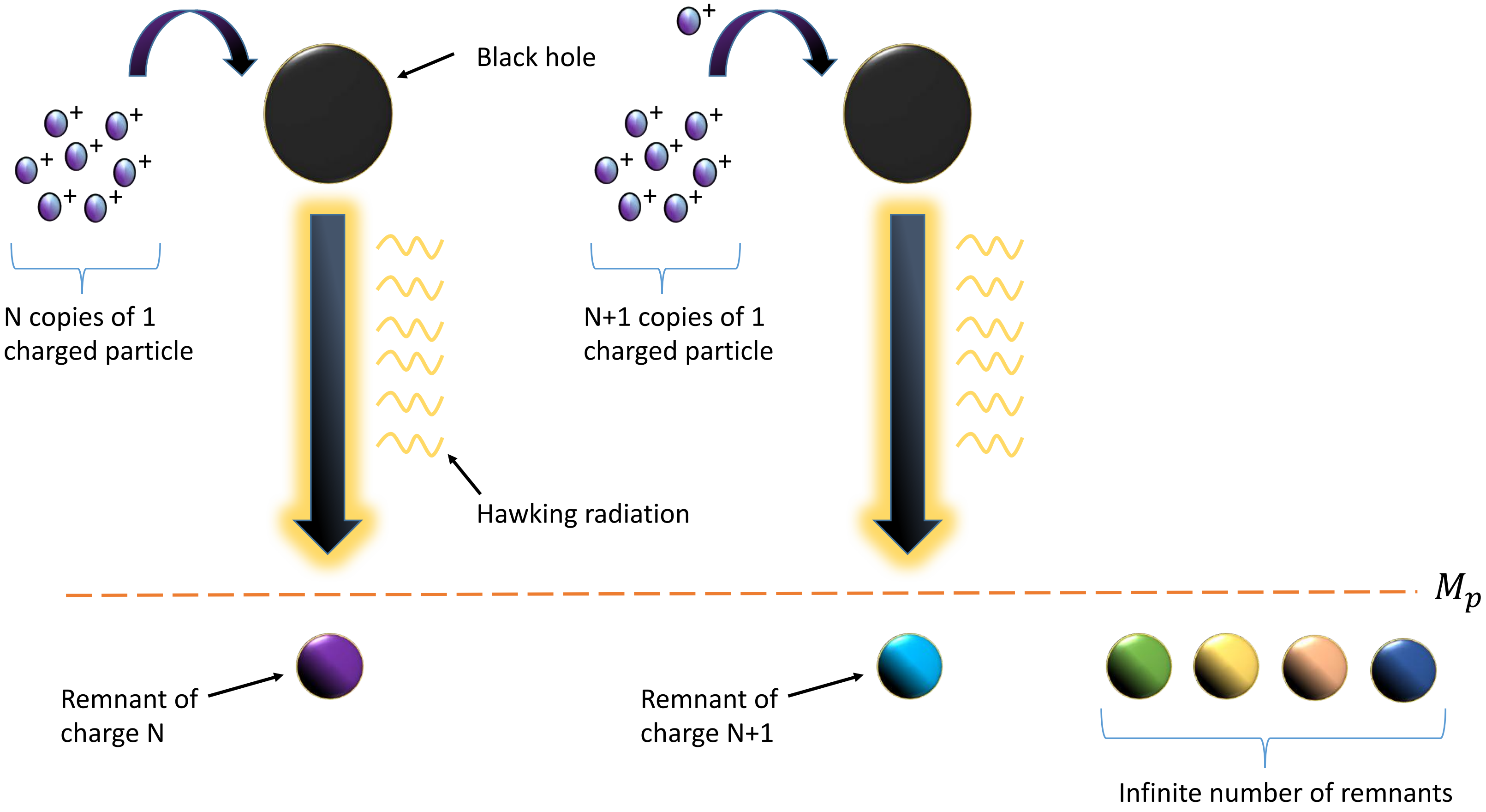
[Vafa ’05]

An old idea is that in quantum gravity there are no U(1) global symmetries

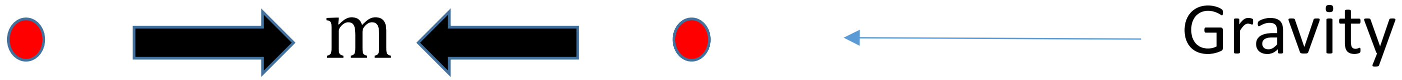
The simplest argument is that a black hole does not reflect any global symmetry charge on its horizon

Therefore, there are an infinite number of microstates associated to a black hole – in contradiction with the expected finite entropy

It is possible to rephrase this argument in terms of an infinite number of stable black hole remnants

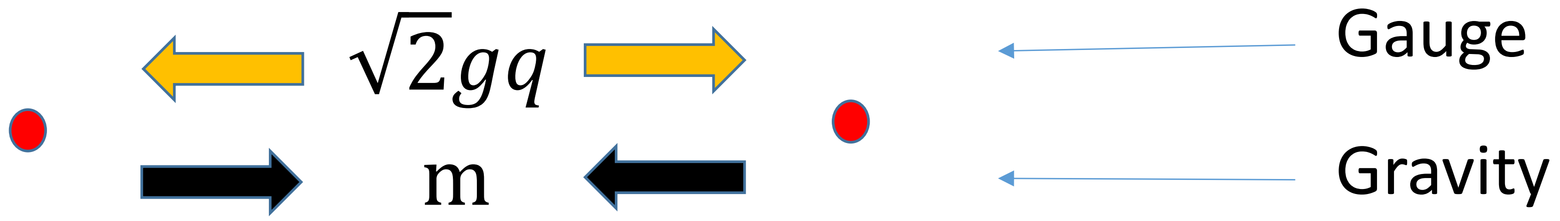


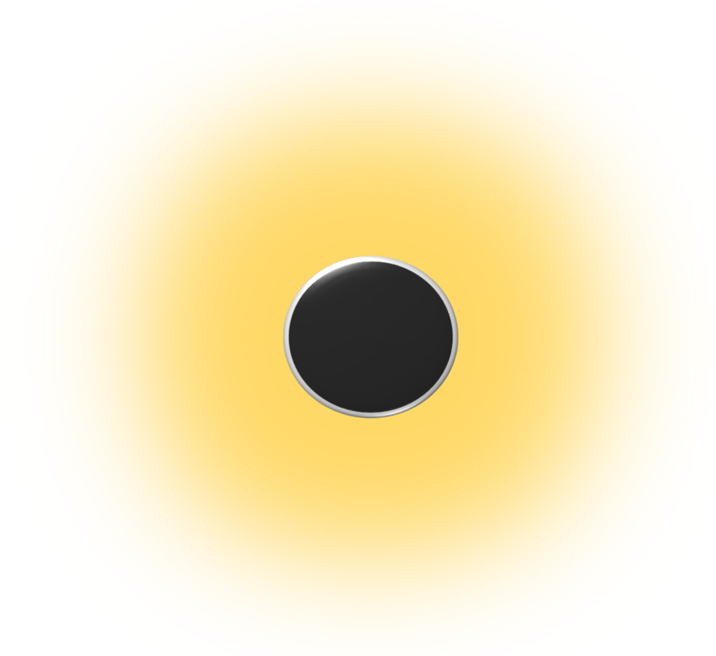
And we can rephrase it again in terms of an infinite number of stable bound states



$$\left(\frac{m}{q}\right)_{\text{Bound}} < \left(\frac{m}{q}\right)_{\text{Particle}} < \left(\frac{m}{q}\right)_{\text{Anything else}}$$

Gauging the U(1) modifies the story





$$m \geq \sqrt{2}\, g\, q\, M_p$$

But we recover the same physics in the $g \to 0$ limit

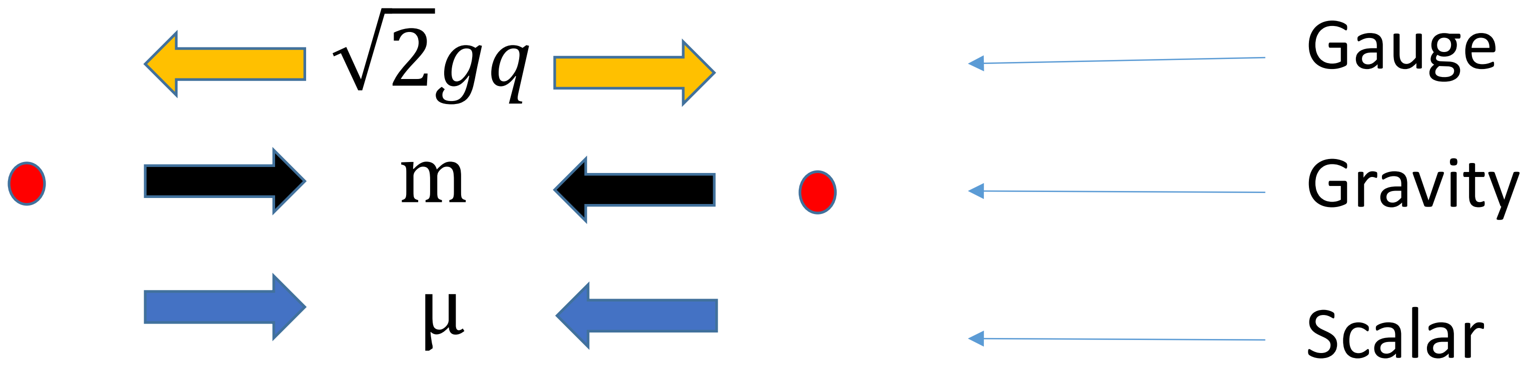
The Weak Gravity Conjecture proposes that we still should not have the stable bound states / stable black holes (not clear that should be true)

[Arkani-Hamed, Motl, Nicolis, Vafa ’06]

If so, then we must have a charged particle with mass smaller than charge

$$\sqrt{2}\, g\, q\, M_p \geq m$$

The logic naturally leads to the expectation that in the case of massless scalar fields we should demand the particle is self-repulsive under all forces

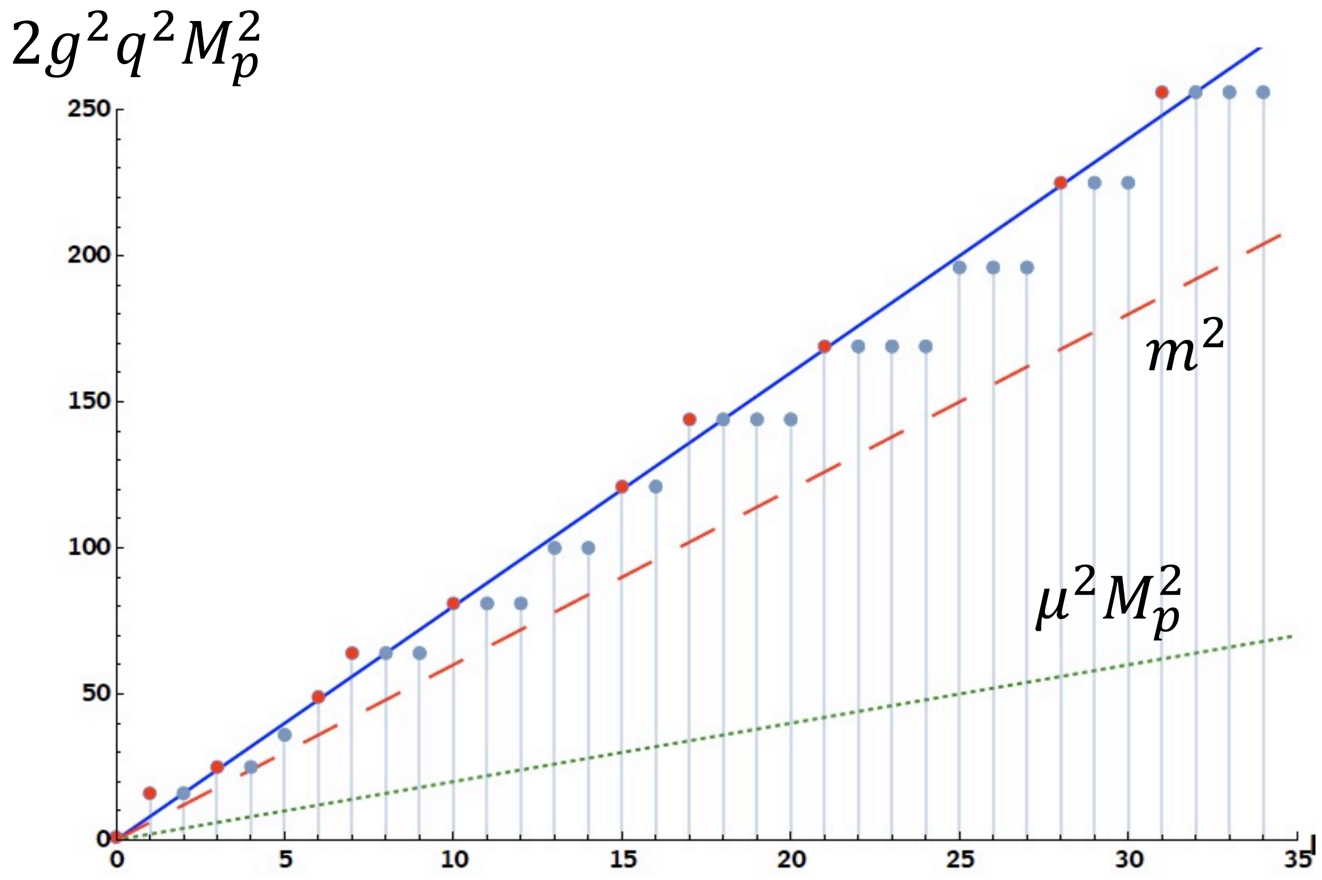


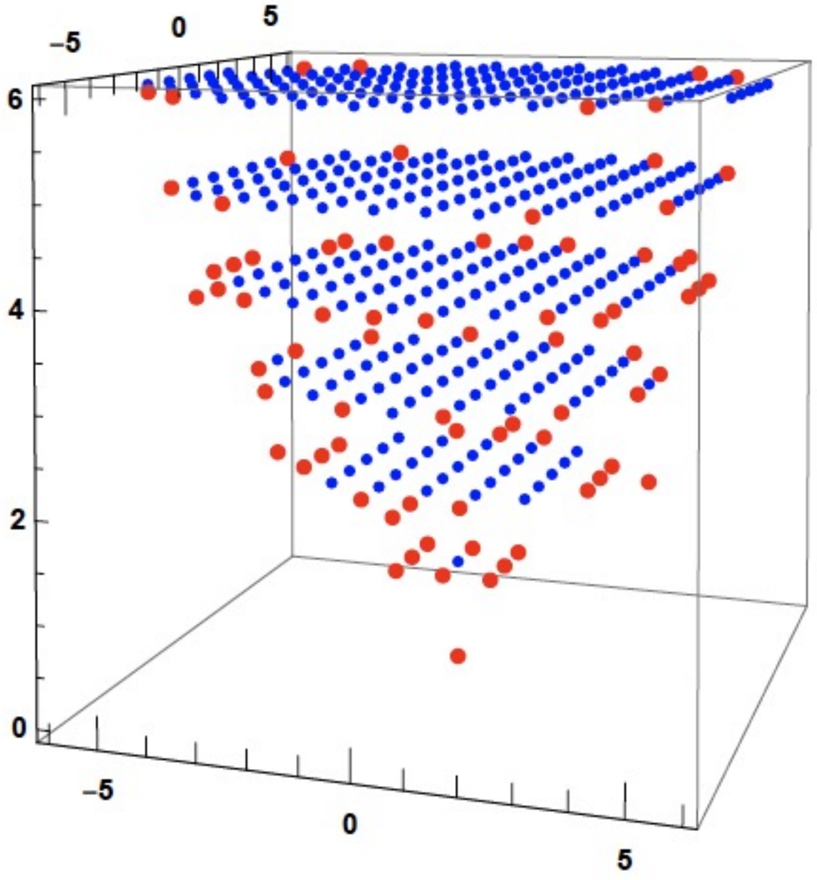
So should formulate the WGC as

$$2g^2q^2M_p^2 \geq m^2 + \mu^2 M_p^2$$

[EP ’17]

Argument receives support from string theory





[..., Lee, Lerche, Weigand '18,...]

No violation found to date (but difficult to test in non-supersymmetric settings)

In anti-de Sitter space (AdS), the separation of a particle from its copy is bounded by the radius

It is more natural/accurate to formulate the conjecture as:

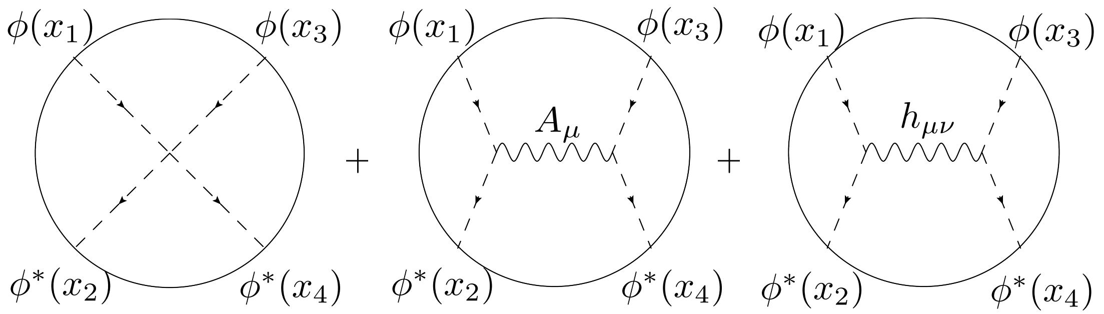
Positive Binding Conjecture: *For a (weakly coupled) gravitational theory with a $U(1)$ gauge field, there should exist at least one charged particle in the theory, with charge of order one, which has a non-negative self-binding energy.*

For example, consider 5-dimensional AdS with charged scalar

$$S = \frac{1}{\kappa^2} \int d^5x \sqrt{-g} \left(\frac{1}{2} R + 6 - \frac{1}{4g^2} F_{\mu\nu}^2 + |D_\mu \varphi|^2 - m^2 |\varphi|^2 - a |\varphi|^4 + b |\varphi|^2 |D_\mu \varphi|^2 \right)$$

Binding energy is calculated (at tree level) as [Fitzpatrick, Shih '11]

$$\gamma_{\varphi^2} = \gamma_{\varphi^2}^{\text{photon}} + \gamma_{\varphi^2}^{\text{graviton}} + \gamma_{\varphi^2}^{\text{quartic}}$$



with:

$$\gamma_{\varphi^2}^{\text{photon}} = \frac{\pi^2 N_\Delta^4 g^2 q^2}{2\Delta - 1} ,$$

$$\gamma_{\varphi^2}^{\text{graviton}} = -\frac{2\pi^2 N_\Delta^4 \Delta^2 (\Delta - 2)}{3(\Delta - 1)(2\Delta - 1)} ,$$

$$\gamma_{\varphi^2}^{\text{quartic}} = \frac{\pi^2 N_\Delta^4 (a + b\Delta(2 - \Delta))}{(\Delta - 1)(2\Delta - 1)} .$$

$$m^2 = \Delta(\Delta - 4) , \quad N_\Delta = \sqrt{\frac{\Delta - 1}{2\pi^2}} .$$

Positive Binding Conjecture demands: $\gamma_{\phi^2} \geq 0$

The binding energy is dual to the anomalous dimension of the operator ϕ^2

$$\gamma_{\phi^2} = \Delta(\phi^2) - 2\Delta(\phi)$$

Where ϕ^2 is the leading operator in the Operator Product Expansion of $\phi \times \phi$

So, if we write the dimension of operators as a function of charge q we have the conjecture:

$$\Delta(2q) - 2\Delta(q) \geq 0$$

We can then formulate this more generally as:

Abelian Convex Charge Conjecture: *Consider any CFT with a* $U(1)$ *global symmetry. Denote by* $\Delta(q)$ *the dimension of the lowest dimension operator of charge* q*. Then this must satisfy a convex-like constraint*

$$\Delta(n_1 q_0 + n_2 q_0) \geq \Delta(n_1 q_0) + \Delta(n_2 q_0) \ , \tag{1.1}$$

for any positive integers n_1*,* n_2*, for some* q_0 *of order one.*

What is q_0? From the bound-states intuition we expect it to be the charge of the operator with the smallest dimension-to-charge ratio in the theory

It is clear that if we define it that way, the conjecture follows almost trivially:

$$\frac{\Delta(n\, q_0)}{n\, q_0} > \frac{\Delta(q_0)}{q_0}$$

So, the non-trivial statement is: $\boldsymbol{q_0}$ **is of order one**

This is capturing the WGC: if the operator could form ‘bound states’ then the minimum dimension-to-charge ratio would occur at large q

Now we make a ‘wild leap’:

Since the formulation seems unrelated to the existence of a weakly-curved gravity dual, we propose that **the conjecture holds for any CFT**

For example, the formulation has no problem handling a gravity dual with an infinite number of massless higher spin fields (it makes no reference to gravitons or gauge fields)

Before testing the conjecture, we formulate a more general version:

Convex Charge Conjecture: *Consider any CFT with a continuous global symmetry group G, and consider a simple factor $G_0 \subset G$. Denote by $\Delta(r)$ the dimension of the lowest dimension operator in the representation r of G. Then, there is always some representation r_0, which is non-trivial in G_0 and has weights of order one, such that the dimensions $\tilde{\Delta}(q) \equiv \Delta(\mathrm{Sym}^q(r_0))$ satisfy a convex-like constraint*

$$\tilde{\Delta}(n_1 + n_2) \geq \tilde{\Delta}(n_1) + \tilde{\Delta}(n_2) \ , \tag{1.2}$$

for any positive integers n_1, n_2.[1]

Tests of the Conjecture

At large charge the spectrum is convex:

$$\Delta(q) \sim A\, q^{\frac{d}{d-1}}$$

[Hellerman, Orlando, Reffert, Watanabe ‘15]
[Monin, Pirtskhalava, Rattazi, Seibold ‘16]
[Alvarez-Gaume, Loukas, Orlando, Reffert ‘16]

For BPS states in supersymmetric theories, or for free scalar theories, the spectrum is marginally convex:

$$\Delta(q) \sim A\, q$$

In 2-dimensional CFTs a U(1) symmetry implies 2 symmetries, each one of which can be gauged to yield a coset CFT, with spectrum

$$\Delta(q) \sim \Delta_{\frac{CFT}{U(1)}} + a\, q^2$$

For free fermionic theories, the spectrum is not even marginally convex if we take q_0 equal to the number of components of the fermion

This is because Pauli's exclusion principle requires derivatives to be inserted in operators with multiple fermions

For example, in 3 dimensions we have the spectrum:

$$\Delta(1) = 1 \qquad \Delta(4) = 6$$
$$\Delta(2) = 2 \qquad \Delta(5) = 8$$
$$\Delta(3) = 4 \qquad \Delta(6) = 10$$

[Komargodski, Mezei, Pal, Raviv-Moshe '21]

Therefore, any small perturbative interaction will maintain convexity

Way out: Large number of fermions

Testing in explicit theories requires some expansion parameter which leads to some sense of weak-coupling.

We can then identify specific fields associated to the operators, and measure convexity by

$$\gamma_{n_1,n_2} \equiv \Delta\left(\phi^{n_1+n_2}\right) - \Big(\Delta\left(\phi^{n_1}\right) + \Delta\left(\phi^{n_2}\right)\Big)$$

Requiring:

$$\gamma_{n_1,n_2} \geq 0$$

The simplest explicit theory to test it in is Wilson-Fisher U(1) in $4-\epsilon$ dimensions

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi + \frac{\lambda}{4}\left(\phi^\dagger \phi\right)^2 \qquad \frac{\lambda_\star}{(4\pi)^2} = \frac{\epsilon}{5} + \frac{3}{25}\epsilon^2 + \mathcal{O}\left(\epsilon^3\right)$$

Not a rigorous test, because the theory contains non-unitary operators

[Hogervorst, Rychkov, van Rees ‘15]

$$\Delta\left(\phi^n\right) = \frac{d-2}{2}n + \frac{\epsilon}{10}n\left(n-1\right) - \frac{\epsilon^2}{100}n\left(2n^2-8n+5\right) + \mathcal{O}\left(\epsilon^3\right)$$

✓ $$\gamma_{n_1,n_2} = \frac{n_1 n_2}{5}\epsilon - \frac{n_1 n_2}{50}\left(3n_1+3n_2-8\right)\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \ ,$$

[Badel, Cuomo, Monin, Rattazzi ‘19]

Convex, and remains convex for any value of (ϵn) for large n

[Cuomo, Komargodski (Private Comm.)]

$O(N)$ (quartic) model in $4-\epsilon$ dimensions:

✓ $$\gamma_{n_1,n_2} = \frac{2n_1n_2}{N+8}\epsilon - \frac{n_1n_2}{(N+8)^3}\epsilon^2\Big(48n_1+48n_2 +6n_1N+6n_2N-16N+N^2-132\Big)+\mathcal{O}\left(\epsilon^3\right)$$

[Jack, Jones '21]

$U(1)$ and $O(N)$ (sextic) model in $3-\epsilon$ dimensions:

✓ $$\gamma_{n_1,n_2} = \left(\frac{\lambda_\star}{8\pi}\right)^2\frac{n_1n_2\left(n_1+n_2-2\right)}{3} - \left(\frac{\lambda_\star}{8\pi}\right)^4\frac{n_1n_2}{72}\Big[8\left(-182+123n_1-32n_1^2+5n_1^3 +123n_2-48n_1n_2+10n_1^2n_2-32n_2^2+10n_1n_2^2+5n_2^3\right) +(16+N)\left(11-9n_1+2n_1^2-9n_2+3n_1n_2+2n_2^2\right)\pi^2\Big]+\mathcal{O}(\lambda_\star^6)\,.$$

✓ $$\gamma_{1,1} = 2\left(\frac{\lambda_\star}{8\pi}\right)^4+\mathcal{O}(\lambda_\star^6)$$

[Badel, Cuomo, Monin, Rattazzi '19]
[Jack, Jones '21]

Can also analyse theories using large N expansion

$O(N)$ (quartic) model in 3 dimensions:

✓ $$\Delta(q) = N\left[\frac{q}{2N} + \frac{4}{\pi^2}\left(\frac{q}{N}\right)^2 + \mathcal{O}\left(\left(\frac{q}{N}\right)^3\right)\right]$$

[Alvarez-Gaume, Orlando, Reffert ‘19]
[Giombi, Hyman ‘20]

$O(N)$ (quartic) model in 5 dimensions (non-unitary):

✗ $$\Delta(q) = N\left[\frac{3q}{2N} - \frac{32}{3\pi^2}\left(\frac{q}{N}\right)^2 + \mathcal{O}\left(\left(\frac{q}{N}\right)^3\right)\right]$$

[Giombi, Hyman ‘20]
[Giombi, Huang, Klebanov, Pufu, Tarnopolsky ‘19]
[Arias-Tamargo, Rodriguez-Gomez, Russo ‘20]

$O(N)$ (quartic) model in $6-\epsilon$ dimensions (non-unitary):

✗ $$\gamma_{n_1,n_2} = -264\epsilon\frac{n_1 n_2}{N^2} + \mathcal{O}\left(\left(\frac{\epsilon n}{N}\right)^2\right)$$

[Arias-Tamargo, Rodriguez-Gomez, Russo ‘20]
[Antipin,Bersini,Sannino,Wang,Zhang ’21]

Gauge theories, with perturbative approach:

4-dimensional $SU(N_c)$ gauge theory with N_f massless fermions and N_s massless scalars has a perturbative (Banks-Zaks type) fixed point at large N

This needs to coincide with the fixed point of the quartic scalar terms

$$V(\phi) = \tilde{h}\mathrm{Tr}(\phi^\dagger\phi\phi^\dagger\phi) + \tilde{f}(\mathrm{Tr}(\phi^\dagger\phi))^2$$

[Hansen, Janowski, Langaeble, Mann, Sannino, Steele, Wang ’17]
[Benini, Iossa, Serone ‘19]

Scalar mesons are charged under $SU(N_s)$ global symmetry $\qquad \mathcal{O}_n^\phi = (\phi^*\phi)^n$

Find (at 1-loop): $\gamma_{1,1}^\phi = \Delta\left(\mathcal{O}_2^\phi\right) - 2\Delta\left(\mathcal{O}_1^\phi\right) = A(\tilde{h} + \tilde{f}) \qquad A > 0$

At fixed points have $\left(\tilde{h} + \tilde{f}\right) > 0$, so spectrum is convex ✓

Gauge theories with semi-classical methods:

3-dimensional $U(N_c)$ gauge theory with N_f fermions flows to CFT in the IR

Has global symmetry $U(1)_{\text{top}}$ with monopole operators charged under it

At large N_f the dimensions of these can be computed as

q	Δ_q
1	$0.265\, N_f - 0.0383 + \mathcal{O}(1/N_f)$
2	$0.673\, N_f - 0.194 + \mathcal{O}(1/N_f)$
3	$1.186\, N_f - 0.422 + \mathcal{O}(1/N_f)$
4	$1.786\, N_f - 0.706 + \mathcal{O}(1/N_f)$
5	$2.462\, N_f - 1.04 + \mathcal{O}(1/N_f)$

✓

[Pufu ‘13]
[Dyer, Mezei, Pufu ’13]
[Chester, Iliesiu, Mezei, Pufu ‘17]

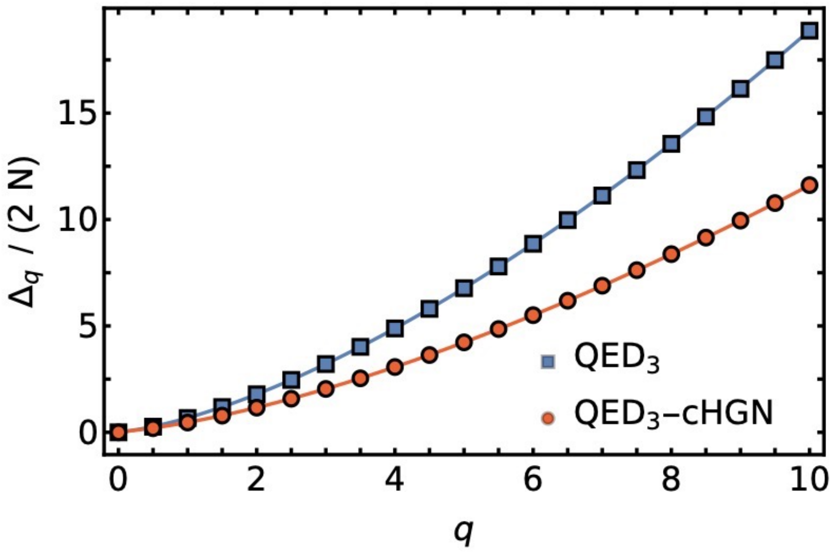
Can do similar computations for large N_f scalars (with or without quartic terms), and also for fermions with quartic couplings (Gross-Neveu type)

✓

q	$\mathbb{CP}^{N_f}$	tri-critical
1	0.125	0.097
2	0.311	0.226
3	0.544	0.384
4	0.816	0.567
5	1.121	0.771

[Dyer, Mezei, Pufu ’13]
[Dyer, Mezei, Pufu, Sachdev ’15]
[Chester – Private Communication]

✓



[Dupuis, Paranjape, Witczak-Krempa ’19]

Can also add Chern-Simons term with level k, and find that monopole spectrum becomes more convex monotonically with increasing k

✓

[Chester, Iliesiu, Mezei, Pufo ’17] [Chester ‘21] [Chester – Private Communication]

A simple supersymmetric counter-example to $q_0 = 1$:

Can consider single chiral superfield: $W = g\Phi^3$

In 3 dimensions flows to IR interacting fixed point (in 4 dimensions IR free):

Global $U(1)_R$ symmetry with charges and dimensions:

Scalar: $q_\phi = \frac{2}{3}$, $\Delta(2) = \frac{2}{3}$ Fermion: $q_\psi = \frac{1}{3}$, $\Delta(1) = \frac{7}{6}$

Requires taking $q_0 = 2$, and then convexity seems natural (but not proven) from BPS bound:

$$\Delta(2n) > n\,\Delta(2)$$

All our examples are consistent with strong versions of conjecture:

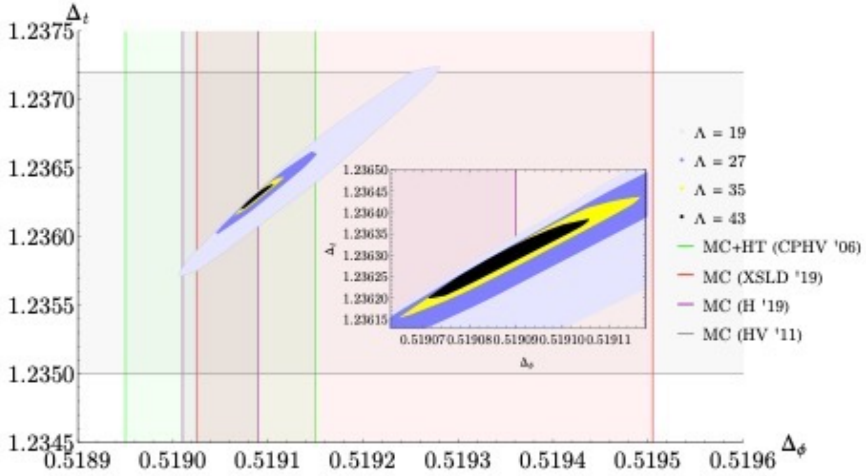
1) q_0 *is charge of lightest charged operator*
2) q_0 *is smallest scalar operator charge*

Have proposed counter examples on gravity side in flat space for supersymmetric theories with multiple U(1)s: [Heidenreich, Reece, Rudelius '16]

Can test in completely strongly-coupled theories: O(2) model in 3D

Q	ϵ^5	λ^6	MC	bootstrap
1	0.518(1)	-	0.5190(1)	0.5190(1)
2	1.234(3)	1.23(2)	1.236(1)	1.236(3)
3	2.10(1)	2.10(1)	2.108(2)	-
4	3.114(4)	3.103(8)	3.108(6)	-

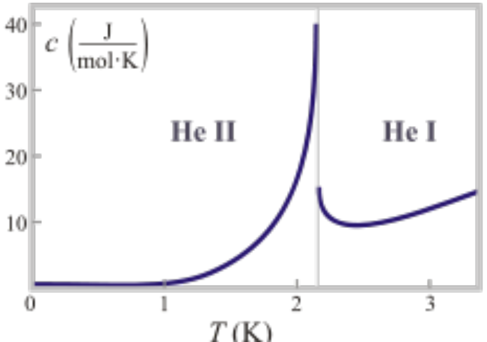
Table from: [Banerjee, Chandrasekharan, Orlando '18]



CFT data	method	value
Δ_s	EXP	1.50946(22)
	MC	1.51122(15)
	CB	1.51136(**22**)
Δ_ϕ	MC	0.519050(40)
	CB	0.519088(**22**)
Δ_t	MC	1.2361(11)
	CB	1.23629(**11**)

[Chester et al. '20]

Theory describing the superfluid transition in ^4He



Summary

Proposed that the natural formulation of the WGC is in terms of the self-binding energy of a particle

This leads to a CFT dual statement which is that the spectrum of charged operators should be convex

Seems to hold in all the examples we tested so far

In the absence of a general argument/proof, we need to keep testing it, searching for counter-examples (and learning what is special about them if they exist…)

Experimental predictions/tests?

Thank You

A simple supersymmetric counter-example to $q_0 = 1$:

Can consider single chiral superfield: $W = g\Phi^3$

In 3 dimensions flows to IR interacting fixed point, in 4 dimensions IR free

Global $U(1)_R$ symmetry with charges: Scalar $q_\phi = \frac{2}{3}$, fermion $q_\psi = -\frac{1}{3}$

Consider 4-dimensional free theory, then have

$$\Delta(2) = 1, \quad \Delta(1) = \frac{3}{2}$$

Requires taking $q_0 = 2$, and then convexity seems natural (but not proven) from BPS bound:

$$\Delta(2n) > n\,\Delta(2)$$